Noname manuscript No.
(will be inserted by the editor)

Robin's criterion on divisibility

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the date of receipt and acceptance should be inserted later

Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers $n > 5040$ that are not divisible by some prime between 2 and 1771559. We prove that the Robin inequality holds when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ for some $n > 5040$ where n' is the square free kernel of the natural number n. The possible smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$, $1 < \frac{(1+\frac{1.2762}{\log q_m})\times\log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$ and $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers · Riemann zeta function

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real

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part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [9].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [4]. We extend the indivisibility property on the following result:

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are not divisible by some prime between* 3 *and* 1771559.

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.3 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [4].*

In addition, we show that $\mathsf{Robins}(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ such that n' is the square free kernel of the natural number n. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Theorem 1.4 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Theorem 1.5 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

Suppose that $n > 5040$ is the possible smallest counterexample of the Robin inequality, then we prove that $q_m > e^{31.018189471}$, $1 < \frac{(1+\frac{1.2762}{\log q_m})\times\log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$ and $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n is an Hardy-Ramanujan integer of the form $\prod_{i=1}^{m} q_i^{a_i}$.

2 A Central Lemma

These are known results:

Lemma 2.1 *[4]. For* $n > 1$*:*

$$f(n) < \prod_{q|n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let* $n > 1$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

Proof Putting together the lemmas 2.1 and 2.2 yields the proof:

$$f(n) < \prod_{i=1}^{m} \left(\frac{q_i}{q_i-1}\right) = \prod_{i=1}^{m} \left(\frac{q_i+1}{q_i} \times \frac{1}{1-\frac{1}{q_i^2}}\right) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

3 Robin on Divisibility

We know the following lemmas:

Lemma 3.1 *[7]. Let* $n > e^{e^{23.762143}}$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i-1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Lemma 3.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $10^{10^{13.11485}} \geq n > 5040$ *[8].*

Theorem 3.3 *Suppose* $n > 5040$*. If there exists a prime* $q \leq 1771559$ *with* $q \nmid n$*, then* $\mathsf{Robins}(n)$ *holds.*

Proof We have that $f(n) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log(n)$ for any number $n > 10^{10^{13.11485}}$ since the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q|n} \frac{q}{q-1}$

from the lemma 2.1, where $\varphi(x)$ is the Euler's totient function. Suppose that n is not divisible by some prime $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \times q}{\varphi(n \times q)} \times \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times e^{\gamma} \times \log\log(n \times q) \end{aligned}$$

and

$$\begin{aligned} \frac{f(n)}{e^{\gamma} \times \log\log(n)} &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n \times q)}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n) + \log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)} \\ &= \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right) \end{aligned}$$

So

$$\frac{f(n)}{e^{\gamma} \times \log\log(n)} < \frac{1771561}{1771560} \times \frac{q-1}{q} \times \left(1 + \frac{\log(1 + \frac{\log(q)}{\log(n)})}{\log\log(n)}\right)$$

for $n \geq 10^{10^{13.11485}}$. The right hand side is less than 1 for $q \leq 1771559$ and $n \geq 10^{10^{13.11485}}$. Therefore, $\mathsf{Robins}(n)$ holds.

4 On the Greatest Prime Divisor

We know that

Lemma 4.1 *[6]. For $x \geq 2973$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{0.2}{\log(x)}).$$

Theorem 4.2 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $q_m > e^{31.018189471}$.*

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 3.3, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{0.2}{\log(q_m)})$$

because of the lemmas 2.1 and 4.1. Hence,

$$\log\log n - \frac{0.2}{\log(q_m)} < \log q_m.$$

However, from the lemma 3.2 and theorem 3.3, we would obtain that

$$\log\log n - \frac{0.2}{\log(q_m)} \geq 13.11485 \times \log(10) + \log\log 10 - \frac{0.2}{\log(1771559)}$$
$$> 31.018189471.$$

Since, we have that

$$\log q_m > \log\log n - \frac{0.2}{\log(q_m)} > 31.018189471$$

then, we would obtain that $q_m > e^{31.018189471}$ under the assumption that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

5 Some Feasible Cases

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 5.1 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 7$*, where* q *is the largest prime divisor of* n*.*

Proof This is an immediate consequence of theorem 3.3.

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

Theorem 5.2 *Let* $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ *for some* $n > 5040$ *such that* n' *is the square free kernel of the natural number* n*. Then* $\mathsf{Robins}(n)$ *holds.*

Proof Let n' be the square free kernel of the natural number n, that is the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [4]. However, $\mathsf{Robins}(n)$ holds for all $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the lemma 5.1. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma} \times \log\log n'$$

because of the theorem 1.3. By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

So,

$$\begin{aligned} f(n) &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \\ &= \frac{\pi^2}{6} \times f(n') \\ &< \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n' \\ &\leq e^{\gamma} \times \log\log n \end{aligned}$$

according to the formula $f(x)$ for the square free numbers [4].

6 On Possible Counterexample

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{0.2}{\log^2(p_n)}}}{(1 - \frac{1}{\log(p_n)})}$.

Lemma 6.1 *As the prime number p_n increases, the sequence Y_n is strictly decreasing.*

Proof This lemma is obvious.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. We know that

Lemma 6.2 *[10]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Lemma 6.3 *[3]. For $x \geq 2278382$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{0.2}{\log^2(x)}).$$

We will prove another important inequality:

Lemma 6.4 *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 2278382$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

Proof From the lemma 6.2, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned} \log\left(Y_m \times \theta(q_m)\right) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right). \end{aligned}$$

We know that

$$\begin{aligned} \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{0.2}{\log^2(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{0.2}{\log^2(q_m)}}\right) \\ &= \frac{0.2}{\log^2(q_m)}. \end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{0.2}{\log^2(q_m)}).$$

Due to the lemma 6.3, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{0.2}{\log^2(q_m)}) < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right)$$

when $q_m > 2278382$.

We use the following lemma:

Lemma 6.5 *[7]. Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i + 1}}\right).$$

The following theorems have a great significance, because these mean that the possible smallest counterexample of the Robin inequality greater than 5040 must be very close to its square free kernel.

Theorem 6.6 *Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$*. If* $n > 5040$ *is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then* $(\log n)^{\beta} < Y_m \times \log(N_m)$*, where* $N_m = \prod_{i=1}^{m} q_i$ *is the primorial number of order m and* $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 4.2, we know that necessarily $q_m > e^{31.018189471}$. From the lemma 6.5, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right)$$

because of the lemma 6.4 when $q_m > 2278382$. If we multiply by $\prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$ the both sides of the previous inequality, then we obtain that

$$f(n) < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

If n is the smallest integer exceeding 5040 that does not satisfy the Robin inequality, then

$$e^{\gamma} \times \log\log n < e^{\gamma} \times \log\left(Y_m \times \log(N_m)\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

because of

$$e^{\gamma} \times \log\log n \leq f(n).$$

That is the same as

$$\prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \times \log\log n < \log\left(Y_m \times \log(N_m)\right)$$

which is equivalent to

$$(\log n)^{\beta} < Y_m \times \log(N_m)$$

where $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$. Therefore, the proof is done.

Theorem 6.7 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and $\beta = \prod_{i=1}^{m} \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof From the theorem 4.2, we know that necessarily $q_m > e^{31.018189471}$. Using the theorem 6.6, we obtain that

$$(\log n)^{\beta} < 1.03352795481 \times \log(N_m)$$

due to the lemma 6.1 since we have that $Y_m < 1.03352795481$ when $q_m > e^{31.018189471}$.

Theorem 6.8 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold, then $n < (2.82915040011)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

Proof According to the theorems 1.4 and 1.5, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. From the lemma 6.4, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right) = e^{\gamma} \times \log\log(N_m^{Y_m})$$

for $q_m > 2278382$. In this way, if $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $n < N_m^{Y_m}$ since by the lemma 2.1 we have that

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m - 1} \times N_m$. We can check that $q_m^{Y_m - 1}$ is monotonically decreasing for all primes $q_m > e^{31.018189471}$. Certainly, the derivative of the function

$$g(x) = x^{\left(\frac{e^{\frac{0.2}{\log^2(x)}}}{(1 - \frac{1}{\log(x)})} - 1\right)}$$

is less than zero for all real numbers $x \geq e^{31.018189471}$. Consequently, we would have that

$$q_m^{Y_m - 1} < g(e^{31.018189471}) < 2.82915040011$$

for all primes $q_m > e^{31.018189471}$. Moreover, we would obtain that

$$q_m^{Y_m - 1} > q_j^{Y_m - 1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (2.82915040011)^m \times N_m$ since $N_m^{Y_m - 1} < (2.82915040011)^m$ when $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

We know the following results:

Lemma 6.9 *[5]. For $x > 1$:*

$$\pi(x) \leq (1 + \frac{1.2762}{\log x}) \times \frac{x}{\log x}$$

where $\pi(x)$ is the prime counting function.

Lemma 6.10 *If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $p < \log n$ where p is the largest prime divisor of n [4].*

Theorem 6.11 *Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. If $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold, then $1 < \frac{(1+\frac{1.2762}{\log q_m}) \times \log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

Proof Note that $n < (2.82915040011)^m \times N_m$ when n is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. If we apply the logarithm to the both sides, then

$$\log n < m \times \log(2.82915040011) + \log N_m.$$

According to the lemma 6.9, we have that

$$\log n < (1 + \frac{1.2762}{\log q_m}) \times \frac{q_m}{\log q_m} \times \log(2.82915040011) + \log N_m.$$

From the lemma 6.10, we would have

$$\log n < (1 + \frac{1.2762}{\log q_m}) \times \frac{\log n}{\log\log n} \times \log(2.82915040011) + \log N_m.$$

which is the same as

$$1 < \frac{(1 + \frac{1.2762}{\log q_m}) \times \log(2.82915040011)}{\log\log n} + \frac{\log N_m}{\log n}$$

after of dividing by $\log n$.

Acknowledgments

The author would like to thank Richard J. Lipton and Craig Helfgott for helpful comments and his mother, maternal brother and his friend Sonia for their support. The author also wishes to thank the referees for their constructive comments and suggestions.

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