Equations of Prolate Spheroid in Spherical Polar Coordinates and its Surface Area by Analytical Method

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Abstract

As you know we do not have any exact equation for calculation Prolate Spheroid surface area. Here we will obtain this equation by analytical method. Moreover, here we will obtain the prolate spheroid equation, when the center is a focal point of prolate spheroid.

Keywords: Prolate spheroid, Ellipsoid, Oblate spheroid, Surface area

Prolate Spheroid Equation

As you know, a prolate spheroid with scales $a = a_0 > (c_0 = b_0)$ is obtained from the whole rotation of an ellipse with semi-major axis a_0 and semi minor axis b_0 around the a_0 axis. Now we want to obtain the equation of this prolate spheroid in spherical coordinates. Consider Figure 1. In this figure, a prolate spheroid will draw from the complete rotation of the blue ellipse with scales a_0 and b_0 around the x-axis.
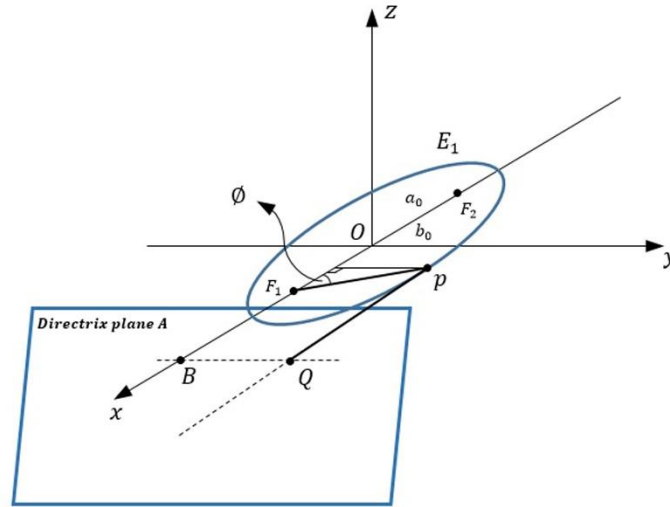

Fig. 1. As shown in the figure, the blue ellipse is on the horizontal plane xy. From the rotation of the blue ellipse around the x-axis, a prolate spheroid will build. The plane A is located at a vertical distance $\frac{a}{\epsilon}$ from the origin O and is perpendicular to the x axis and we call it the *spheroid Directrix plane*.

In the ellipse in figure 1 we have [1][2]:

$$\frac{F_1 P}{QP} = \frac{r}{d + r Sin\theta Cos\emptyset} = \epsilon_0 \tag{1}$$

Where ϵ_0 is eccentricity of ellipse. In this equation, QP is the vertical distance from point P to the *spheroid directrix plane*, i.e. plane A. This plane is perpendicular to the x-axis and its vertical distance to point O is equal to a_0 / ϵ_0. d is the distance between F_1 and B (point B is

on the plane A) and we have: $d = \frac{a_0}{\epsilon_0} - \epsilon_0 a_0$. Equation 1 stands for all the points P on the surface of the prolate spheroid, due to we can build a prolate spheroid by rotation of the ellipse in Fig. 1 around x-axis. In equation 1, $r\sin\theta\cos\emptyset$ is the shadow of $F_1 P$ line on the x axis and is written in spherical polar coordinates $(r.\theta.\emptyset)$ based on figure 2, because the spheroid is a three-dimensional object. $d = \frac{a_0}{\epsilon_0} - \epsilon_0 a_0$ and therefore, from 1 we have:

$$r - r\epsilon_0 Sin\theta Cos\emptyset = a_0\left(1 - \epsilon_0^2\right) \tag{2}$$

This is equation of prolate spheroid in spherical polar coordinates with center of F_1.
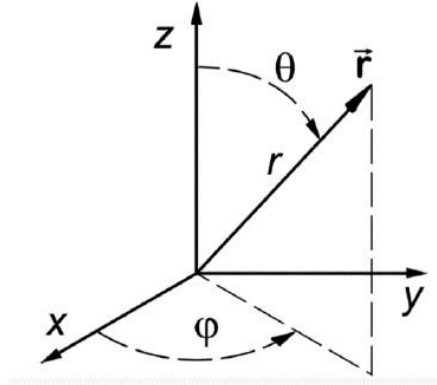

Fig. 2. Spherical polar coordinates

Prolate Spheroid Surface Area

Consider a prolate spheroid with scales $a = a_0 > (c_0 = b_0)$. In the same way as the recent preprint [3] can be shown that:

$$b_0 \rightarrow 0 \ and \ a_0 > c_0 \gg b_0 \Rightarrow S = 4\pi(\frac{a_0}{\sqrt{2}})^2$$

$$a_0 = b_0 = c_0 \Rightarrow S = 4\pi a_0^2$$

$$a_0 > (c_0 = b_0) \Rightarrow S = 4\pi(\frac{a_0}{\sqrt{2}^{\epsilon_0^2}})^2$$

In situation $b \rightarrow 0 \ and \ a_0 > c_0 \gg b_0$ our prolate spheroid is a flattened spheroid. In figure 3 we show a prolate spheroid with $a_0 = 5 \ cm, b_0 = c_0 = 3 \ cm$ and its associated sphere. As you can observe its clear the surface area of our prolate spheroid is equal to its associated sphere.

$$\epsilon_0 = \frac{\sqrt{a_0^2 - b_0^2}}{a_0} = \frac{\sqrt{25 - 9}}{5} = 0.8 \Rightarrow S_{Prolate \ spheroid} = S_{Associated \ sphere} = 4\pi(\frac{a_0}{\sqrt{2}^{\epsilon_0^2}})^2 = 204 \ cm^2$$

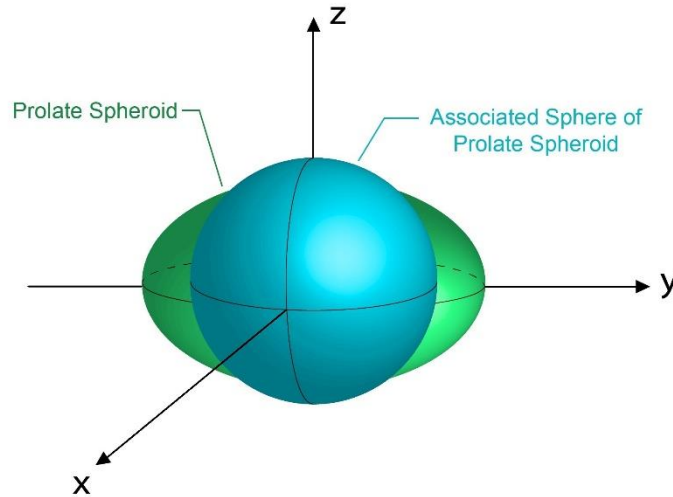The radius of associated sphere is $\frac{a_0}{\sqrt{2}^{\epsilon_0^2}} = 0.8 \ a_0 = 04.03 \ cm$

Fig. 3. A prolate spheroid and its associated sphere. Center of spheroid and sphere are on the center of cartesian coordinates system.

Therefore, the final form of prolate spheroid surface area is:

$$S = 4\pi\left(\frac{a_0}{\sqrt{2}^{\frac{c_0^2}{c_0^2}}}\right)^2 \qquad (3)$$

Conclusion

In this article we used from the concept of *Directrix Plane of a prolate spheroid* to obtain the equation of surface area of prolate spheroid. It's clear this concept is useable for ellipsoid and subsequently oblate spheroid. *Directrix Plane* seems to be entering Geometry and Mathematics for the first time.

References:

[1]. Silverman, R. *Calculus with Analytic Geometry* (Prentice-Hall, Inc. New Jersey. 1985), pp. 628-641

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[3]. Soltani, A. Exact Equation of Ellipse Perimeter by Analytical Method, *Cambridge Open Engage*, DOI: https://doi.org/10.33774/coe-2022-rskdp-v3