Exact Calculation of Ellipse Perimeter by Analytical Method

Abolfazl Soltani *

Department of Physics, University of Birjand, Birjand, Iran

***Email**: soltani.a.physics@gmail.com*

Abstract

As you know, we do not have any exact equation for calculating the perimeter of an ellipse. In this article, we obtain this equation analytically.

Keywords: Ellipse perimeter, Ellipse, Area of ellipsoid

Introduction

Consider the following equation:

$$r = \frac{ed}{1 - eCos\theta} \tag{1}$$

This equation is the equation of a conic section in the polar coordinate system, which is obtained using the directrix-focus property of conic sections based on Figure 1 [1],[2].

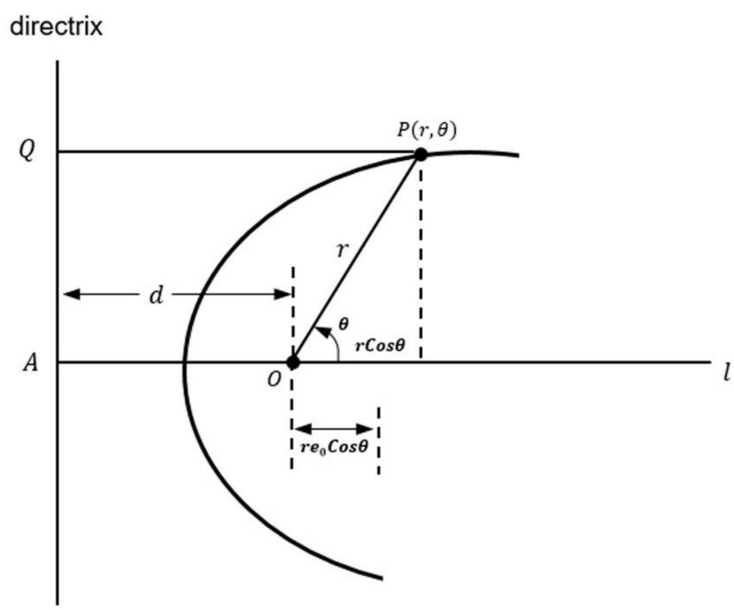

Fig. 1. This figure shows an ellipse as a conic section. Based on the directrix-focus property, we have: $|QP|/|OP| = e$. In an ellipse because of $0 < e < 1$ we have: $re_0 Cos\theta < rCos\theta$, which you can observe it in the figure.

In equation 1, d is the distance between a conic section directrix line and the focal point close to it [1],[2]. For an ellipse with Semi major axis $a = a_0$ and $e = e_0$ we have: $d = {a_0}/{e_0} - e_0 a_0$ [1]. substituting d, equation 1 for this ellipse is as follows [1]:

$$r = \frac{a_0(1 - {e_0}^2)}{1 - e_0 Cos\,\theta} \tag{2}$$

Now we want to obtain the ellipse perimeter using equation 2. Using the formula

$$L = \int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta \tag{3}$$

Which is the equation of calculation of the length of a polar curve [1],[3] and according to equation 2, we have:

$$L_{Ellipse} = a_0(1 - {e_0}^2) \int_0^{2\pi} \sqrt{\frac{(1 - e_0 Cos\theta)^2 + {e_0}^2 Sin^2\theta}{(1 - e_0 Cos\theta)^4}}\ d\theta \tag{4}$$

As you can see, the above integral is clearly difficult to solve (probably the above integral is a kind of elliptic integral). So we need to find the perimeter of the ellipse in another way. First we write equation 2 as follows:

$$r - re_0 Cos\,\theta = a_0(1 - {e_0}^2) \tag{5}$$

If we consider the left side of equation 5 equal to η, then we have

$$\eta = a_0(1 - {e_0}^2) \tag{6}$$

It is clear, in the above equation, η is a function of θ: $\eta = \eta(\theta)$ like $r = r(\theta)$. Inasmuch as $r^2(Sin^2\theta + Cos^2\theta) = a^2$ or $r = a$ is the equation of a circle, we can conclude that equation 6 is the equation of a circle with radius $a_0(1 - {e_0}^2)$. This means that equation 2 is both the equation of an ellipse and the equation of a circle. So to calculate the perimeter of an ellipse, we compute the perimeter of its equivalent circle namely equation 6, instead of calculating integral 4. Using integral 3 and equation 6, we have:

$$L_{Ellipse} = L_{Equivalent\ Circle} = \int_0^{2\pi} \eta d\theta = 2\pi a_0(1 - {e_0}^2) = 2\pi a_0 - 2\pi a_0 e_0^2 \tag{7}$$

The above equation is our ellipse perimeter equation. If you look at equation 7, according to figure 2, you will see that the first sentence of the right side (namely $2\pi a_0$), is the perimeter of a circle with radius a_0 (L_{c_2}). This circle surrounds the ellipse as shown in figure 2, and its center coincides with the center of the ellipse. Equation 7 shows that the perimeter of the ellipse is less than the perimeter of the circle C_2, as expected.

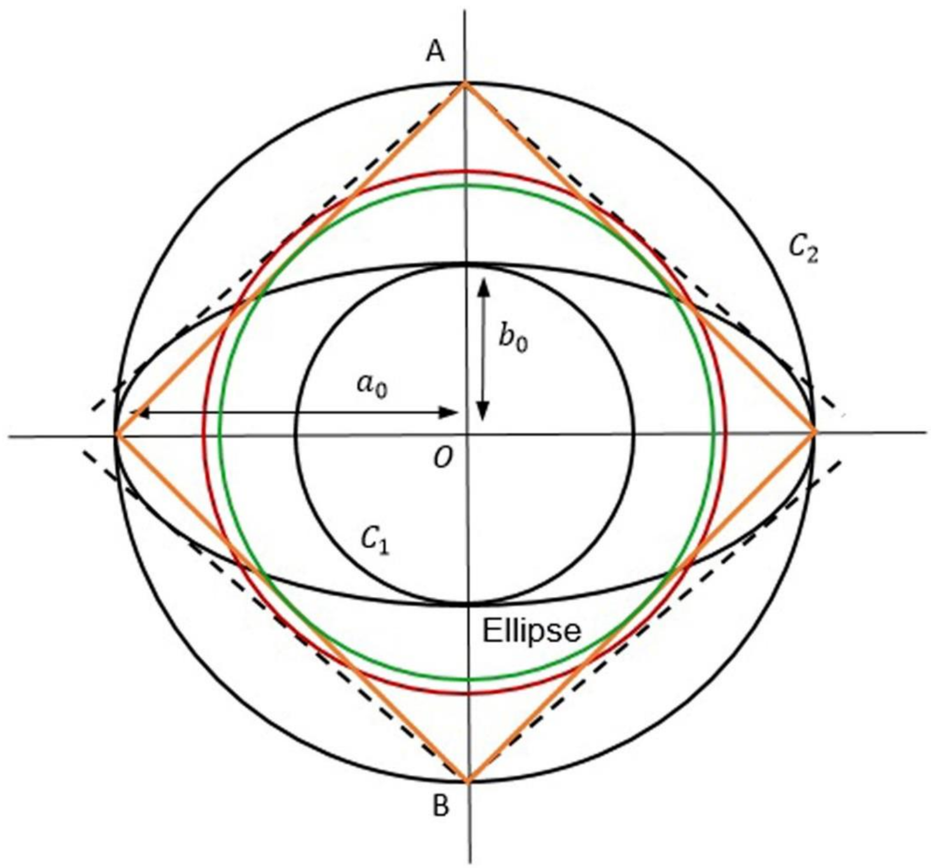

Fig. 2.

We can write equation 7 in another way

$$b_0 = a_0\sqrt{1-e_0^2} \implies L_{Ellipse} = 2\pi \frac{b_0^2}{a_0} \qquad (8)$$

Where b_0 is the Semi minor axis of the ellipse.

Where is the location of $C_{equivalent}$ in Fig. 2? In an ellipse we have: $0 < e_0 < 1$. Therefore:

$$a_0\sqrt{1-e_0^2} < a_0(1-{e_0}^2) < a_0$$

$$\overset{\times 2\pi}{\implies} 2\pi a_0\sqrt{1-e_0^2} < 2\pi a_0(1-{e_0}^2) < 2\pi a_0 \implies L_{c_1} < (L_{c_{equivalent}} = L_{Ellipse}) < L_{c_2} \qquad (9)$$

As shown in Figure 2, four tangent lines to the ellipse can be drawn from points A and B. These four lines are also tangent to the red circle. On the other hand, there is a green circle that is tangent to the orange square faces. As shown in Figure 2, the perimeters of both green and red circles are greater than the perimeter of circle C_1 and less than the perimeter of circle C_2. So the inequality 9 is true about them, and therefore, **probably**, **one of** the red or green circles is the circle $C_{equivalent}$.

Of course, **maybe none** of them. Many other circles can be drawn with center of O to satisfy inequality 9. We only guessed here that maybe one of the two green and red circles is our circle $C_{equivalent}$.

Finally, I need to point out that the area of the circle $C_{equivalent}$ and its corresponding ellipse are not equal:

$$A_E = \pi a_0 b_0 = \pi a_0^2 \sqrt{1 - e_0^2} \quad and \quad A_{C_{equivalent}} = \pi \eta^2 = \pi a_0^2 (1 - e_0^2)^2 \quad \Longrightarrow A_{C_{equivalent}} \neq A_E$$

Conclusion

It seems that after more than 300 years, we have been able to obtain the exact equation of the perimeter of an ellipse. I think using the method of this article can also lead us to the exact equation of the area of an ellipsoid.

References:

[1]. Silverman, R. *Calculus with Analytic Geometry* (Prentice-Hall, Inc. New Jersey. 1985), pp. 628-641
[2]. Johnson, R. *Calculus with Analytic Geometry* (Allyn and Bacon. Inc. Boston, ed. 4), pp. 536-539
[3]. Sherman, S. *Calculus with Analytic Geometry* (McGraw-Hill, Inc., ed. 3, 1982), pp. 548-554