

# On Robin's criterion for the Riemann Hypothesis

Frank Vega

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**Abstract** Robin's criterion states that the Riemann hypothesis is true if and only if the inequality  $\sigma(n) < e^\gamma \times n \times \log \log n$  holds for all natural numbers  $n > 5040$ , where  $\sigma(n)$  is the sum-of-divisors function of  $n$  and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. In 2022, Vega stated that the possible existence of the smallest counterexample  $n > 5040$  of the Robin inequality implies that  $q_m > e^{31.018189471}$  and  $(\log n)^\beta < 1.03352795481 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$ ,  $q_m$  is the largest prime divisor of  $n$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$  when  $n$  must be an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$ . Based on that result, we obtain a contradiction just assuming the existence of such possible smallest counterexample  $n > 5040$  for the Robin inequality. By contraposition, we show that the Riemann hypothesis should be true.

**Keywords** Riemann hypothesis · Robin inequality · Sum-of-divisors function · Prime numbers · Counterexample

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## 1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$ :

$$\sum_{d|n} d$$

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F. Vega  
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France  
ORCID: 0000-0001-8210-4126  
E-mail: vega.frank@gmail.com

where  $d \mid n$  means the integer  $d$  divides  $n$  and  $d \nmid n$  means the integer  $d$  does not divide  $n$ . Define  $f(n)$  to be  $\frac{\sigma(n)}{n}$ . Say Robins( $n$ ) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\log$  is the natural logarithm. The importance of this property is:

**Theorem 1.1** Robins( $n$ ) holds for all natural numbers  $n > 5040$  if and only if the Riemann hypothesis is true [3].

It is known that Robins( $n$ ) holds for many classes of numbers  $n$ . We recall that an integer  $n$  is said to be square free if for every prime divisor  $q$  of  $n$  we have  $q^2 \nmid n$ .

**Theorem 1.2** Robins( $n$ ) holds for all natural numbers  $n > 5040$  that are square free [1].

Let  $q_1 = 2, q_2 = 3, \dots, q_m$  denote the first  $m$  consecutive primes, then an integer of the form  $\prod_{i=1}^m q_i^{a_i}$  with  $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$  is called an Hardy-Ramanujan integer [1]. Now, we are able to use this recently result:

**Theorem 1.3** The possible existence of the smallest counterexample  $n > 5040$  of the Robin inequality implies that  $q_m > e^{31.018189471}$  and  $(\log n)^\beta < 1.03352795481 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$ ,  $q_m$  is the largest prime divisor of  $n$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$  when  $n$  must be an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$  [4].

Putting all together yields a proof for the Riemann hypothesis using the Theorem 1.3 as the principal argument.

## 2 Known Results

These are known results:

**Lemma 2.1** For every  $x > -1$  [2]:

$$\log(1+x) \geq \frac{x}{x+1}.$$

**Lemma 2.2** For every real number  $x$  [2]:

$$e^x \geq 1+x.$$

**Lemma 2.3** For every  $x > -1$  [2]:

$$\frac{\log(1+x)}{x} \geq \frac{2}{x+2}.$$

### 3 A Central Lemma

The following is a key Lemma.

**Lemma 3.1** *If the natural number  $n > 5040$  is an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$ , then  $\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$  where  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ .*

*Proof* If we apply the logarithm to the value of

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$$

then we obtain that

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right).$$

For some  $1 \leq j \leq m$ , we know that

$$\frac{q_j^{a_j+1}}{q_j^{a_j+1}-1} = 1 + \frac{1}{q_j^{a_j+1}-1}.$$

We use the Lemma 2.1 to show that

$$\begin{aligned} \log\left(1 + \frac{1}{q_j^{a_j+1}-1}\right) &\geq \frac{\frac{1}{q_j^{a_j+1}-1}}{\frac{1}{q_j^{a_j+1}-1} + 1} \\ &= \frac{1}{(q_j^{a_j+1}-1) \times \left(\frac{1}{q_j^{a_j+1}-1} + 1\right)} \\ &= \frac{1}{1 + (q_j^{a_j+1}-1)} \\ &= \frac{1}{q_j^{a_j+1}}. \end{aligned}$$

So,

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right) \geq \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and thus,

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \geq e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}}.$$

Using the Lemma 2.2, we have that

$$e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and therefore,

$$\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}.$$

#### 4 Main Insight

This is the main insight.

**Lemma 4.1** *Suppose that  $n > 5040$  is an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$  and  $q_m > e^{31.018189471}$ . Then  $(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1.03352795481$ .*

*Proof* If we apply the logarithm to the both sides of the inequality, then

$$\left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \log \log n \geq \log(1.03352795481).$$

Let's multiply the both sides of the inequality by  $e^\gamma$ ,

$$\left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times e^\gamma \times \log \log n \geq e^\gamma \times \log(1.03352795481).$$

From the Theorem 1.2, we know that

$$\begin{aligned} e^\gamma \times \log \log n &\geq e^\gamma \times \log \log N_m \\ &> f(N_m) \\ &= \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \end{aligned}$$

since  $n > 5040$  is an Hardy-Ramanujan integer,  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$  and thus,  $n \geq N_m$  and  $N_m$  is square free. Hence, we would have that

$$\left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \geq e^\gamma \times \log(1.03352795481).$$

If we apply the logarithm to the both sides again, then

$$\log \left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) + \sum_{i=1}^m \log \left(1 + \frac{1}{q_i}\right) \geq \log(e^\gamma \times \log(1.03352795481)).$$

We use the Lemma 2.3 to show that

$$\begin{aligned}
 \log \left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) &= \log \left( 1 + (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) \right) \\
 &\geq \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{(-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) + 2} \\
 &= \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \\
 &> 2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) \\
 &= -2 + 2 \times \left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right)
 \end{aligned}$$

since

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} > -1.$$

For some  $1 \leq j \leq m$ , we know that

$$\begin{aligned}
 \log(1 + \frac{1}{q_j}) &\geq \frac{\frac{1}{q_j}}{\frac{1}{q_j} + 1} \\
 &= \frac{1}{q_j \times (\frac{1}{q_j} + 1)} \\
 &= \frac{1}{1 + q_j}
 \end{aligned}$$

according to the Lemma 2.1. However, we note that

$$-2 + 2 \times \left( \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) + \sum_{i=1}^m \frac{1}{1 + q_i} > 0$$

when  $q_m > e^{31.018189471}$ . In addition, we have that

$$0 > \log(e^\gamma \times \log(1.03352795481))$$

and finally, the proof is complete.

## 5 Main Theorem

We conclude with the following statement:

**Theorem 5.1** *The Riemann hypothesis is true.*

*Proof* Suppose that  $n > 5040$  is the possible smallest number such that  $\text{Robins}(n)$  does not hold. By the Theorem 1.3, we know that  $q_m > e^{31.018189471}$  and  $(\log n)^\beta < 1.03352795481 \times \log(N_m)$ , where  $N_m = \prod_{i=1}^m q_i$  is the primorial number of order  $m$ ,  $q_m$  is the largest prime divisor of  $n$  and  $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$  when  $n$  must be an Hardy-Ramanujan integer of the form  $\prod_{i=1}^m q_i^{a_i}$ . From the Lemma 3.1, we know that

$$(\log n)^\beta \geq (\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)}$$

and therefore, we would have that

$$(\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)} < 1.03352795481 \times \log(N_m)$$

when  $n > 5040$  is the possible smallest number such that  $\text{Robins}(n)$  does not hold. Thus, we would obtain that

$$(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} < 1.03352795481$$

since  $n$  must be an Hardy-Ramanujan integer and so,  $\log n \geq \log N_m$ . However, we know the previous inequality cannot be satisfied because of the Lemma 4.1. By contraposition, we show that the Riemann hypothesis is true, since we obtain a contradiction just assuming the possible smallest counterexample for the Robin inequality greater than 5040. Certainly, this is a direct consequence of the Theorem 1.1.

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