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# Robin's criterion on divisibility

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**Abstract** Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of $n$ and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers $n > 5040$ that are not divisible by some prime between 2 and 1771559. We prove that the Robin inequality holds when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ for some $n > 5040$ where $n'$ is the square free kernel of the natural number $n$. The possible smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{30.99733785}$, $1 < \frac{1.25 \times \log(4.7312714399)}{\log q_m} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.0501395952 \times \log(N_m)$ and $n < (4.7312714399)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order $m$, $q_m$ is the largest prime divisor of $n$ and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when $n$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

**Keywords** Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers · Riemann zeta function

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## 1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [3]. As usual $\sigma(n)$ is the sum-of-divisors function of $n$ [4]:

$$\sum_{d|n} d$$

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where $d \mid n$ means the integer $d$ divides $n$ and $d \nmid n$ means the integer $d$ does not divide $n$. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

**Theorem 1.1** $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [8].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers $n$. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [4]. On the one hand, we prove that $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by some prime between 3 and 1771559. We recall that an integer $n$ is said to be square free if for every prime divisor $q$ of $n$ we have $q^2 \nmid n$ [4].

**Theorem 1.2** $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [4].*

In addition, we show that $\mathsf{Robins}(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ such that $n'$ is the square free kernel of the natural number $n$. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first $m$ consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4]. A natural number $n$ is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

**Theorem 1.3** *If $n$ is superabundant, then $n$ is an Hardy-Ramanujan integer [2].*

**Theorem 1.4** *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

On the other hand, suppose that $n > 5040$ is the possible smallest counterexample of the Robin inequality, then we prove that $q_m > e^{30.99733785}$, $1 < \frac{1.25 \times \log(4.7312714399)}{\log q_m} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.0501395952 \times \log(N_m)$ and $n < (4.7312714399)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order $m$, $q_m$ is the largest prime divisor of $n$ and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ when $n$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

## 2 A Central Lemma

These are known results:

**Lemma 2.1** *[4]. For $n > 1$:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

**Lemma 2.2** *[5].*

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers $n$. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers $n$. Further the bound only uses the primes that divide $n$ and not how many times they divide $n$.

**Lemma 2.3** *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

*Proof* Putting together the lemmas 2.1 and 2.2 yields the proof:

$$f(n) < \prod_{i=1}^{m} \left(\frac{q_i}{q_i-1}\right) = \prod_{i=1}^{m} \left(\frac{q_i+1}{q_i} \times \frac{1}{1-\frac{1}{q_i^2}}\right) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

## 3 Robin on Divisibility

We know the following lemmas:

**Lemma 3.1** *[6]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i-1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

**Lemma 3.2** $\mathsf{Robins}(n)$ *holds for all natural numbers $10^{10^{13.11485}} \geq n > 5040$ [7].*

**Theorem 3.3** *Suppose $n > 5040$. If there exists a prime $q \leq 1771559$ with $q \nmid n$, then* $\mathsf{Robins}(n)$ *holds.*

*Proof* We have that $f(n) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log(n)$ for any number $n > 10^{10^{13.11485}}$ since the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q|n} \frac{q}{q-1}$ from the lemma 2.1, where $\varphi(x)$ is the Euler's totient function. Suppose that $n$ is not divisible by some prime $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \times q}{\varphi(n \times q)} \times \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times e^{\gamma} \times \log\log(n \times q) \end{aligned}$$

and

$$\begin{aligned}\frac{f(n)}{e^{\gamma} \times \log\log(n)} &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n \times q)}{\log\log(n)} \\ &\leq \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n \times q)}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n) + \log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right)\end{aligned}$$

So

$$\frac{f(n)}{e^{\gamma} \times \log\log(n)} < \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right)$$

for $n \geq 10^{10^{13.11485}}$. The right hand side is less than 1 for $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Therefore, $\mathsf{Robins}(n)$ holds.

## 4 On the Greatest Prime Divisor

We know that

**Lemma 4.1** *[9]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

**Theorem 4.2** *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $q_m > e^{30.99733785}$.*

*Proof* According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first $m$ consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 3.3, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the lemmas 2.1 and 4.1. Hence,

$$\log\log n - \frac{1}{2 \times \log(q_m)} < \log q_m.$$

However, from the lemma 3.2 and theorem 3.3, we would obtain that

$$\log\log n - \frac{1}{2\times\log(q_m)} \geq 13.11485\times\log(10)+\log\log 10 - \frac{1}{2\times\log(1771559)} > 30.99733785.$$

Since, we have that

$$\log q_m > \log\log n - \frac{1}{2\times\log(q_m)} > 30.99733785$$

then, we would obtain that $q_m > e^{30.99733785}$ under the assumption that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

## 5 Some Feasible Cases

In basic number theory, for a given prime number $p$, the $p$-adic order of a natural number $n$ is the highest exponent $\nu_p \geq 0$ such that $p^{\nu_p}$ divides $n$. This is a known result:

**Lemma 5.1** *In general, we know that* $\mathsf{Robins}(n)$ *holds for a natural number* $n > 5040$ *that satisfies* $\nu_2(n) \leq 19$*, where* $\nu_p(n)$ *is the p-adic order of n [6].*

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

**Lemma 5.2** $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 7$*, where q is the largest prime divisor of n.*

*Proof* This is an immediate consequence of theorem 3.3.

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

**Theorem 5.3** *Let* $\frac{\pi^2}{6}\times\log\log n' \leq \log\log n$ *for some* $n > 5040$ *such that* $n'$ *is the square free kernel of the natural number n. Then* $\mathsf{Robins}(n)$ *holds.*

*Proof* Let $n'$ be the square free kernel of the natural number $n$, that is the product of the distinct primes $q_1,\ldots,q_m$. By assumption we have that

$$\frac{\pi^2}{6}\times\log\log n' \leq \log\log n.$$

For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [4]. However, $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the lemma 5.2. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma}\times\log\log n'$$

because of the theorem 1.2. By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6}\times\prod_{i=1}^{m}\frac{q_i+1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^{\gamma} \times \log\log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}$$

according to the formula $f(x)$ for the square free numbers [4].

## 6 On Possible Counterexample

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2 \times \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$.

**Lemma 6.1** *For every prime number $p_n > 2$, the sequence $Y_n$ is strictly decreasing.*

*Proof* This lemma is obvious.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers $p$ that are less than or equal to $x$.

**Lemma 6.2** *[9]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

We will prove another important inequality:

**Lemma 6.3** *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 286$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

*Proof* From the lemma 6.2, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned}\log\left(Y_m \times \theta(q_m)\right) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right).\end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{1}{2\times\log(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{1}{2\times\log(q_m)}}\right) \\ &= \frac{1}{2\times\log(q_m)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{1}{2\times\log(q_m)}).$$

Due to the lemma 4.1, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{1}{2\times\log(q_m)}) < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right)$$

when $q_m > 286$.

We use the following lemma:

**Lemma 6.4** *[6]. Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

The following theorems have a great significance, because these mean that the possible smallest counterexample of the Robin inequality greater than 5040 must be very close to its square free kernel.

**Theorem 6.5** *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $(\log n)^{\beta} < Y_m \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

*Proof* According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first $m$ consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 4.2, we know that necessarily $q_m > e^{30.99733785}$. From the lemma 6.4, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right)$$

because of the lemma 6.3 when $q_m > e^{30.99733785}$. If we multiply by $\prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$ the both sides of the previous inequality, then we obtain that

$$f(n) < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

If $n$ is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^{\gamma} \times \log\log n < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

because of

$$e^{\gamma} \times \log\log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \times \log\log n < \log\left(Y_m \times \log(N_m)\right)$$

which is equivalent to

$$(\log n)^{\beta} < Y_m \times \log(N_m)$$

where $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$. Therefore, the proof is done.

**Theorem 6.6** *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $(\log n)^{\beta} < 1.0501395952 \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

*Proof* From the theorem 4.2, we know that necessarily $q_m > e^{30.99733785}$. Using the theorem 6.5, we obtain that

$$(\log n)^{\beta} < 1.0501395952 \times \log(N_m)$$

due to the lemma 6.1 since we have that $Y_m < 1.0501395952$ when $q_m > e^{30.99733785}$.

**Theorem 6.7** *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $n < (4.7312714399)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

*Proof* According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first $m$ consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the lemma 6.3, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log(Y_m \times \theta(q_m)) = e^{\gamma} \times \log\log(N_m^{Y_m})$$

for $q_m > e^{30.99733785}$. In this way, if $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $n < N_m^{Y_m}$ since by the lemma 2.1 we have that

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m - 1} \times N_m$. We can check that $q_m^{Y_m - 1}$ is monotonically decreasing for all primes $q_m > e^{30.99733785}$. Certainly, the function

$$g(x) = x^{\left(\frac{e^{\frac{1}{2 \times \log(x)}}}{(1 - \frac{1}{\log(x)})} - 1\right)}$$

complies that its derivative is lesser than zero for all real numbers $x \geq e^{30.99733785}$. Consequently, we would have that

$$q_m^{Y_m - 1} < g(e^{30.99733785}) < 4.7312714399$$

for all primes $q_m > e^{30.99733785}$. Moreover, we would obtain that

$$q_m^{Y_m - 1} > q_j^{Y_m - 1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (4.7312714399)^m \times N_m$ since $N_m^{Y_m - 1} < (4.7312714399)^m$ when $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

We know the following results:

**Lemma 6.8** *[9]. For $x \geq 114$:*

$$\pi(x) < 1.25 \times \frac{x}{\log x}$$

*where $\pi(x)$ is the prime counting function.*

**Lemma 6.9** *If* $n > 5040$ *is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then* $p < \log n$ *where* $p$ *is the largest prime divisor of* $n$ *[4].*

**Theorem 6.10** *Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of* $n$ *as a product of primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$*. If* $n > 5040$ *is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then* $1 < \frac{1.25 \times \log(4.7312714399)}{\log q_m} + \frac{\log N_m}{\log n}$*, where* $N_m = \prod_{i=1}^{m} q_i$ *is the primorial number of order* $m$*.*

*Proof* Note that $n < (4.7312714399)^m \times N_m$ when $n$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. If we apply the logarithm to the both sides, then

$$\log n < \log(4.7312714399)^m + \log N_m$$

which is equivalent to

$$\log n < m \times \log(4.7312714399) + \log N_m.$$

According to the lemma 6.8, we have that

$$\log n < 1.25 \times \frac{q_m}{\log q_m} \times \log(4.7312714399) + \log N_m.$$

From the lemma 6.9, we would have

$$\log n < 1.25 \times \frac{\log n}{\log q_m} \times \log(4.7312714399) + \log N_m.$$

which is the same as

$$1 < \frac{1.25 \times \log(4.7312714399)}{\log q_m} + \frac{\log N_m}{\log n}$$

after of dividing by $\log n$.

## 7 Another Bound

This is a known result:

**Lemma 7.1** *[9]. For* $x > 1$*:*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x} \tag{7.1}$$

*where*

$$B = 0.2614972128\cdots$$

*denotes the (Meissel-)Mertens constant [9].*

We show another result:

**Lemma 7.2** *For* $x \geq 11$*, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12.$$

*Proof* Let's define $H = \gamma - B$. The lemma 7.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

## 8 On a Square Free Number

We know the following results:

**Lemma 8.1** *[4]. For $0 < a < b$:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{8.1}$$

**Lemma 8.2** *[4]. For $q > 0$:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{8.2}$$

We know from the theorem 1.2 that $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are square free.

**Lemma 8.3** *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

*such that $q_1 < q_2 < \cdots < q_m$ are odd prime numbers, $q_m \geq 11$ and $3 \nmid n$, then:*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

*Proof* By induction with respect to $\omega(n)$, that is the number of distinct prime factors of $n$ [4]. Put $\omega(n) = m$ [4]. We need to prove the assertion for those integers with $m = 1$. From a square free number $n$, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{8.3}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [4]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{8.4}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (8.4) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m-1$, with $m \geq 2$ and let us consider the assertion for those square free $n$ with $\omega(n) = m$ [4]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

*Case 1:* $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1 + 1) \times \cdots \times (q_{m-1} + 1) \times (q_m + 1) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by $(q_m + 1)$. We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m + 1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 8.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [4].

*Case 2:* $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note that $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 8.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

but this is true according to the lemma 7.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied.

## 9 Main Insight

The next result is a main insight.

**Theorem 9.1** *Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m$. When $q_m \geq 11$, $3 \nmid n$ and $2^{20} \mid n$, then*

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n.$$

*Proof* We need to prove that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n.$$

Using the formula (8.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of the natural number $n$ [4]. We know that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (8.3) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 8.3 when $3 \nmid \frac{n'}{2}$ and $q_m \geq 11$. To sum up, the proof is complete.

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