A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract. We prove the Riemann hypothesis. We also prove that every zero of $\zeta(s)$ is simple.

1. Introduction

The Riemann zeta function $\zeta(s)$ is defined on $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$. It is analytically continued to a meromorphic function on the whole plane with a pole at $s = 1$. It is well-known that negative even integers are zeros of $\zeta(s)$. Other zeros of $\zeta(s)$ are called complex zeros or nontrivial zeros.

Riemann stated the following statement, the so-called Riemann hypothesis, in [6] in 1859.

The Riemann hypothesis. All nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$.

Hilbert listed it as the eighth problem of his 23 problems in his 1900 address to the Paris International Congress of Mathematicians. This is one of the most important unsolved problems in the twenty-first century.

Nobody has succeeded to prove it up to the present, but many computational results are known. In the early part of the twentieth century, they were obtained by hand computation ([1], [4], [5] and [7]). Numerical computations by computers have permitted us to check the truth of the Riemann hypothesis to extremely large t. We refer to [3] for a history of numerical verifications.

We will obtain further results with the development of computers and numerical methods. However, we can never reach to the goal in

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this way. We have to analyze the behaviour of the zeta function or the xi function for larger t than a fixed t_0. For this purpose, we expand the real part and the imaginary part of the integration in a representation of the xi function into sums of powers of $1/t$ in Section 3. The precise representations of these parts are given in Section 6. We derive an equation which is satisfied if s is a zero of the xi function. We give a new zero-free region of $\zeta(s)$ by the above expansions and equation in Section 9. Let B be the supremum of the real parts of nontrivial zeros. Combining our new zero-free region with the theorem of de la Vallée Poussin, we first show $\frac{1}{2} \leq B < 1$. In Section 10 we give a lemma on Dirichlet series. It plays an important role to show that if $\frac{1}{2} < B < 1$, then there is no zero of $\zeta(s)$ on the line $\sigma = B$. We conclude $B = \frac{1}{2}$ by this fact and our new zero-free region. In the last section, we also prove the simpleness of zeros.

2. Preliminaries

The xi function $\xi(s)$ of Riemann is defined by

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$. It also satisfies $\xi(\overline{s}) = \overline{\xi(s)}$. The set of nontrivial zeros of $\zeta(s)$ coincides with the set of zeros of $\xi(s)$. Therefore, if s is a nontrivial zero of $\zeta(s)$, then $\overline{s}, 1-s$ and $1-\overline{s}$ are also zeros of $\zeta(s)$.

The function $\xi(s)$ has the following integral representation

$$\xi(s) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx, \tag{2.1}$$

where $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ (for example, see p.22 in [8]).

Let

$$\Phi(s) = \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx.$$

We denote by $R(s) = \mathrm{Re}\Phi(s)$ and $I(s) = \mathrm{Im}\Phi(s)$ the real part and the imaginary part of $\Phi(s)$, respectively. Since

$$\begin{cases} \mathrm{Re}\xi(s) = \dfrac{1}{2} + \dfrac{1}{2}\mathrm{Re}\left(s(s-1)\Phi(s)\right), \\ \mathrm{Im}\xi(s) = \dfrac{1}{2}\mathrm{Im}\left(s(s-1)\Phi(s)\right), \end{cases}$$

we have that $\xi(s) = 0$ if and only if

$$\begin{cases} 1 + (\sigma(\sigma-1) - t^2)R(s) - t(2\sigma-1)I(s) = 0, \\ \quad t(2\sigma-1)R(s) + (\sigma(\sigma-1) - t^2)I(s) = 0. \end{cases} \tag{2.2}$$

Solving (2.2) for $R(s)$ and $I(s)$, we obtain

$$\begin{cases} R(s) = \dfrac{t^2 - \sigma(\sigma-1)}{(\sigma(\sigma-1) - t^2)^2 + t^2(2\sigma-1)^2}, \\ I(s) = \dfrac{t(2\sigma-1)}{(\sigma(\sigma-1) - t^2)^2 + t^2(2\sigma-1)^2}. \end{cases} \tag{2.3}$$

Therefore, $\xi(s) = 0$ if and only if (2.3) holds.

We also consider $\xi'(s) = 0$. Differentiating (2.1), we obtain

$$\xi'(s) = \left(s - \frac{1}{2}\right)\Phi(s) + \frac{s}{2}(s-1)\Phi'(s), \tag{2.4}$$

where

$$\Phi'(s) = \frac{1}{2}\int_1^\infty \log x \left(x^{\frac{s}{2}-1} - x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)dx. \tag{2.5}$$

If we set $U(s) = \mathrm{Re}\Phi'(s)$ and $V(s) = \mathrm{Im}\Phi'(s)$, then we have

$$\begin{cases} \mathrm{Re}\xi'(s) = \left(\sigma - \dfrac{1}{2}\right) R(s) - tI(s) + \dfrac{1}{2}\left\{\sigma(\sigma-1) - t^2\right\} U(s) \\ \qquad\qquad - \dfrac{1}{2}t(2\sigma-1)V(s), \\ \mathrm{Im}\xi'(s) = tR(s) + \left(\sigma - \dfrac{1}{2}\right) I(s) + \dfrac{1}{2}t(2\sigma-1)U(s) \\ \qquad\qquad + \dfrac{1}{2}\left\{\sigma(\sigma-1) - t^2\right\} V(s) \end{cases}$$

by (2.4). Therefore, $\xi'(s) = 0$ if and only if

$$\begin{cases} \left(\sigma - \dfrac{1}{2}\right) R(s) - tI(s) + \dfrac{1}{2}\left\{\sigma(\sigma-1) - t^2\right\} U(s) \\ \qquad - \dfrac{1}{2}t(2\sigma-1)V(s) = 0, \\ tR(s) + \left(\sigma - \dfrac{1}{2}\right) I(s) + \dfrac{1}{2}t(2\sigma-1)U(s) \\ \qquad + \dfrac{1}{2}\left\{\sigma(\sigma-1) - t^2\right\} V(s) = 0. \end{cases} \tag{2.6}$$

3. Expansions of $R(s)$, $I(s)$ and $V(s)$

We have

$$R(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{-\frac{\sigma}{2}-\frac{1}{2}}) \cos\left(\frac{t}{2}\log x\right)\psi(x)dx$$

and

$$I(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} - x^{-\frac{\sigma}{2}-\frac{1}{2}}) \sin\left(\frac{t}{2}\log x\right)\psi(x)dx.$$

First we expand $R(s)$. We set

$$f_0(x) := (x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}})\psi(x).$$

Using integral by parts and

$$\frac{d}{dx}\left(\frac{2}{t}\sin\left(\frac{t}{2}\log x\right)\right) = x^{-1}\cos\left(\frac{t}{2}\log x\right),$$

we obtain

$$R(s) = -\frac{2}{t}\int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_0'(x)dx.$$

Let $f_1(x) := xf_0'(x)$. Then we have

$$\begin{aligned}\int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_0'(x)dx &= \int_1^\infty x^{-1}\sin\left(\frac{t}{2}\log x\right) f_1(x)dx \\ &= \frac{2}{t}f_1(1) + \frac{2}{t}\int_1^\infty \cos\left(\frac{t}{2}\log x\right) f_1'(x)dx\end{aligned}$$

by integral by parts. We inductively define $f_k(x) := xf_{k-1}'(x)$ for $k = 1, 2, \ldots$. By the same way as above, we obtain

$$\begin{aligned}R(s) = &-\left(\frac{2}{t}\right)^2 f_1(1) + \left(\frac{2}{t}\right)^4 f_3(1) - \left(\frac{2}{t}\right)^6 f_5(1) \\ &+ \left(\frac{2}{t}\right)^7 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_6'(x)dx.\end{aligned} \tag{3.1}$$

We use the same argument for $I(s)$. We define $g_0(x) := (x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}})\psi(x)$ and $g_k(x) := xg_{k-1}'(x)$ for $k = 1, 2, \ldots$. Then, we similarly obtain

$$\begin{aligned}I(s) = &-\left(\frac{2}{t}\right)^3 g_2(1) + \left(\frac{2}{t}\right)^5 g_4(1) - \left(\frac{2}{t}\right)^7 g_6(1) \\ &+ \left(\frac{2}{t}\right)^8 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) g_7'(x)dx\end{aligned} \tag{3.2}$$

by integral by parts.

We define $h_0(x) = f_0(x)\log x$ and $h_k(x) = xh_{k-1}'(x)$ for $k = 1, 2, 3$. Then, we also obtain

$$V(s) = \frac{2}{t^3} + \frac{2^3}{t^4}\int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx. \tag{3.3}$$

4. Functions $f_k(x), f_k'(x), g_k(x), g_k'(x)$ and $h_3'(x)$

We collect precise forms of $f_k(x), f_k'(x), g_k(x), g_k'(x)$ and $h_3'(x)$ in this section. By a direct calculation we obtain the following formulas

$$
\begin{aligned}
f_0'(x) = &\left\{\frac{\sigma}{2}x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\
&+ \left(x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}}\right)\psi^{(1)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_1(x) = &\left\{\frac{\sigma}{2}x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) \\
&+ \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(1)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_1'(x) = &\left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\
&+ \left\{(\sigma+1)x^{\frac{\sigma}{2}} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(2)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_2(x) = &\left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) \\
&+ \left\{(\sigma+1)x^{\frac{\sigma}{2}+1} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(1)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(2)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_2'(x) = &\left\{\left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) \\
&+ \left\{\left(\frac{3}{4}\sigma^2+\frac{3}{2}\sigma+1\right)x^{\frac{\sigma}{2}} + \left(\frac{3}{4}\sigma^2-3\sigma+\frac{13}{4}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) \\
&+ \left\{\left(\frac{3}{2}\sigma+3\right)x^{\frac{\sigma}{2}+1} + \left(-\frac{3}{2}\sigma+\frac{9}{2}\right)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(2)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(3)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_3(x) = &\left\{ \left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1\right) x^{\frac{\sigma}{2}+1} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma + 3\right) x^{\frac{\sigma}{2}+2} + \left(-\frac{3}{2}\sigma + \frac{9}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+3} + x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(3)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_3'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^4 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^4 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. + \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7\right) x^{\frac{\sigma}{2}+1} + \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ (2\sigma+6) x^{\frac{\sigma}{2}+2} + (-2\sigma+8) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+3} + x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_4(x) = &\left\{ \left(\frac{\sigma}{2}\right)^4 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^4 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. + \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7\right) x^{\frac{\sigma}{2}+2} + \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ (2\sigma+6) x^{\frac{\sigma}{2}+3} + (-2\sigma+8) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+4} + x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(4)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_4'(x) = & \left\{ \left(\frac{\sigma}{2}\right)^5 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^5 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. + \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15\right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. + \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma^2 + 15\sigma + 25\right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. + \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma + 10\right) x^{\frac{\sigma}{2}+3} + \left(-\frac{5}{2}\sigma + \frac{25}{2}\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left(x^{\frac{\sigma}{2}+4} + x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(5)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_5(x) = &\left\{ \left(\frac{\sigma}{2}\right)^5 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^5 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. + \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. + \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^2 + 15\sigma + 25\right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. + \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2}\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma + 10\right) x^{\frac{\sigma}{2}+4} + \left(-\frac{5}{2}\sigma + \frac{25}{2}\right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(4)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+5} + x^{-\frac{\sigma}{2}+\frac{11}{2}} \right) \psi^{(5)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_5'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^6 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^6 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. + \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. + \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. + \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65\right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. + \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4}\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
&+ \left\{ (3\sigma+15) x^{\frac{\sigma}{2}+4} + (-3\sigma+18) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+5} + x^{-\frac{\sigma}{2}+\frac{11}{2}} \right) \psi^{(6)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_6(x) = &\left\{ \left(\frac{\sigma}{2}\right)^6 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^6 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. + \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. + \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90\right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. + \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65\right) x^{\frac{\sigma}{2}+4} \right. \\
&\quad \left. + \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4}\right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(4)}(x) \\
&+ \left\{ (3\sigma+15) x^{\frac{\sigma}{2}+5} + (-3\sigma+18) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(5)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+6} + x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(6)}(x),
\end{aligned}
$$

$$
\begin{aligned}
f_6'(x) = & \left\{ \left(\frac{\sigma}{2}\right)^7 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^7 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64}\sigma^6 + \frac{21}{32}\sigma^5 + \frac{35}{16}\sigma^4 + \frac{35}{8}\sigma^3 + \frac{21}{4}\sigma^2 \right.\right. \\
& \quad \left. + \frac{7}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}} + \left(\frac{7}{64}\sigma^6 - \frac{21}{16}\sigma^5 + \frac{455}{64}\sigma^4 - \frac{175}{8}\sigma^3 \right. \\
& \quad \left.\left. + \frac{2541}{64}\sigma^2 - \frac{637}{16}\sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32}\sigma^5 + \frac{105}{16}\sigma^4 + \frac{245}{8}\sigma^3 + \frac{315}{4}\sigma^2 + \frac{217}{2}\sigma \right.\right. \\
& \quad + 63) \, x^{\frac{\sigma}{2}+1} + \left(-\frac{21}{32}\sigma^5 + \frac{315}{32}\sigma^4 - \frac{1015}{16}\sigma^3 + \frac{3465}{16}\sigma^2 \right. \\
& \quad \left.\left. - \frac{12397}{32}\sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16}\sigma^4 + \frac{105}{4}\sigma^3 + \frac{525}{4}\sigma^2 + 315\sigma + 301 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. + \left(\frac{35}{16}\sigma^4 - 35\sigma^3 + \frac{1785}{8}\sigma^2 - 665\sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^3 + \frac{105}{2}\sigma^2 + \frac{455}{2}\sigma + 350 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. + \left(-\frac{35}{8}\sigma^3 + \frac{525}{8}\sigma^2 - \frac{2765}{8}\sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4}\sigma^2 + \frac{105}{2}\sigma + 140 \right) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. + \left(\frac{21}{4}\sigma^2 - 63\sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2}\sigma + 21 \right) x^{\frac{\sigma}{2}+5} + \left(-\frac{7}{2}\sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2}+6} + x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(7)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_0'(x) = & \left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right) x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left(x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}} \right) \psi^{(1)}(x),
\end{aligned}
$$

$$g_1(x) = \left\{\frac{\sigma}{2}x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) + \left(x^{\frac{\sigma}{2}+1} - x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(1)}(x),$$

$$g_1'(x) = \left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) + \left\{(\sigma+1)x^{\frac{\sigma}{2}} - (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) + \left(x^{\frac{\sigma}{2}+1} - x^{-\frac{\sigma}{2}+\frac{3}{2}}\right)\psi^{(2)}(x),$$

$$g_2(x) = \left\{\left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) + \left\{(\sigma+1)x^{\frac{\sigma}{2}+1} - (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(1)}(x) + \left(x^{\frac{\sigma}{2}+2} - x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(2)}(x),$$

$$g_2'(x) = \left\{\left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}}\right\}\psi(x) + \left\{\left(\frac{3}{4}\sigma^2+\frac{3}{2}\sigma+1\right)x^{\frac{\sigma}{2}} - \left(\frac{3}{4}\sigma^2-3\sigma+\frac{13}{4}\right)x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi^{(1)}(x) + \left\{\left(\frac{3}{2}\sigma+3\right)x^{\frac{\sigma}{2}+1} - \left(-\frac{3}{2}\sigma+\frac{9}{2}\right)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(2)}(x) + \left(x^{\frac{\sigma}{2}+2} - x^{-\frac{\sigma}{2}+\frac{5}{2}}\right)\psi^{(3)}(x),$$

$$g_3(x) = \left\{\left(\frac{\sigma}{2}\right)^3 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^3 x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) + \left\{\left(\frac{3}{4}\sigma^2+\frac{3}{2}\sigma+1\right)x^{\frac{\sigma}{2}+1} - \left(\frac{3}{4}\sigma^2-3\sigma+\frac{13}{4}\right)x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(1)}(x) + \left\{\left(\frac{3}{2}\sigma+3\right)x^{\frac{\sigma}{2}+2} - \left(-\frac{3}{2}\sigma+\frac{9}{2}\right)x^{-\frac{\sigma}{2}+\frac{5}{2}}\right\}\psi^{(2)}(x) + \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}}\right)\psi^{(3)}(x),$$

$$
\begin{aligned}
g_3'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^4 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^4 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. - \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7\right) x^{\frac{\sigma}{2}+1} - \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ (2\sigma+6)x^{\frac{\sigma}{2}+2} - (-2\sigma+8)x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_4(x) = &\left\{ \left(\frac{\sigma}{2}\right)^4 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^4 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. - \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7\right) x^{\frac{\sigma}{2}+2} - \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ (2\sigma+6)x^{\frac{\sigma}{2}+3} - (-2\sigma+8)x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+4} - x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(4)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_4'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^5 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^5 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1\right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. - \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. - \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^2 + 15\sigma + 25\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. - \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma + 10\right) x^{\frac{\sigma}{2}+3} - \left(-\frac{5}{2}\sigma + \frac{25}{2}\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+4} - x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(5)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_5(x) = &\left\{\left(\frac{\sigma}{2}\right)^5 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^5 x^{-\frac{\sigma}{2}+\frac{1}{2}}\right\}\psi(x) \\
&+ \left\{\left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. - \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}}\right\}\psi^{(1)}(x) \\
&+ \left\{\left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. - \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}}\right\}\psi^{(2)}(x) \\
&+ \left\{\left(\frac{5}{2}\sigma^2 + 15\sigma + 25\right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. - \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2}\right) x^{-\frac{\sigma}{2}+\frac{7}{2}}\right\}\psi^{(3)}(x) \\
&+ \left\{\left(\frac{5}{2}\sigma + 10\right) x^{\frac{\sigma}{2}+4} - \left(-\frac{5}{2}\sigma + \frac{25}{2}\right) x^{-\frac{\sigma}{2}+\frac{9}{2}}\right\}\psi^{(4)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+5} - x^{-\frac{\sigma}{2}+\frac{11}{2}}\right)\psi^{(5)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_5'(x) = &\left\{ \left(\frac{\sigma}{2}\right)^6 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^6 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
&\quad \left. - \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31 \right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. - \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90 \right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. - \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190 \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65 \right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. - \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
&+ \left\{ (3\sigma+15)x^{\frac{\sigma}{2}+4} - (-3\sigma+18)x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+5} - x^{-\frac{\sigma}{2}+\frac{11}{2}} \right) \psi^{(6)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_6(x) = &\left\{ \left(\frac{\sigma}{2}\right)^6 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^6 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
&+ \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1\right) x^{\frac{\sigma}{2}+1} \right. \\
&\quad \left. - \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8}\right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
&+ \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31\right) x^{\frac{\sigma}{2}+2} \right. \\
&\quad \left. - \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16}\right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
&+ \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90\right) x^{\frac{\sigma}{2}+3} \right. \\
&\quad \left. - \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190\right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
&+ \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65\right) x^{\frac{\sigma}{2}+4} \right. \\
&\quad \left. - \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4}\right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(4)}(x) \\
&+ \left\{ (3\sigma+15) x^{\frac{\sigma}{2}+5} - (-3\sigma+18) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(5)}(x) \\
&+ \left(x^{\frac{\sigma}{2}+6} - x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(6)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_6'(x) = & \left\{ \left(\frac{\sigma}{2}\right)^7 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^7 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64}\sigma^6 + \frac{21}{32}\sigma^5 + \frac{35}{16}\sigma^4 + \frac{35}{8}\sigma^3 + \frac{21}{4}\sigma^2 + \frac{7}{2}\sigma \right.\right. \\
& \quad \left. +1\right) x^{\frac{\sigma}{2}} - \left(\frac{7}{64}\sigma^6 - \frac{21}{16}\sigma^5 + \frac{455}{64}\sigma^4 - \frac{175}{8}\sigma^3 \right. \\
& \quad \left.\left. + \frac{2541}{64}\sigma^2 - \frac{637}{16}\sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32}\sigma^5 + \frac{105}{16}\sigma^4 + \frac{245}{8}\sigma^3 + \frac{315}{4}\sigma^2 + \frac{217}{2}\sigma \right.\right. \\
& \quad \left. +63\right) x^{\frac{\sigma}{2}+1} - \left(-\frac{21}{32}\sigma^5 + \frac{315}{32}\sigma^4 - \frac{1015}{16}\sigma^3 + \frac{3465}{16}\sigma^2 \right. \\
& \quad \left.\left. -\frac{12397}{32}\sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16}\sigma^4 + \frac{105}{4}\sigma^3 + \frac{525}{4}\sigma^2 + 315\sigma + 301 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. - \left(\frac{35}{16}\sigma^4 - 35\sigma^3 + \frac{1785}{8}\sigma^2 - 665\sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^3 + \frac{105}{2}\sigma^2 + \frac{455}{2}\sigma + 350 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. - \left(-\frac{35}{8}\sigma^3 + \frac{525}{8}\sigma^2 - \frac{2765}{8}\sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4}\sigma^2 + \frac{105}{2}\sigma + 140 \right) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. - \left(\frac{21}{4}\sigma^2 - 63\sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2}\sigma + 21 \right) x^{\frac{\sigma}{2}+5} - \left(-\frac{7}{2}\sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2}+6} - x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(7)}(x),
\end{aligned}
$$

$$
\begin{aligned}
g_7(x) = & \left\{ \left(\frac{\sigma}{2}\right)^7 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2}\right)^7 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64}\sigma^6 + \frac{21}{32}\sigma^5 + \frac{35}{16}\sigma^4 + \frac{35}{8}\sigma^3 + \frac{21}{4}\sigma^2 + \frac{7}{2}\sigma \right.\right. \\
& \left. +1\right) x^{\frac{\sigma}{2}+1} - \left(\frac{7}{64}\sigma^6 - \frac{21}{16}\sigma^5 + \frac{455}{64}\sigma^4 - \frac{175}{8}\sigma^3 \right. \\
& \left.\left. + \frac{2541}{64}\sigma^2 - \frac{637}{16}\sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32}\sigma^5 + \frac{105}{16}\sigma^4 + \frac{245}{8}\sigma^3 + \frac{315}{4}\sigma^2 + \frac{217}{2}\sigma \right.\right. \\
& \left. +63\right) x^{\frac{\sigma}{2}+2} - \left(-\frac{21}{32}\sigma^5 + \frac{315}{32}\sigma^4 - \frac{1015}{16}\sigma^3 + \frac{3465}{16}\sigma^2 \right. \\
& \left.\left. -\frac{12397}{32}\sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16}\sigma^4 + \frac{105}{4}\sigma^3 + \frac{525}{4}\sigma^2 + 315\sigma + 301 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \left. - \left(\frac{35}{16}\sigma^4 - 35\sigma^3 + \frac{1785}{8}\sigma^2 - 665\sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^3 + \frac{105}{2}\sigma^2 + \frac{455}{2}\sigma + 350 \right) x^{\frac{\sigma}{2}+4} \right. \\
& \left. - \left(-\frac{35}{8}\sigma^3 + \frac{525}{8}\sigma^2 - \frac{2765}{8}\sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4}\sigma^2 + \frac{105}{2}\sigma + 140 \right) x^{\frac{\sigma}{2}+5} \right. \\
& \left. - \left(\frac{21}{4}\sigma^2 - 63\sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2}\sigma + 21 \right) x^{\frac{\sigma}{2}+6} - \left(-\frac{7}{2}\sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2}+\frac{13}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2}+7} - x^{-\frac{\sigma}{2}+\frac{15}{2}} \right) \psi^{(7)}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
g_7'(x) = & \left\{ \left(\frac{\sigma}{2}\right)^8 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2}+\frac{1}{2}\right)^8 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& \left\{ \left(\frac{1}{16}\sigma^7 + \frac{7}{16}\sigma^6 + \frac{7}{4}\sigma^5 + \frac{35}{8}\sigma^4 + 7\sigma^3 + 7\sigma^2 + 4\sigma \right.\right. \\
& \quad \left. +1\right) x^{\frac{\sigma}{2}} - \left(-\frac{1}{16}\sigma^7 + \frac{7}{8}\sigma^6 - \frac{91}{16}\sigma^5 + \frac{175}{8}\sigma^4 \right. \\
& \quad \left.\left. - \frac{847}{16}\sigma^3 + \frac{637}{8}\sigma^2 - \frac{1093}{16}\sigma + \frac{205}{8} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{7}{16}\sigma^6 + \frac{21}{4}\sigma^5 + \frac{245}{8}\sigma^4 + 105\sigma^3 + 217\sigma^2 + 252\sigma \right.\right. \\
& \quad \left. +127\right) x^{\frac{\sigma}{2}+1} - \left(\frac{7}{16}\sigma^6 - \frac{63}{8}\sigma^5 + \frac{1015}{16}\sigma^4 - \frac{1155}{4}\sigma^3 \right. \\
& \quad \left.\left. + \frac{12397}{16}\sigma^2 - \frac{9219}{8}\sigma + \frac{11797}{16} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{7}{4}\sigma^5 + \frac{105}{4}\sigma^4 + 175\sigma^3 + 630\sigma^2 + 1204\sigma \right.\right. \\
& \quad \left. +966\right) x^{\frac{\sigma}{2}+2} - \left(-\frac{7}{4}\sigma^5 + 35\sigma^4 - \frac{595}{2}\sigma^3 + 1330\sigma^2 \right. \\
& \quad \left.\left. - \frac{12411}{4}\sigma + 3003 \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^4 + 70\sigma^3 + 455\sigma^2 + 1400\sigma + 1701 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. - \left(\frac{35}{8}\sigma^4 - \frac{175}{2}\sigma^3 + \frac{2765}{4}\sigma^2 - \frac{5075}{2}\sigma + \frac{29043}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (7\sigma^3 + 105\sigma^2 + 560\sigma + 1050) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. - (-7\sigma^3 + 126\sigma^2 - 791\sigma + 1722) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ (7\sigma^2 + 84\sigma + 266) x^{\frac{\sigma}{2}+5} \right. \\
& \quad \left. - (7\sigma^2 - 98\sigma + 357) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left\{ (4\sigma + 28) x^{\frac{\sigma}{2}+6} - (-4\sigma + 32) x^{-\frac{\sigma}{2}+\frac{13}{2}} \right\} \psi^{(7)}(x) \\
& + \left(x^{\frac{\sigma}{2}+7} - x^{-\frac{\sigma}{2}+\frac{15}{2}} \right) \psi^{(8)}(x).
\end{aligned}
$$

We also have

$$h_3'(x) = \left(f_0^{(1)}(x) + 7xf_0^{(2)}(x) + 6x^2 f_0^{(3)}(x) + x^3 f_0^{(4)}(x)\right)\log x + 4f_0^{(1)}(x) + 12xf_0^{(2)}(x) + 4x^2 f_0^{(3)}(x). \tag{4.1}$$

5. Formulas of $\psi^{(k)}(x)$

An important property of the series $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ is the reciprocal formula

$$2\psi(x) + 1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right) + 1\right) \tag{5.1}$$

for $x > 0$ (for example, see (2.6.3) in [8]). The integral representation (2.1) of $\xi(s)$ is derived from the above reciprocal formula.

It is rewritten as

$$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}. \tag{5.2}$$

Differentiating (5.2) one after another, we obtain the following equalities

$$\psi^{(1)}(x) = -\frac{1}{2}x^{-\frac{3}{2}}\psi\left(\frac{1}{x}\right) - x^{-\frac{5}{2}}\psi^{(1)}\left(\frac{1}{x}\right) - \frac{1}{4}x^{-\frac{3}{2}}, \tag{5.3}$$

$$\psi^{(2)}(x) = \frac{3}{4}x^{-\frac{5}{2}}\psi\left(\frac{1}{x}\right) + 3x^{-\frac{7}{2}}\psi^{(1)}\left(\frac{1}{x}\right) + x^{-\frac{9}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + \frac{3}{8}x^{-\frac{5}{2}}, \tag{5.4}$$

$$\psi^{(3)}(x) = -\frac{15}{8}x^{-\frac{7}{2}}\psi\left(\frac{1}{x}\right) - \frac{45}{4}x^{-\frac{9}{2}}\psi^{(1)}\left(\frac{1}{x}\right) - \frac{15}{2}x^{-\frac{11}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - x^{-\frac{13}{2}}\psi^{(3)}\left(\frac{1}{x}\right) - \frac{15}{16}x^{-\frac{7}{2}}, \tag{5.5}$$

$$\begin{aligned}\psi^{(4)}(x) &= \frac{105}{16}x^{-\frac{9}{2}}\psi\left(\frac{1}{x}\right) + \frac{105}{2}x^{-\frac{11}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ &\quad + \frac{105}{2}x^{-\frac{13}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + 14x^{-\frac{15}{2}}\psi^{(3)}\left(\frac{1}{x}\right)\\ &\quad + x^{-\frac{17}{2}}\psi^{(4)}\left(\frac{1}{x}\right) + \frac{105}{32}x^{-\frac{9}{2}},\end{aligned} \tag{5.6}$$

$$\begin{aligned}\psi^{(5)}(x) &= -\frac{945}{32}x^{-\frac{11}{2}}\psi\left(\frac{1}{x}\right) - \frac{4725}{16}x^{-\frac{13}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ &\quad - \frac{1575}{4}x^{-\frac{15}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - \frac{315}{2}x^{-\frac{17}{2}}\psi^{(3)}\left(\frac{1}{x}\right)\\ &\quad - \frac{45}{2}x^{-\frac{19}{2}}\psi^{(4)}\left(\frac{1}{x}\right) - x^{-\frac{21}{2}}\psi^{(5)}\left(\frac{1}{x}\right)\\ &\quad - \frac{945}{64}x^{-\frac{11}{2}},\end{aligned} \tag{5.7}$$

$$\begin{aligned}\psi^{(6)}(x) &= \frac{10395}{64}x^{-\frac{13}{2}}\psi\left(\frac{1}{x}\right) + \frac{31185}{16}x^{-\frac{15}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ &\quad + \frac{51975}{16}x^{-\frac{17}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + \frac{3465}{2}x^{-\frac{19}{2}}\psi^{(3)}\left(\frac{1}{x}\right)\\ &\quad + \frac{1485}{4}x^{-\frac{21}{2}}\psi^{(4)}\left(\frac{1}{x}\right) + 33x^{-\frac{23}{2}}\psi^{(5)}\left(\frac{1}{x}\right)\\ &\quad + x^{-\frac{25}{2}}\psi^{(6)}\left(\frac{1}{x}\right) + \frac{10395}{128}x^{-\frac{13}{2}},\end{aligned} \tag{5.8}$$

$$\begin{aligned}\psi^{(7)}(x) &= -\frac{135135}{128}x^{-\frac{15}{2}}\psi\left(\frac{1}{x}\right) - \frac{945945}{64}x^{-\frac{17}{2}}\psi^{(1)}\left(\frac{1}{x}\right)\\ &\quad - \frac{945945}{32}x^{-\frac{19}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - \frac{315315}{16}x^{-\frac{21}{2}}\psi^{(3)}\left(\frac{1}{x}\right)\\ &\quad - \frac{45045}{8}x^{-\frac{23}{2}}\psi^{(4)}\left(\frac{1}{x}\right) - \frac{3003}{4}x^{-\frac{25}{2}}\psi^{(5)}\left(\frac{1}{x}\right)\\ &\quad - \frac{91}{2}x^{-\frac{27}{2}}\psi^{(6)}\left(\frac{1}{x}\right) - x^{-\frac{29}{2}}\psi^{(7)}\left(\frac{1}{x}\right)\\ &\quad - \frac{135135}{256}x^{-\frac{15}{2}}\end{aligned} \tag{5.9}$$

and

$$
\begin{aligned}
\psi^{(8)}(x) = {} & \frac{2027025}{256} x^{-\frac{17}{2}} \psi\left(\frac{1}{x}\right) + \frac{2027025}{16} x^{-\frac{19}{2}} \psi^{(1)}\left(\frac{1}{x}\right) \\
& + \frac{4729725}{16} x^{-\frac{21}{2}} \psi^{(2)}\left(\frac{1}{x}\right) + \frac{945945}{4} x^{-\frac{23}{2}} \psi^{(3)}\left(\frac{1}{x}\right) \\
& + \frac{675675}{8} x^{-\frac{25}{2}} \psi^{(4)}\left(\frac{1}{x}\right) + 15015 x^{-\frac{27}{2}} \psi^{(5)}\left(\frac{1}{x}\right) \\
& + 1365 x^{-\frac{29}{2}} \psi^{(6)}\left(\frac{1}{x}\right) + 60 x^{-\frac{31}{2}} \psi^{(7)}\left(\frac{1}{x}\right) \\
& + x^{-\frac{33}{2}} \psi^{(8)}\left(\frac{1}{x}\right) + \frac{2027025}{512} x^{-\frac{17}{2}}.
\end{aligned} \tag{5.10}
$$

Next we give formulas of $\psi^{(k)}(1)$. Substituting $x = 1$ in (5.3), we have

$$\frac{1}{4}\psi(1) + \psi^{(1)}(1) = -\frac{1}{8}. \tag{5.11}$$

Similarly, we obtain

$$\frac{15}{4}\psi^{(2)}(1) + \psi^{(3)}(1) = \frac{15}{32}\psi(1) + \frac{15}{64} \tag{5.12}$$

by (5.5) and (5.11). We also have

$$
\begin{aligned}
\frac{45}{4}\psi^{(4)}(1) + \psi^{(5)}(1) = {} & -\frac{315}{64}\psi(1) - \frac{945}{16}\psi^{(1)}(1) - \frac{1575}{16}\psi^{(2)}(1) \\
& - \frac{105}{2}\psi^{(3)}(1) - \frac{315}{128}
\end{aligned}
$$

by (5.8). Using (5.11) and (5.12), we obtain

$$\frac{45}{4}\psi^{(4)}(1) + \psi^{(5)}(1) = -\frac{945}{64}\psi(1) + \frac{1575}{16}\psi^{(2)}(1) - \frac{945}{128}. \tag{5.13}$$

We substitute $x = 1$ in (5.9). Then we have

$$
\begin{aligned}
\frac{91}{4}\psi^{(6)}(1) + \psi^{(7)}(1) = {} & -\frac{135135}{256}\psi(1) - \frac{945945}{128}\psi^{(1)}(1) \\
& - \frac{945945}{64}\psi^{(2)}(1) - \frac{315315}{32}\psi^{(3)}(1) \\
& - \frac{45045}{16}\psi^{(4)}(1) - \frac{3003}{8}\psi^{(5)}(1) \\
& - \frac{135135}{512}.
\end{aligned}
$$

Therefore, it follows from (5.11), (5.12) and (5.13) that

$$\begin{aligned}\frac{91}{4}\psi^{(6)}(1)+\psi^{(7)}(1) &= \frac{2297295}{1024}\psi(1)-\frac{945945}{64}\psi^{(2)}(1)\\ &\quad+\frac{45045}{32}\psi^{(4)}(1)+\frac{2297295}{2048}.\end{aligned} \tag{5.14}$$

We can consider $\psi^{(k)}$ for negative integers k. For any $k \in \mathbb{Z}$ we set

$$\psi^{(k)}(x) := (-1)^k \pi^k \sum_{n=1}^{\infty} n^{2k} e^{-\pi n^2 x}.$$

Then we have $\psi(x) = \psi^{(0)}(x)$ and $\frac{d}{dx}\psi^{(k)}(x) = \psi^{(k+1)}(x)$. By integral by parts, we obtain

$$\int_1^{\infty} \psi(x)dx = -\psi^{(-1)}(1) \tag{5.15}$$

and

$$\int_1^{\infty} x^2\psi(x)dx = -\psi^{(-1)}(1) + 2\psi^{(-2)}(1) - 2\psi^{(-3)}(1). \tag{5.16}$$

Furthermore we have

$$\int_1^{\infty} x^{2k-1}|\psi^{(2k-1)}(x)|dx = \sum_{j=0}^{2k-1}(-1)^j \frac{(2k-1)!}{(2k-1-j)!}\psi^{(2k-2-j)}(1) \tag{5.17}$$

and

$$\int_1^{\infty} x^{2k}\psi^{(2k)}(x)dx = \sum_{j=0}^{2k}(-1)^{j+1} \frac{(2k)!}{(2k-j)!}\psi^{(2k-1-j)}(1) \tag{5.18}$$

for $k = 1, 2, \ldots$.

6. Precise representations of $R(s)$ and $I(s)$

We give representations of $R(s)$ and $I(s)$ more precisely, determining $f_k(1)$ and $g_k(1)$ in (3.1) and (3.2) respectively. We use results in Sections 4 and 5.

We obtain

$$\begin{aligned}f_1(1) &= \left\{\frac{\sigma}{2} + \left(-\frac{\sigma}{2}+\frac{1}{2}\right)\right\}\psi(1) + 2\psi^{(1)}(1)\\ &= 2\left(\frac{1}{4}\psi(1)+\psi^{(1)}(1)\right)\\ &= -\frac{1}{4}\end{aligned}$$

by the representation of $f_1(x)$ in Section 4 and (5.11). Then we have

$$-\left(\frac{2}{t}\right)^2 f_1(1) = \frac{1}{t^2}. \tag{6.1}$$

It follows from the formula of $f_3(x)$ in Section 4, (5.11) and (5.12) that

$$\begin{aligned}
f_3(1) &= \frac{1}{8}(3\sigma^3 - 3\sigma + 1)\psi(1) + \left(\frac{3}{2}\sigma^2 - \frac{3}{2}\sigma + \frac{17}{4}\right)\psi^{(1)}(1) \\
&\quad + \frac{15}{2}\psi^{(2)}(1) + 2\psi^{(3)}(1) \\
&= \frac{1}{2}(3\sigma^2 - 3\sigma + 1)\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) + \frac{15}{4}\psi^{(1)}(1) \\
&\quad + 2\left(\frac{15}{4}\psi^{(2)}(1) + \psi^{(3)}(1)\right) \\
&= -\frac{1}{2^4}(3\sigma^2 - 3\sigma + 1) + \frac{15}{4}\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) + \frac{15}{32} \\
&= -\frac{1}{2^4}(3\sigma^2 - 3\sigma + 1),
\end{aligned}$$

therefore

$$\left(\frac{2}{t}\right)^4 f_3(1) = -\frac{1}{t^4}(3\sigma^2 - 3\sigma + 1). \tag{6.2}$$

We have

$$\begin{aligned}
f_5(1) &= \left(\frac{5}{32}\sigma^4 - \frac{5}{16}\sigma^3 + \frac{5}{16}\sigma^2 - \frac{5}{32}\sigma + \frac{1}{32}\right)\psi(1) \\
&\quad + \left(\frac{5}{8}\sigma^4 - \frac{5}{4}\sigma^3 + \frac{85}{8}\sigma^2 - 10\sigma + \frac{137}{16}\right)\psi^{(1)}(1) \\
&\quad + \left(\frac{75}{4}\sigma^2 - \frac{75}{4}\sigma + \frac{225}{4}\right)\psi^{(2)}(1) \\
&\quad + \left(5\sigma^2 - 5\sigma + \frac{135}{2}\right)\psi^{(3)}(1) \\
&\quad + \frac{45}{2}\psi^{(4)}(1) + 2\psi^{(5)}(1)
\end{aligned}$$

by the formula of $f_5(x)$. Using (5.13), we can rewrite it as

$$\begin{aligned} f_5(1) &= \frac{5}{8}\sigma^4\left(\frac{1}{4}\psi(1)+\psi^{(1)}(1)\right)-\frac{5}{4}\sigma^3\left(\frac{1}{4}\psi(1)+\psi^{(1)}(1)\right)\\ &\quad+\sigma^2\left(\frac{5}{16}\psi(1)+\frac{85}{8}\psi^{(1)}(1)+\frac{75}{4}\psi^{(2)}(1)+5\psi^{(3)}(1)\right)\\ &\quad-\sigma\left(\frac{5}{32}\psi(1)+10\psi^{(1)}(1)+\frac{75}{4}\psi^{(2)}(1)+5\psi^{(3)}(1)\right)\\ &\quad-\frac{59}{2}\psi(1)+\frac{137}{16}\psi^{(1)}(1)+\frac{2025}{8}\psi^{(2)}(1)\\ &\quad+\frac{135}{2}\psi^{(3)}(1)-\frac{945}{64}. \end{aligned}$$

Then we obtain

$$f_5(1)=-\frac{5}{64}\sigma^4+\frac{5}{32}\sigma^3-\frac{5}{32}\sigma^2+\frac{5}{64}\sigma-\frac{1}{64}$$

by (5.11) and (5.12). Hence we have

$$-\left(\frac{2}{t}\right)^6 f_5(1)=\frac{1}{t^6}(5\sigma^4-10\sigma^3+10\sigma^2-5\sigma+1). \tag{6.3}$$

Therefore, it follows from (6.1), (6.2) and (6.3) that

$$\begin{aligned} R(s) &= \frac{1}{t^2}-\frac{1}{t^4}(3\sigma^2-3\sigma+1)\\ &\quad+\frac{1}{t^6}(5\sigma^4-10\sigma^3+10\sigma^2-5\sigma+1)\\ &\quad+\frac{1}{t^7}\alpha_1(s), \end{aligned} \tag{6.4}$$

where

$$\alpha_1(s)=2^7\int_1^\infty \sin\left(\frac{t}{2}\log x\right)f_6'(x)dx.$$

We use the same argument as above for $I(s)$. By results in Sections 4 and 5, we obtain

$$\begin{aligned} I(s) &= \frac{1}{t^3}(2\sigma-1)-\frac{1}{t^5}(2\sigma-1)(2\sigma^2-2\sigma+1)\\ &\quad+\frac{1}{t^7}(2\sigma-1)(3\sigma^4-6\sigma^3+7\sigma^2-4\sigma+1)\\ &\quad+\frac{1}{t^8}\beta_1(s), \end{aligned} \tag{6.5}$$

where

$$\beta_1(s)=2^8\int_1^\infty \sin\left(\frac{t}{2}\log x\right)g_7'(x)dx.$$

7. Estimates

We use inequalities $3.14159265 < \pi < 3.14159266$ and $0.04321391 < e^{-\pi} < 0.04321392$ to estimate $|\psi^{(k)}(1)|$.

Since

$$\psi(1) = \sum_{n=1}^{\infty} e^{-n^2\pi} < \sum_{n=1}^{\infty} (e^{-\pi})^n = \frac{e^{-\pi}}{1 - e^{-\pi}},$$

we have

$$\psi(1) < 0.04516571. \tag{7.1}$$

We also have

$$\begin{aligned} |\psi^{(-1)}(1)| &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi} \\ &< \frac{1}{\pi}\psi(1) < 0.01437670. \end{aligned} \tag{7.2}$$

To estimate $|\psi^{(1)}(1)| = \pi \sum_{n=1}^{\infty} n^2 e^{-n^2\pi}$, we need a function $\varphi_1(x) = x^2 e^{-\pi x^2}$. Since $\varphi_1'(x) = 2x(1 - \pi x^2)e^{-\pi x^2}$, $\varphi_1(x)$ is monotonously decreasing on $(1, \infty)$. Then we have

$$\sum_{n=1}^{\infty} n^2 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_1(x) dx.$$

Noting

$$\int_1^{\infty} e^{-\pi x^2} dx < \int_1^{\infty} e^{-\pi x} dx = \frac{1}{\pi} e^{-\pi},$$

we obtain

$$\int_1^{\infty} \varphi_1(x) dx < \left(1 + \frac{1}{\pi}\right) \frac{1}{2\pi} e^{-\pi}$$

by integral by parts. Therefore we have

$$\begin{aligned} |\psi^{(1)}(1)| &< \pi e^{-\pi} + (1 + \pi) \frac{1}{2\pi} e^{-\pi} \\ &< 0.16424611. \end{aligned} \tag{7.3}$$

We consider a function $\varphi_2(x) = x^4 e^{-\pi x^2}$ for $\psi^{(2)}(1) = \pi^2 \sum_{n=1}^{\infty} n^4 e^{-n^2\pi}$. By the same way as above, we obtain

$$\sum_{n=1}^{\infty} n^4 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_2(x) dx$$

for $\varphi_2'(x) = 2x^3(2 - \pi x^2)e^{-\pi x^2}$. Since

$$\int_1^{\infty} \varphi_2(x) dx = \frac{1}{2\pi} e^{-\pi} + \frac{3}{2\pi} \int_1^{\infty} \varphi_1(x) dx,$$

we have

$$\sum_{n=1}^{\infty} n^4 e^{-n^2\pi} < e^{-\pi} + \left\{1 + \frac{3}{2\pi}\left(1 + \frac{1}{\pi}\right)\right\} \frac{1}{2\pi} e^{-\pi}.$$

Hence we obtain

$$\begin{aligned} \psi^{(2)}(1) &< \left\{\pi^2 + \frac{\pi}{2}\left(1 + \frac{3}{2\pi}\left(1 + \frac{1}{\pi}\right)\right)\right\} e^{-\pi} \\ &< 0.53711157. \end{aligned} \tag{7.4}$$

We estimate $|\psi^{(3)}(1)| = \pi^3 \sum_{n=1}^{\infty} n^6 e^{-n^2\pi}$ using a function $\varphi_3(x) = x^6 e^{-\pi x^2}$. Since $\varphi_3'(x) = 2x^5(3 - \pi x^2)e^{-\pi x^2}$, we obtain

$$\sum_{n=1}^{\infty} n^6 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_3(x)dx.$$

We have

$$\int_1^{\infty} \varphi_3(x)dx = \frac{1}{2\pi} e^{-\pi} + \frac{5}{2\pi} \int_1^{\infty} \varphi_2(x)dx.$$

Then we obtain

$$\sum_{n=1}^{\infty} n^6 e^{-n^2\pi} < e^{-\pi} + \left\{1 + \frac{5}{2\pi} + \frac{5 \cdot 3}{(2\pi)^2}\left(1 + \frac{1}{\pi}\right)\right\} \frac{1}{2\pi} e^{-\pi}.$$

Therefore, we have

$$\begin{aligned} |\psi^{(3)}(1)| &< \left\{\pi^3 + \frac{\pi^2}{2}\left(1 + \frac{5}{2\pi} + \frac{5 \cdot 3}{(2\pi)^2}\left(1 + \frac{1}{\pi}\right)\right)\right\} e^{-\pi} \\ &< 1.82967310. \end{aligned} \tag{7.5}$$

We can estimate other $|\psi^{(k)}(1)|$ by the same way. We do not give their estimation furthermore, because we do not need their real values in our argument.

We show the following lemma using the above estimates.

Lemma 1. *If $t > 41.232345$, then we have*

$$0 < R(s) < 0.001176398$$

for $\frac{1}{2} \le \sigma \le 1$.

Proof. We have already shown in Sections 3 and 5 that

$$R(s) = \frac{1}{t^2} + \left(\frac{2}{t}\right)^3 \int_1^{\infty} \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx.$$

From the precise form of $f_2'(x)$ in Section 4, it follows that

$$\begin{aligned}|f_2'(x)| &\le \frac{1}{2^3}(3\sigma^2-3\sigma+1)\psi(x)\\ &\quad+\left(\frac{3}{2}\sigma^2-\frac{3}{2}\sigma+\frac{17}{4}\right)x^{\frac{\sigma}{2}}|\psi^{(1)}(x)|\\ &\quad+\frac{15}{2}x^{\frac{\sigma}{2}+1}\psi^{(2)}(x)+2x^{\frac{\sigma}{2}+2}|\psi^{(3)}(x)|\\ &\le \frac{1}{2^3}\psi(x)+\frac{17}{4}x|\psi^{(1)}(x)|+\frac{15}{2}x^2\psi^{(2)}(x)+2x^3|\psi^{(3)}(x)|\end{aligned}$$

for $\frac{1}{2} \le \sigma \le 1$ and $1 \le x$. Then we obtain

$$\begin{aligned}2^3\left|\int_1^\infty \sin\left(\frac{t}{2}\log x\right)f_2'(x)dx\right| &\le \int_1^\infty \psi(x)dx+34\int_1^\infty x|\psi^{(1)}(x)|dx\\ &\quad+60\int_1^\infty x^2\psi^{(2)}(x)dx\\ &\quad+16\int_1^\infty x^3|\psi^{(3)}(x)|dx\\ &= -\psi^{(-1)}(1)+34(\psi(1)-\psi^{(-1)}(1))\\ &\quad+60(-\psi^{(1)}(1)+2\psi(1)-2\psi^{(-1)}(1))\\ &\quad+16(\psi^{(2)}(1)-3\psi^{(1)}(1)+6\psi(1)\\ &\quad-6\psi^{(-1)}(1))\\ &= -251\psi^{(-1)}(1)+250\psi(1)-108\psi^{(1)}(1)\\ &\quad+16\psi^{(2)}(1)\\ &< 41.232345\end{aligned}$$

by (5.15), (5.16), (5.17), (5.18), (7.1), (7.2), (7.3) and (7.4). Therefore we have

$$\frac{1}{t^2}\left(1-41.232345\frac{1}{|t|}\right) < R(s) < \frac{1}{t^2}\left(1+41.232345\frac{1}{|t|}\right).$$

If $t > 41.232345$, then $1-41.232345\dfrac{1}{t} > 0$ and

$$\frac{1}{t^2}\left(1+41.232345\frac{1}{t}\right) < \frac{2}{(41.232345)^2} < 0.001176398.$$

□

We estimate $\alpha_1(s)$ in (6.4) by the same argument as the proof of Lemma 1.

Lemma 2. *There exists a constant* M_1 *such that*

$$|\alpha_1(s)| < M_1 \tag{7.6}$$

for $\frac{1}{2} \leq \sigma \leq 1$ *and* $t \in \mathbb{R}$.

We also have a similar result for $\beta_1(s)$ in (6.5).

Lemma 3. *There exists a constant* L_1 *such that*

$$|\beta_1(s)| < (2\sigma - 1)L_1 \tag{7.7}$$

for $\frac{1}{2} < \sigma \leq 1$ *and* $t \in \mathbb{R}$.

Proof. We rewrite $g_7'(x)$ in Section 4 as

$$\begin{aligned} g_7'(x) = &\sum_{j=0}^{7} \left(A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right) \psi^{(j)}(x) \\ &+ \left(x^{\frac{\sigma}{2}+7} - x^{-\frac{\sigma}{2}+\frac{15}{2}} \right) \psi^{(8)}(x). \end{aligned} \tag{7.8}$$

Now we have

$$\begin{aligned} &A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &\quad = \left(A_j(\sigma) - B_j(\sigma)\right) x^{\frac{\sigma}{2}-1+j} + B_j(\sigma) \left(x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right). \end{aligned} \tag{7.9}$$

It holds that

$$\begin{aligned} &x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &\quad = \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \left(\frac{\sigma}{2} - 1 + j \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^n \right\} (\log x)^n. \end{aligned}$$

We have

$$\begin{aligned} &\left(\frac{\sigma}{2} - 1 + j \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^n \\ &\quad = \left(\sigma - \frac{1}{2} \right) \sum_{k=0}^{n-1} \left(\frac{\sigma}{2} - 1 + j \right)^{n-1-k} \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^k. \end{aligned} \tag{7.10}$$

Since

$$\left| \frac{\sigma}{2} - 1 \right| < \frac{3}{4} < 1 \quad \text{and} \quad \left| -\frac{\sigma}{2} - \frac{1}{2} \right| \leq 1$$

for σ with $\frac{1}{2} < \sigma \leq 1$, we obtain

$$\left| \left(\frac{\sigma}{2} - 1 \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} \right)^n \right| \leq \left(\sigma - \frac{1}{2} \right) n$$

by (7.10). Therefore, we have

$$\begin{aligned}
&\left|x^{\frac{\sigma}{2}-1}-x^{-\frac{\sigma}{2}-\frac{1}{2}}\right| \\
&\quad\le \left(\sigma-\frac{1}{2}\right)(\log x)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}(\log x)^{n-1} \\
&\quad= \left(\sigma-\frac{1}{2}\right)(\log x)x.
\end{aligned}$$

Noting

$$\left|\frac{\sigma}{2}\right| \le \frac{1}{2} \quad \text{and} \quad \left|-\frac{\sigma}{2}-\frac{1}{2}+1\right| < \frac{1}{4} < \frac{1}{2},$$

we similarly obtain

$$\left|x^{\frac{\sigma}{2}}-x^{-\frac{\sigma}{2}-\frac{1}{2}+1}\right| \le \left(\sigma-\frac{1}{2}\right)(\log x)x^{\frac{1}{2}}.$$

Furthermore, we obtain by the same way as above

$$\left|x^{\frac{\sigma}{2}-1+j}-x^{-\frac{\sigma}{2}-\frac{1}{2}+j}\right| \le \left(\sigma-\frac{1}{2}\right)(\log x)x^{-\frac{1}{2}+j}$$

for $j=2,3,\ldots,7$.

Moreover, we have the following estimates for any σ with $\frac{1}{2}<\sigma\le 1$

$$\begin{aligned}
|A_0(\sigma)-B_0(\sigma)| &= \frac{1}{2^8}(2\sigma-1)\left|4\sigma^6-12\sigma^5+22\sigma^4-24\sigma^3+16\sigma^2-6\sigma+1\right| \\
&\le \frac{1}{2^8}(2\sigma-1),
\end{aligned}$$

$$|B_0(\sigma)| < B_0\left(\frac{1}{2}\right) = \frac{1}{2^{16}},$$

$$\begin{aligned}
|A_1(\sigma)-B_1(\sigma)| &= \frac{1}{2^8}(2\sigma-1)\left|16\sigma^6-48\sigma^5+928\sigma^4-1776\sigma^3+6784\sigma^2\right. \\
&\quad \left.-5904\sigma+6304\right| \\
&\le \frac{197}{2^3}(2\sigma-1),
\end{aligned}$$

$$|B_1(\sigma)| < B_1\left(\frac{1}{2}\right) = \frac{12207}{2^{11}},$$

$$\begin{aligned}
|A_2(\sigma)-B_2(\sigma)| &= \frac{1680}{2^8}(2\sigma-1)\left|\sigma^4-2\sigma^3+29\sigma^2-28\sigma+93\right| \\
&\le \frac{3905}{2^6}(2\sigma-1),
\end{aligned}$$

$$|B_2(\sigma)| < B_2\left(\frac{1}{2}\right) = \frac{330199}{2^{10}},$$

$$
\begin{aligned}
|A_3(\sigma) - B_3(\sigma)| &= \frac{448}{2^8}(2\sigma-1)\left|\sigma^4 - 2\sigma^3 + 134\sigma^2 - 133\sigma + 1164\right| \\
&\le 2037(2\sigma-1), \\
|B_3(\sigma)| &< B_3\left(\frac{1}{2}\right) = \frac{223881}{2^7}, \\
|A_4(\sigma) - B_4(\sigma)| &= \frac{1}{2^3}(2\sigma-1)\left|630\sigma^2 - 630\sigma + 15435\right| \\
&\le \frac{15435}{2^3}(2\sigma-1), \\
|B_4(\sigma)| &< B_4\left(\frac{1}{2}\right) = \frac{323043}{2^7}, \\
|A_5(\sigma) - B_5(\sigma)| &= (2\sigma-1)\left|7\sigma^2 - 7\sigma + 672\right| \\
&\le 672(2\sigma-1), \\
|B_5(\sigma)| &< B_5\left(\frac{1}{2}\right) = \frac{10857}{2^3}, \\
|A_6(\sigma) - B_6(\sigma)| &= 91(2\sigma-1), \quad |B_6(\sigma)| < B_6\left(\frac{1}{2}\right) = \frac{1239}{2^2}, \\
|A_7(\sigma) - B_7(\sigma)| &= 4(2\sigma-1) \quad \text{and} \quad |B_7(\sigma)| < B_7\left(\frac{1}{2}\right) = 30.
\end{aligned}
$$

We have $x^{\frac{\sigma}{2}} \le x$, $(\log x)x^{\frac{1}{2}} \le x$, $x^{\frac{\sigma}{2}-1} \le 1$ and $(\log x)x \le x^2$ for x with $1 \le x$. Then, it follows from (7.8) and the above inequalities that

$$
\begin{aligned}
\frac{2^8}{2\sigma-1}|g_7'(x)| < \psi(x) &+ \frac{1}{2^9}x^2\psi(x) + \left(2^5 \cdot 197 + \frac{12207}{24}\right) x|\psi^{(1)}(x)| \\
&+ \left(2^2 \cdot 3905 + \frac{330199}{2^3}\right) x^2\psi^{(2)}(x) \\
&+ (2^8 \cdot 2037 + 223881)x^3|\psi^{(3)}(x)| \\
&+ (2^5 \cdot 15435 + 323043)x^4\psi^{(4)}(x) \\
&+ (2^{13} \cdot 21 + 2^4 \cdot 10857)x^5|\psi^{(5)}(x)| \\
&+ (2^8 \cdot 91 + 2^5 \cdot 1239)x^6\psi^{(6)}(x) \\
&+ (2^{10} + 2^8 \cdot 15)x^7|\psi^{(7)}(x)| \\
&+ 2^7 x^8\psi^{(8)}(x).
\end{aligned}
$$

Integrating it, we obtain the conclusion by the results in Section 5 and the estimates of $|\psi^{(k)}(1)|$. □

Furthermore, we have the following lemma.

Lemma 4. *There exist constants M_1' and L_1' such that*

$$\left|\frac{\partial\alpha_1}{\partial t}(s)\right| < M_1' \tag{7.11}$$

and

$$\left|\frac{\partial\beta_1}{\partial t}(s)\right| < (2\sigma-1)L_1' \tag{7.12}$$

for $\frac{1}{2} < \sigma \leq 1$ and $t \in \mathbb{R}$, respectively.

Proof. We have

$$\frac{\partial\alpha_1}{\partial t}(s) = 2^6 \int_1^\infty \log x \cos\left(\frac{t}{2}\log x\right) f_6'(x)dx$$

and

$$\frac{\partial\beta_1}{\partial t}(s) = 2^7 \int_1^\infty \log x \cos\left(\frac{t}{2}\log x\right) g_7'(x)dx.$$

Since $\log x < x^{\frac{1}{2}}$ for $1 \leq x$, we obtain the statement by the same argument as the proofs of Lemmas 2 and 3. □

8. Mean value theorem

We apply the mean value theorem to a function $\sqrt{1+x}$. For any x with $0 < |x| < 1$, there exists θ_x with $0 < \theta_x < 1$ such that

$$\sqrt{1+x} = 1 + \frac{1}{2}\frac{x}{\sqrt{1+\theta_x x}}. \tag{8.1}$$

In general, θ_x is not uniquely determined in the mean value theorem. However, we can explicitly write it in this case as follows:

$$\theta_x = \frac{x^2 - 4(\sqrt{1+x}-1)^2}{4x(\sqrt{1+x}-1)^2}.$$

We need the following lemma in the next section.

Lemma 5. *It holds that* $\lim_{x\to 0}\theta_x = \frac{1}{2}$ *and* $\lim_{x\to 0}(d\theta_x/dx)x = -\frac{1}{8}$.

Proof. If we set $\mu(x) = x^2 - 4(\sqrt{1+x}-1)^2$ and $\lambda(x) = 4x(\sqrt{1+x}-1)^2$, then we have

$$\lim_{x\to 0}\theta_x = \lim_{x\to 0}\frac{\mu(x)}{\lambda(x)} = \frac{\mu'''(0)}{\lambda'''(0)} = \frac{1}{2}$$

by l'Hôpital's rule.

We consider the next limit. We have

$$\begin{aligned}\frac{d\theta_x}{dx}x &= \frac{x\left(\mu'(x)\lambda(x) - \mu(x)\lambda'(x)\right)}{\lambda(x)^2} \\ &= \frac{x^2\left(\sqrt{1+x}-1\right) - x^3(1+x)^{-\frac{1}{2}} + 4\left(\sqrt{1+x}-1\right)^3}{4x\left(\sqrt{1+x}-1\right)^3}.\end{aligned}$$

We set

$$\nu(x) = x^2\left(\sqrt{1+x}-1\right) - x^3(1+x)^{-\frac{1}{2}} + 4\left(\sqrt{1+x}-1\right)^3$$

and $\tau(x) = 4x\left(\sqrt{1+x}-1\right)^3$. Then we obtain

$$\lim_{x\to 0}\frac{d\theta_x}{dx}x = \lim_{x\to 0}\frac{\nu(x)}{\tau(x)} = \frac{\nu^{(4)}(0)}{\tau^{(4)}(0)} = -\frac{1}{8}$$

using l'Hôpital's rule again. □

9. New zero-free region

In Section 2 we showed that $\xi(s) = 0$ if and only if (2.3) holds. The first equality of (2.3) is equivalent to

$$R(s)t^4 + \left\{(2\sigma^2 - 2\sigma + 1)R(s) - 1\right\}t^2 + \sigma^2(\sigma-1)^2R(s) + \sigma(\sigma-1) = 0.$$

If we set $T = t^2$, then it is

$$R(s)T^2 + B(s)T + C(s) = 0, \tag{9.1}$$

where we set

$$\begin{cases} B(s) = (2\sigma^2 - 2\sigma + 1)R(s) - 1, \\ C(s) = \sigma^2(\sigma-1)^2R(s) + \sigma(\sigma-1). \end{cases}$$

We assume $t > 41.232345$ in this section. Since we had known the truth of the Riemann hypothesis for $|t| \le 41.232345$ even in the early stage of numerical verifications ([1], [5] and [7]), it is not a restriction. Then $R(s) > 0, B(s) < 0$ and $C(s) < 0$ by Lemma 1. The equation (9.1) means

$$T = \frac{-B(s) + \sqrt{B(s)^2 - 4R(s)C(s)}}{2R(s)}$$

for $T > 0$. We set $D(s) = B(s)^2 - 4R(s)C(s)$ and $A(s) = -B(s) + \sqrt{D(s)}$. Then, we have

$$t = \sqrt{\frac{A(s)}{2R(s)}}. \tag{9.2}$$

The second equation of (2.3) is equivalent to

$$\left\{t^4 + (2\sigma^2 - 2\sigma + 1)t^2 + \sigma^2(\sigma - 1)^2\right\} I(s) = (2\sigma - 1)t.$$

We substitute (9.2) into the above equation. Then we have

$$\left\{\left(\frac{A(s)}{2R(s)}\right)^2 + (2\sigma^2 - 2\sigma + 1)\frac{A(s)}{2R(s)} + \sigma^2(\sigma - 1)^2\right\} I(s) = (2\sigma - 1)\sqrt{\frac{A(s)}{2R(s)}}.$$

Multiplying it squared by $(2R(s))^4$, we obtain

$$\left\{A(s)^2 + 2(2\sigma^2 - 2\sigma + 1)A(s)R(s) + 4\sigma^2(\sigma - 1)^2 R(s)^2\right\}^2 I(s)^2 = 8(2\sigma - 1)^2 A(s)R(s)^3. \tag{9.3}$$

For the sake of simplicity, we set $\alpha(\sigma) = 4\sigma^2 - 4\sigma + 1$, $\beta(\sigma) = 2\sigma^2 - 2\sigma + 1$ and $\gamma(\sigma) = 3\sigma^2 - 3\sigma + 1$. Since

$$\begin{aligned}
&A(s)^2 + 2(2\sigma^2 - 2\sigma + 1)A(s)R(s) + 4\sigma^2(\sigma - 1)^2 R(s)^2 \\
&= \{A(s) + (2\sigma^2 - 2\sigma + 1)R(s)\}^2 + \left\{4\sigma^2(\sigma - 1)^2 - (2\sigma^2 - 2\sigma + 1)^2\right\} R(s)^2 \\
&= \left(1 + \sqrt{D(s)}\right)^2 - (4\sigma^2 - 4\sigma + 1)R(s)^2 \\
&= 1 + 2\sqrt{D(s)} + D(s) - (4\sigma^2 - 4\sigma + 1)R(s)^2 \\
&= 2\left(1 + \sqrt{D(s)} - \alpha(\sigma)R(s)\right),
\end{aligned}$$

(9.3) is equivalent to

$$\left\{1 + 2\sqrt{D(s)} + D(s) - 2\left(1 + \sqrt{D(s)}\right)\alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2\right\} I(s)^2 = 2(2\sigma - 1)^2\left(-B(s) + \sqrt{D(s)}\right) R(s)^3.$$

Then we have

$$\begin{aligned}
&\left\{2\left(1 - \alpha(\sigma)R(s)\right) I(s)^2 - 2(2\sigma - 1)^2 R(s)^3\right\}\sqrt{D(s)} \\
&\qquad = -\left(1 + D(s) - 2\alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2\right) I(s)^2 \\
&\qquad\qquad - 2(2\sigma - 1)^2 B(s)R(s)^3.
\end{aligned}$$

We multiply it by itself. Then we have

$$\begin{aligned}(9.4)\quad & 4\left\{(1-\alpha(\sigma)R(s))I(s)^2-(2\sigma-1)^2R(s)^3\right\}^2 D(s)\\ &= \left\{\left(1+D(s)-2\alpha(\sigma)R(s)+\alpha(\sigma)^2R(s)^2\right)I(s)^2\right.\\ &\qquad \left. +2(2\sigma-1)^2B(s)R(s)^3\right\}^2.\end{aligned}$$

Since

$$\begin{aligned}D(s) &= B(s)^2-4R(s)C(s)\\ &= (4\sigma^2-4\sigma+1)R(s)^2-2(4\sigma^2-4\sigma+1)R(s)+1\\ &= \alpha(\sigma)R(s)^2-2\alpha(\sigma)R(s)+1,\end{aligned}$$

we have

$$\begin{aligned}&1+D(s)-2\alpha(\sigma)R(s)+\alpha(\sigma)^2R(s)^2\\ &\qquad = 2\left(\alpha(\sigma)\beta(\sigma)R(s)^2-2\alpha(\sigma)R(s)+1\right).\end{aligned}$$

Then, it follows from (9.4) that

$$\begin{aligned}(9.5)\quad & \left\{(1-\alpha(\sigma)R(s))^2 I(s)^4-2(2\sigma-1)^2\left(1-\alpha(\sigma)R(s)\right)R(s)^3I(s)^2\right.\\ &\quad \left.+(2\sigma-1)^4R(s)^6\right\}\left(\alpha(\sigma)R(s)^2-2\alpha(\sigma)R(s)+1\right)\\ &= \left(\alpha(\sigma)\beta(\sigma)R(s)^2-2\alpha(\sigma)R(s)+1\right)^2 I(s)^4\\ &\quad +2(2\sigma-1)^2\left(\alpha(\sigma)\beta(\sigma)R(s)^2-2\alpha(\sigma)R(s)+1\right)B(s)R(s)^3I(s)^2\\ &\quad +(2\sigma-1)^4B(s)^2R(s)^6.\end{aligned}$$

The left side minus the right side of (9.5) equals

$$\begin{aligned}&\alpha(\sigma)\left(1+\alpha(\sigma)-2\beta(\sigma)\right)R(s)^2I(s)^4\\ &\quad -2\alpha(\sigma)^2\left(1+\alpha(\sigma)-2\beta(\sigma)\right)R(s)^3I(s)^4\\ &\quad +\alpha(\sigma)^2\left(\alpha(\sigma)-\beta(\sigma)^2\right)R(s)^4I(s)^4\\ &\quad +2(2\sigma-1)^2\alpha(\sigma)\left(-1-2\alpha(\sigma)+3\beta(\sigma)\right)R(s)^5I(s)^2\\ &\quad +2(2\sigma-1)^2\alpha(\sigma)\left(\alpha(\sigma)-\beta(\sigma)^2\right)R(s)^6I(s)^2\\ &\quad +(2\sigma-1)^4\left(\alpha(\sigma)-\beta(\sigma)^2\right)R(s)^8\\ &\quad +2(2\sigma-1)^2\left(\alpha(\sigma)-\beta(\sigma)\right)R(s)^4I(s)^2\\ &\quad +2(2\sigma-1)^4\left(\beta(\sigma)-\alpha(\sigma)\right)R(s)^7.\end{aligned}$$

Dividing it by $R(s)^2$, we set

$$\begin{aligned}
F(s) :=\ & \alpha(\sigma)\left(1+\alpha(\sigma)-2\beta(\sigma)\right) I(s)^4 \\
& - 2\alpha(\sigma)^2\left(1+\alpha(\sigma)-2\beta(\sigma)\right) R(s)I(s)^4 \\
& + \alpha(\sigma)^2\left(\alpha(\sigma)-\beta(\sigma)^2\right) R(s)^2 I(s)^4 \\
& + 2(2\sigma-1)^2\alpha(\sigma)\left(-1-2\alpha(\sigma)+3\beta(\sigma)\right) R(s)^3 I(s)^2 \\
& + 2(2\sigma-1)^2\alpha(\sigma)\left(\alpha(\sigma)-\beta(\sigma)^2\right) R(s)^4 I(s)^2 \\
& + (2\sigma-1)^4\left(\alpha(\sigma)-\beta(\sigma)^2\right) R(s)^6 \\
& + 2(2\sigma-1)^2\left(\alpha(\sigma)-\beta(\sigma)\right) R(s)^2 I(s)^2 \\
& + 2(2\sigma-1)^4\left(\beta(\sigma)-\alpha(\sigma)\right) R(s)^5.
\end{aligned}$$

Since $1+\alpha(\sigma)-2\beta(\sigma)=0$, $\alpha(\sigma)=(2\sigma-1)^2$, $\alpha(\sigma)-\beta(\sigma)^2=-4\sigma^2(\sigma-1)^2$, $\alpha(\sigma)-\beta(\sigma)=2\sigma(\sigma-1)$ and $-1-2\alpha(\sigma)+3\beta(\sigma)=\beta(\sigma)-\alpha(\sigma)=-2\sigma(\sigma-1)$, we obtain

$$\begin{aligned}
F(s) =\ & -4\sigma^2(\sigma-1)^2(2\sigma-1)^4 R(s)^2 I(s)^4 \\
& - 4\sigma(\sigma-1)(2\sigma-1)^4 R(s)^3 I(s)^2 \\
& - 8\sigma^2(\sigma-1)^2(2\sigma-1)^4 R(s)^4 I(s)^2 \\
& - 4\sigma^2(\sigma-1)^2(2\sigma-1)^4 R(s)^6 \\
& + 4\sigma(\sigma-1)(2\sigma-1)^2 R(s)^2 I(s)^2 \\
& - 4\sigma(\sigma-1)(2\sigma-1)^4 R(s)^5.
\end{aligned}$$

Let

$$\begin{aligned}
G(s) :=\ & \sigma(\sigma-1)(2\sigma-1)^2 I(s)^4 + (2\sigma-1)^2 R(s) I(s)^2 \\
& + 2\sigma(\sigma-1)(2\sigma-1)^2 R(s)^2 I(s)^2 + \sigma(\sigma-1)(2\sigma-1)^2 R(s)^4 \\
& - I(s)^2 + (2\sigma-1)^2 R(s)^3.
\end{aligned}$$

Then we have

$$F(s) = -4\sigma(\sigma-1)(2\sigma-1)^2 R(s)^2 G(s). \tag{9.6}$$

Proposition 1. *If $s=\sigma+it$ with $\frac{1}{2}<\sigma<1$ and $t>41.232345$ is a zero of $\xi(s)$, then we have $G(s)=0$.*

Proof. In the above, we showed $F(s)=0$ if $\xi(s)=0$. By Lemma 1, $-4\sigma(\sigma-1)(2\sigma-1)^2R(s)^2>0$ for any s with $\frac{1}{2}<\sigma<1$ and $t>41.232345$. Then $G(s)=0$ by (9.6). □

Theorem 1. *There exists $T_0>0$ such that any $s=\sigma+it$ with $\frac{1}{2}<\sigma<1$ and $t>T_0$ is not a zero of $\zeta(s)$.*

Proof. It suffices to find a region in which $G(s) \neq 0$, by Proposition 1.

Adding to $\alpha(\sigma), \beta(\sigma)$ and $\gamma(\sigma)$, we set

$$\begin{cases} C_1(\sigma) = 5\sigma^4 - 10\sigma^3 + 10\sigma^2 - 5\sigma + 1, \\ C_2(\sigma) = 3\sigma^4 - 6\sigma^3 + 7\sigma^2 - 4\sigma + 1. \end{cases}$$

From (6.4) and (6.5), it follows that

$$R(s)^2 = \frac{1}{t^4} - \frac{2}{t^6}\gamma(\sigma) + \frac{1}{t^8}\left(\gamma(\sigma)^2 + 2C_1(\sigma)\right) + \frac{1}{t^9}\alpha_2(s) \tag{9.7}$$

and

$$\begin{aligned} I(s)^2 &= \frac{1}{t^6}(2\sigma - 1)^2 - \frac{2}{t^8}(2\sigma - 1)^2\beta(\sigma) \\ &\quad + \frac{1}{t^{10}}(2\sigma - 1)^2\left(\beta(\sigma)^2 + 2C_2(\sigma)\right) + \frac{1}{t^{11}}\beta_2(s), \end{aligned} \tag{9.8}$$

where we set

$$\begin{aligned} \alpha_2(s) = 2\alpha_1(s) &- \frac{2}{t}\gamma(\sigma)C_1(\sigma) - \frac{2}{t^2}\gamma(\sigma)\alpha_1(s) + \frac{1}{t^3}C_1(\sigma)^2 \\ &+ \frac{2}{t^4}C_1(\sigma)\alpha_1(s) + \frac{1}{t^5}\alpha_1(s)^2 \end{aligned}$$

and

$$\begin{aligned} \beta_2(s) = 2(2\sigma - 1)\beta_1(s) &- \frac{2}{t}(2\sigma - 1)^2\beta(\sigma)C_2(\sigma) \\ &- \frac{2}{t^2}(2\sigma - 1)\beta(\sigma)\beta_1(s) + \frac{1}{t^3}(2\sigma - 1)^2 C_2(\sigma)^2 \\ &+ \frac{2}{t^4}(2\sigma - 1)C_2(\sigma)\beta_1(s) + \frac{1}{t^5}\beta_1(s)^2. \end{aligned}$$

We rewrite $G(s)$ as

$$\begin{aligned} G(s) &= \sigma(\sigma - 1)(2\sigma - 1)^2\left(R(s)^2 + I(s)^2\right)^2 \\ &\quad + (2\sigma - 1)^2 R(s)\left(R(s)^2 + I(s)^2\right) - I(s)^2. \end{aligned}$$

The equation $G(s) = 0$ means that $R(s)^2 + I(s)^2$ is a root of the following quadratic equation

$$\sigma(\sigma - 1)(2\sigma - 1)^2 X^2 + (2\sigma - 1)^2 R(s) X - I(s)^2 = 0.$$

Then we have

$$R(s)^2 + I(s)^2 = \frac{-(2\sigma - 1)R(s) \pm \sqrt{(2\sigma - 1)^2 R(s)^2 + 4\sigma(\sigma - 1)I(s)^2}}{2\sigma(\sigma - 1)(2\sigma - 1)}. \tag{9.9}$$

We determine the sign in (9.9). It follows from (9.7) and (9.8) that

$$R(s)^2 + I(s)^2 = \frac{1}{t^4} - \frac{1}{t^6}\beta(\sigma) + \frac{1}{t^8}C_3(\sigma) + \frac{1}{t^9}\gamma_1(s), \tag{9.10}$$

where we set

$$C_3(\sigma) = \gamma(\sigma)^2 + 2C_1(\sigma) - 2(2\sigma-1)^2\beta(\sigma)$$

and

$$\gamma_1(s) = \alpha_2(s) + \frac{1}{t}(2\sigma-1)^2\left(\beta(\sigma)^2 + 2C_2(\sigma)\right) + \frac{1}{t^2}\beta_2(s).$$

The functions $\beta(\sigma)$ and $C_3(\sigma)$ are uniformly bounded on $\frac{1}{2} < \sigma < 1$. By Lemmas 2 and 3, we also see that $\gamma_1(s)$ is uniformly bounded on $\frac{1}{2} < \sigma < 1$ and $t > t_0$, where t_0 is some constant with $t_0 > 41.232345$. Then $t^2\left(R(s)^2 + I(s)^2\right) \longrightarrow 0$ as $t \longrightarrow \infty$. On the other hand, we have $\lim_{t\to\infty} t^2 R(s) = 1$ and

$$\lim_{t\to\infty} t^2\sqrt{(2\sigma-1)^2R(s)^2 + 4\sigma(\sigma-1)I(s)^2} = 2\sigma - 1.$$

Therefore, the sign in (9.9) must be plus. Thus we have

$$\begin{aligned}2\sigma(\sigma-1)(2\sigma-1)\left(R(s)^2 + I(s)^2\right) + (2\sigma-1)R(s)& \\ - \sqrt{(2\sigma-1)^2R(s)^2 + 4\sigma(\sigma-1)I(s)^2} &= 0\end{aligned} \tag{9.11}$$

if s is a zero of $\xi(s)$ with $\frac{1}{2} < \sigma < 1$ and $t > t_0$.

By the same way as (9.10), we obtain

$$\begin{aligned}&(2\sigma-1)^2R(s)^2 + 4\sigma(\sigma-1)I(s)^2 \\ &\quad= \frac{1}{t^4}(2\sigma-1)^2 - \frac{2}{t^6}(2\sigma-1)^2(\sigma^2-\sigma+1) \\ &\qquad + \frac{1}{t^8}(2\sigma-1)^2C_4(\sigma) + \frac{1}{t^9}(2\sigma-1)^2\gamma_2(s),\end{aligned}$$

where we set

$$\begin{aligned}C_4(\sigma) &= \gamma(\sigma)^2 + 2C_1(\sigma) - 8\sigma(\sigma-1)\beta(\sigma) \\ &= 3\sigma^4 - 6\sigma^3 + 11\sigma^2 - 8\sigma + 3\end{aligned}$$

and

$$\begin{aligned}\gamma_2(s) = \alpha_2(s) &+ \frac{4}{t}\sigma(\sigma-1)\left(\beta(\sigma)^2 + 2C_2(\sigma)\right) \\ &+ \frac{4}{t^2}\sigma(\sigma-1)\frac{\beta_2(s)}{(2\sigma-1)^2}.\end{aligned}$$

If we set

$$x = x(s) := -\frac{2}{t^2}(\sigma^2-\sigma+1) + \frac{1}{t^4}C_4(\sigma) + \frac{1}{t^5}\gamma_2(s),$$

then we have

$$\sqrt{(2\sigma-1)^2R(s)^2+4\sigma(\sigma-1)I(s)^2}=\frac{1}{t^2}(2\sigma-1)\sqrt{1+x}. \tag{9.12}$$

Therefore, the equation (9.11) is equivalent to

$$2\sigma(\sigma-1)\left(R(s)^2+I(s)^2\right)+R(s)-\frac{1}{t^2}\sqrt{1+x}=0. \tag{9.13}$$

We apply the results in Section 8 to $\sqrt{1+x}$. Letting $M(\theta_x)=\frac{1}{2}(1+\theta_x x)^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}\sqrt{1+x}&=1-\frac{2}{t^2}M(\theta_x)(\sigma^2-\sigma+1)+\frac{1}{t^4}M(\theta_x)C_4(\sigma)\\&\quad+\frac{1}{t^5}M(\theta_x)\gamma_2(s).\end{aligned} \tag{9.14}$$

From (6.4), (9.10) and (9.14), it follows that

$$\begin{aligned}&2\sigma(\sigma-1)\left(R(s)^2+I(s)^2\right)+R(s)-\frac{1}{t^2}\sqrt{1+x}\\&\quad=\frac{1}{t^4}\left\{2\sigma(\sigma-1)-\gamma(\sigma)+2M(\theta_x)(\sigma^2-\sigma+1)\right\}\\&\quad\quad-\frac{1}{t^6}\left\{2\sigma(\sigma-1)\beta(\sigma)-C_1(\sigma)+M(\theta_x)C_4(\sigma)\right\}+\frac{1}{t^7}E(s),\end{aligned}$$

where we set

$$\begin{aligned}E(s)&=\alpha_1(s)-M(\theta_x)\gamma_2(s)+\frac{2}{t}\sigma(\sigma-1)C_3(\sigma)\\&\quad+\frac{2}{t^2}\sigma(\sigma-1)\gamma_1(s).\end{aligned}$$

Since $0<\theta_x<1$ and $x<0$ for a sufficiently large t, we have

$$\begin{aligned}&2\sigma(\sigma-1)-\gamma(\sigma)+2M(\theta_x)(\sigma^2-\sigma+1)\\&\qquad\qquad=(\sigma^2-\sigma+1)\left\{(1+\theta_x x)^{-\frac{1}{2}}-1\right\}>0\end{aligned}$$

for $\frac{1}{2}<\sigma<1$.

Now we have

$$
\begin{aligned}
(9.15)\quad & 2\sigma(\sigma-1)\left(R(s)^2+I(s)^2\right)+R(s)-\frac{1}{t^2}\sqrt{1+x} \\
&= \frac{1}{t^4}(\sigma^2-\sigma+1)\left\{(1+\theta_x x)^{-\frac{1}{2}}-1\right\} \\
&- \frac{1}{t^6}\left\{2\sigma(\sigma-1)\beta(\sigma)-C_1(\sigma)+M(\theta_x)C_4(\sigma)\right\}+\frac{1}{t^7}E(s) \\
&= \frac{8\left\{(1+\theta_x x)^{-\frac{1}{2}}-1\right\}}{3t^4(\sigma^2-\sigma+1)}\times\Bigg[\frac{3}{8}(\sigma^2-\sigma+1)^2 \\
&- \frac{3}{8}(\sigma^2-\sigma+1)\frac{2\sigma(\sigma-1)\beta(\sigma)-C_1(\sigma)+M(\theta_x)C_4(\sigma)}{t^2\left\{(1+\theta_x x)^{-\frac{1}{2}}-1\right\}} \\
&+ \frac{3}{8t}(\sigma^2-\sigma+1)\frac{E(s)}{t^2\left\{(1+\theta_x x)^{-\frac{1}{2}}-1\right\}}\Bigg].
\end{aligned}
$$

We obtain by a direct calculation

$$
\begin{aligned}
\frac{\partial\gamma_2}{\partial t}(s) &= \frac{\partial\alpha_2}{\partial t}(s)-\frac{4}{t^2}\sigma(\sigma-1)\left(\beta(\sigma)^2+2C_2(\sigma)\right)-\frac{8}{t^3}\sigma(\sigma-1)\frac{\beta_2(s)}{(2\sigma-1)^2} \\
&+ \frac{4}{t^2}\sigma(\sigma-1)\frac{1}{(2\sigma-1)^2}\frac{\partial\beta_2}{\partial t}(s),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\alpha_2}{\partial t}(s) &= 2\frac{\partial\alpha_1}{\partial t}(s)+\frac{1}{t^2}\gamma(\sigma)^2+\frac{6}{t^3}\gamma(\sigma)\alpha_1(s)-\frac{2}{t^2}\gamma(\sigma)\frac{\partial\alpha_1}{\partial t}(s) \\
&+ \frac{3}{t^4}\alpha_1(s)^2-\frac{2}{t^3}\alpha_1(s)\frac{\partial\alpha_1}{\partial t}(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\beta_2}{\partial t}(s) &= 2(2\sigma-1)\frac{\partial\beta_1}{\partial t}(s)+\frac{2}{t^2}(2\sigma-1)^2\beta(\sigma)C_2(\sigma) \\
&+ \frac{6}{t^3}(2\sigma-1)\beta(\sigma)\beta_1(s)-\frac{2}{t^2}(2\sigma-1)\beta(\sigma)\frac{\partial\beta_1}{\partial t}(s) \\
&- \frac{3}{t^4}(2\sigma-1)^2C_2(\sigma)^2-\frac{8}{t^5}(2\sigma-1)C_2(\sigma)\beta_1(s) \\
&+ \frac{2}{t^4}(2\sigma-1)C_2(\sigma)\frac{\partial\beta_1}{\partial t}(s)-\frac{5}{t^6}\beta_1(s)^2 \\
&+ \frac{2}{t^5}\beta_1(s)\frac{\partial\beta_1}{\partial t}(s).
\end{aligned}
$$

Then, we can take positive constants $M_2, M_2', L_2, L_2', N_1, N_2$ and N_2' such that $|\alpha_2(s)| < M_2$, $|\partial\alpha_2/\partial t(s)| < M_2'$, $|\beta_2(s)| < (2\sigma-1)^2L_2$,

$|\partial\beta_2/\partial t(s)| < (2\sigma-1)^2 L_2'$, $|\gamma_1(s)| < N_1$, $|\gamma_2(s)| < N_2$ and $|\partial\gamma_2/\partial t(s)| < N_2'$ for $\frac{1}{2} < \sigma < 1$ and $t > t_0$, by Lemmas 2, 3 and 4. Since

$$\frac{d}{dt}\left\{(1+\theta_x x)^{-\frac{1}{2}} - 1\right\} = -\frac{1}{2}(1+\theta_x x)^{-\frac{3}{2}}\left(\frac{d\theta_x}{dx}x + \theta_x\right)\frac{\partial x}{\partial t},$$

$$\frac{\partial x}{\partial t} = \frac{4}{t^3}(\sigma^2-\sigma+1) - \frac{4}{t^5}C_4(\sigma) - \frac{5}{t^6}\gamma_2(s) + \frac{1}{t^5}\frac{\partial\gamma_2}{\partial t}(s)$$

and $d(1/t^2)/dt = -2/t^3$, we obtain

$$\lim_{t\to\infty} t^2\left\{(1+\theta_x x)^{-\frac{1}{2}} - 1\right\} = \frac{3}{8}(\sigma^2-\sigma+1)$$

by l'Hôpital's rule, Lemma 5 and the above estimates. Then we have

$$\lim_{t\to\infty}\frac{3}{8}(\sigma^2-\sigma+1)\frac{2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + M(\theta_x)C_4(\sigma)}{t^2\left\{(1+\theta_x x)^{-\frac{1}{2}} - 1\right\}}$$
$$= 2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + \frac{1}{2}C_4(\sigma).$$

We note that the convergence is uniformly for $\frac{1}{2} < \sigma < 1$. We have

$$\frac{3}{8}(\sigma^2-\sigma+1)^2 - 2\sigma(\sigma-1)\beta(\sigma) + C_1(\sigma) - \frac{1}{2}C_4(\sigma)$$
$$= -\frac{1}{8}(\sigma^4 - 2\sigma^3 + 3\sigma^2 - 2\sigma + 1) < -\frac{9}{128}$$

and $\frac{3}{4} < \sigma^2 - \sigma + 1 < 1$ for $\frac{1}{2} < \sigma < 1$. Furthermore, there exists a constant $\widetilde{M}$ such that $|E(s)| < \widetilde{M}$ for $\frac{1}{2} < \sigma < 1$ and $t > t_0$. Therefore, there exists $T_0 > 0$ such that if $t > T_0$, then the right side of (9.15) is negative for $\frac{1}{2} < \sigma < 1$. Thus, the proof is complete. □

10. Lemma on Dirichlet series

This section is devoted to the following lemma which is a variant of Lemma 3.12 in [8].

Lemma 6. *We assume that a Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is absolutely convergent for $\sigma > 1$. *Take* $\sigma_0 < 1$ *and* $c > 0$ *with* $\sigma_0 + c > 1$. *Let* $g(s)$ *be an entire function with finite number of zeros* $Z_g = \{\alpha_1, \ldots, \alpha_N\}$ *such that* $\sigma_0 < \mathrm{Re}(\alpha_j) < \sigma_0 + c$ $(j = 1, \ldots, N)$ *and* $|g(s)| \to \infty$ *as* $s \to \infty$. *Then, there exist meromorphic functions* $h_n(s)$

$(1 \leq n < x)$ on $\mathbb{C}$ whose poles are at most $\alpha_1, \ldots, \alpha_N$ and all simple such that

$$\sum_{n<x} \frac{a_n}{n^s}\left(\frac{1}{g(s)} + h_n(s)\right) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{f(s+w)}{g(s+w)}\frac{x^w}{w}dw + \sum_{n<x}\frac{a_n}{n^s}R_n(c,x,T)(s) + \sum_{x<n}\frac{a_n}{n^s}Q_n(c,x,T)(s) \tag{10.1}$$

for any $s \in D(\sigma_0, T) \setminus Z_g$, where a positive number x is not an integer, T is a positive number with $|\mathrm{Im}(\alpha_j)| < \frac{T}{4}$ $(j = 1, \ldots, N)$, $D(\sigma_0, T) = \{\sigma + it; \sigma_0 \leq \sigma, |t| < \frac{T}{4}\}$, and $R_n(c, x, T)(s)$ and $Q_n(c, x, T)(s)$ are holomorphic functions depending only on $g(s), n, c, x, T$ and satisfying

$$|R_n(c,x,T)(s)| < \frac{M}{A\pi T}\left(\frac{x}{n}\right)^c \quad \text{and} \quad |Q_n(c,x,T)(s)| < \frac{M}{A\pi T}\left(\frac{x}{n}\right)^c \tag{10.2}$$

on $D(\sigma_0, T)$ for some constants A and M. Therefore, the right side of (10.1) converges absolutely and uniformly on $D(\sigma_0, T)$.

Proof. Take any $s \in D(\sigma_0, T)$. A function $\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{1}{w}$ of w has poles at $w = 0, \alpha_1 - s, \ldots, \alpha_N - s$. It has the residues $\frac{1}{g(s)}$ and $a_j\left(\frac{x}{n}\right)^{\alpha_j - s}\frac{1}{\alpha_j - s}$ at $w = 0$ and $\alpha_j - s$ respectively, where a_j is the residue of $\frac{1}{g(s)}$ at α_j. We define a meromorphic function

$$h_n(s) := \sum_{j=1}^{N} a_j \left(\frac{x}{n}\right)^{\alpha_j - s}\frac{1}{\alpha_j - s}.$$

If $n < x$, then we obtain

$$\frac{1}{2\pi i}\left(\int_{-\infty-iT}^{c-iT} + \int_{c-iT}^{c+iT} + \int_{c+iT}^{-\infty+iT}\right)\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w \frac{dw}{w} = \frac{1}{g(s)} + h_n(s)$$

by the residue theorem. Let

$$M := \sup\left\{\left|\frac{1}{g(s)}\right|; s = u + it, -\infty < u < \infty, \frac{3}{4}T \leq |t| \leq \frac{5}{4}T\right\}.$$

Then we have $0 < M < \infty$ by the assumption of $g(s)$. Therefore, the following estimation holds

$$\left|\frac{1}{2\pi i}\int_{-\infty-iT}^{c-iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\right| < \frac{M}{2\pi T}\frac{(x/n)^c}{\log(x/n)}.$$

Similarly we have

$$\left| \frac{1}{2\pi i} \int_{c+iT}^{-\infty+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} \right| < \frac{M}{2\pi T} \frac{(x/n)^c}{\log(x/n)}.$$

We define

$$R_n(c,x,T)(s) := \frac{1}{2\pi i} \left(\int_{-\infty-iT}^{c-iT} + \int_{c+iT}^{-\infty+iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w}.$$

Then we obtain

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \frac{1}{g(s)} + h_n(s) - R_n(c,x,T)(s) \tag{10.3}$$

and

$$|R_n(c,x,T)(s)| < \frac{M}{\pi T} \frac{(x/n)^c}{\log(x/n)}$$

for $s \in D(\sigma_0, T)$.

For $x < n$, we similarly obtain

$$\frac{1}{2\pi i} \left(\int_{\infty+iT}^{c+iT} + \int_{c+iT}^{c-iT} + \int_{c-iT}^{\infty-iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = 0,$$

because there is no residue term. If we set

$$Q_n(c,x,T)(s) := \frac{1}{2\pi i} \left(\int_{c+iT}^{\infty+iT} + \int_{\infty-iT}^{c-iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w},$$

then we have

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = -Q_n(c,x,T)(s). \tag{10.4}$$

We also obtain

$$|Q_n(c,x,T)(s)| < \frac{M}{\pi T} \frac{(x/n)^c}{|\log(x/n)|}$$

for $s \in D(\sigma_0, T)$ by the same way as above.

From (10.3) and (10.4), it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{f(s+w)}{g(s+w)}\frac{x^w}{w}dw &= \sum_{n<x}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &\quad+\sum_{x<n}\frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{g(s+w)}\left(\frac{x}{n}\right)^w\frac{dw}{w}\\ &=\sum_{n<x}\frac{a_n}{n^s}\left(\frac{1}{g(s)}+h_n(s)\right)\\ &\quad-\sum_{n<x}\frac{a_n}{n^s}R_n(c,x,T)(s)\\ &\quad-\sum_{x<n}\frac{a_n}{n^s}Q_n(c,x,T)(s).\end{aligned}$$

Then we obtain (10.1). Furthemore, we can take $A>0$ such that $|\log(x/n)|>A$ for any $n\in\mathbb{N}$, by the assumption of x. Hence, we also obtain (10.2). □

11. Proof of the Riemann hypothesis

Let

$$B := \sup\{\beta;\zeta(\beta+i\gamma)=0,\gamma\neq 0\}.$$

The Riemann hypothesis states $B=\frac{1}{2}$. We recall the theorem of de la Vallée Poussin ([2]) which says that there is a constant $A>0$ such that $\zeta(s)$ is not zero for

$$\sigma\geq 1-\frac{A}{\log t}\quad (t>t_0),$$

where t_0 is some positive constant. We may restate it as follows:
if $s=\sigma+it$ $(t>t_0)$ satisfies

$$t\leq\exp\left(\frac{A}{1-\sigma}\right), \tag{11.1}$$

then $\zeta(s)\neq 0$.

Proposition 2. *It holds that* $\frac{1}{2}\leq B<1$.

Proof. Since

$$\exp\left(\frac{A}{1-\sigma}\right)\longrightarrow\infty\quad\text{as}\quad\sigma\longrightarrow 1-0,$$

there exists $\sigma_0<1$ such that $T_0<\exp\left(\frac{A}{1-\sigma}\right)$ for $\sigma_0<\sigma<1$, where T_0 is the constant in Theorem 1. Then, there is no zero of $\zeta(s)$ in a region $\sigma_0<\sigma$ by Theorem 1 and the theorem of de la Vallée Poussin. □

Proposition 3. *If* $\frac{1}{2} < B < 1$*, then there is no zero of* $\zeta(s)$ *on the line* $\sigma = B$*.*

Proof. If the function $\zeta(s)$ has a zero on $\sigma = B$, then the number of zeros of $\zeta(s)$ on $\sigma = B$ is finite by Theorem 1. Let $Z_B = \{\rho_1, \overline{\rho_1}, \dots, \rho_N, \overline{\rho_N}\}$ be the set of zeros of $\zeta(s)$ on the line $\sigma = B$. By Theorem 1, there exists $\delta_0 > 0$ such that there are no zeros of $\zeta(s)$ except Z_B in the set $\{\sigma + it; B - \delta_0 \le \sigma, -\infty < t < \infty\}$.

It is well-known that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1$. We define an entire function $g(s)$ by

$$g(s) := \prod_{j=1}^{N} (s - \rho_j)(s - \overline{\rho_j}).$$

Then, its zeros are Z_B, and $|g(s)| \longrightarrow \infty$ as $s \to \infty$. We take $T > 0$ such that $|\mathrm{Im}(\rho_j)| < \frac{T}{4}$ for $j = 1, \dots, N$. Putting $\sigma_0 = B - \delta_0$ and $c = 2$, we apply Lemma 6. If $x > 1$ is not an integer, then we have

(11.2)

$$\begin{aligned}\sum_{n<x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \\ &\quad + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) \\ &\quad + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s)\end{aligned}$$

for $s \in D(\sigma_0, T) \setminus Z_B$, where $h_n(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple.

We consider the integral of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ along $C = C_0 + C_1 + C_2 + C_3$, where C_0, C_1, C_2 and C_3 are segments from $2 - iT$ to $2 + iT$, from $2 + iT$ to $-\frac{\delta_0}{2} + iT$, from $-\frac{\delta_0}{2} + iT$ to $-\frac{\delta_0}{2} - iT$ and from $-\frac{\delta_0}{2} - iT$ to $2 - iT$, respectively. We set

$$D_0 := \left\{ \sigma + it; B - \frac{1}{4}\delta_0 < \sigma < B + \frac{1}{4}\delta_0, |t| < \frac{T}{4} \right\}.$$

Then $D_0 \subset D(\sigma_0, T)$ and $Z_B \subset D_0$. For any $s \in D_0 \setminus Z_B$, the poles of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ in a domain surrounded by C are $w = 0, \rho_1 - s, \overline{\rho_1} - s, \dots, \rho_N - s$ and $\overline{\rho_N} - s$. The residue of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ at $w = 0$ is $\frac{1}{\zeta(s)g(s)}$. Let a_j and b_j be the residues of $\frac{1}{\zeta(s)g(s)}$ at ρ_j and $\overline{\rho_j}$ respectively.

Then, the residues of $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ at $\rho_j - s$ and $\overline{\rho_j} - s$ are $a_j\frac{x^{\rho_j - s}}{\rho_j - s}$ and $b_j\frac{x^{\overline{\rho_j}-s}}{\overline{\rho_j}-s}$ respectively. We define

$$Q(s) := \sum_{j=1}^{N}\left(a_j\frac{x^{\rho_j-s}}{\rho_j - s} + b_j\frac{x^{\overline{\rho_j}-s}}{\overline{\rho_j}-s}\right).$$

Then, $Q(s)$ is a meromorphic function on $\mathbb{C}$ whose poles are at most Z_B and all simple. We note that $\mathrm{Re}(s+w) > B - \frac{3}{4}\delta_0$ if $s \in D_0$ and w is on C. If $s \in D_0$ and w is on C_2, then $\mathrm{Re}(s+w) < B - \frac{1}{4}\delta_0$. Then, $\frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}$ is holomorphic on C as a function of w for any $s \in D_0$. By the residue theorem, we obtain

$$\frac{1}{2\pi i}\int_C \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)g(s)} + Q(s)$$

for $s \in D_0 \setminus Z_B$. Therefore we have

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw = \frac{1}{\zeta(s)g(s)} + Q(s) + P(s) \tag{11.3}$$

for $s \in D_0 \setminus Z_B$, where

$$P(s) = -\frac{1}{2\pi i}\int_{C_1+C_2+C_3} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw$$

is a holomorphic function on D_0. We set

$$\begin{aligned} E := &\left\{\sigma + it; B - \frac{3}{4}\delta_0 \le \sigma \le B + 2 + \frac{1}{4}\delta_0, \frac{3}{4}T \le |t| \le \frac{5}{4}T\right\} \\ &\bigcup\left\{\sigma + it; B - \frac{3}{4}\delta_0 \le \sigma \le B - \frac{1}{4}\delta_0, |t| \le \frac{5}{4}T\right\}. \end{aligned}$$

Then we have $\{s+w; s \in D_0, w \in C_1 \cup C_2 \cup C_3\} \subset E$. Since there is no pole of $\frac{1}{\zeta(s)g(s)}$ on E, we can take $M_0 > 0$ such that

$$\left|\frac{1}{\zeta(s)g(s)}\right| < M_0$$

on E. Then we obtain

$$\left|\int_{C_1} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw\right| < \frac{M_0 x^2}{T\log x}$$

and

$$\left|\int_{C_3} \frac{1}{\zeta(s+w)g(s+w)}\frac{x^w}{w}dw\right| < \frac{M_0 x^2}{T\log x}$$

for $s \in D_0$. Since we have $w = -\frac{\delta_0}{2} + it$ and $x^w = e^{it \log x} x^{-\frac{\delta_0}{2}}$ on C_2, we obtain

$$\left| \int_{C_2} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \right| < \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}}$$

for $s \in D_0$. It follows from the above estimates that

$$|P(s)| < \frac{1}{2\pi} \left(\frac{2M_0 x^2}{T \log x} + \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}} \right) \tag{11.4}$$

on D_0.

By (11.2) and (11.3), we obtain

$$\begin{aligned} \sum_{n<x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) = \frac{1}{\zeta(s)g(s)} + Q(s) + P(s) \\ + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) \\ + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s) \end{aligned} \tag{11.5}$$

for $s \in D_0 \setminus Z_B$. We see that a function

$$P(s) + \sum_{n<x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) + \sum_{x<n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s)$$

is bounded on D_0 by the properties of $R_n(2, x, T)(s)$ and $Q_n(2, x, T)(s)$, and (11.4). The functions $\frac{1}{g(s)}, h_n(s)$ and $Q(s)$ are meromorphic functions on $\mathbb{C}$ whose poles are at most Z_B and all simple. On the other hand, the function $\frac{1}{\zeta(s)g(s)}$ has poles of order at least 2 at every point in Z_B. This contradicts to the equation (11.5). Hence, there is no zero of $\zeta(s)$ on the line $\sigma = B$. □

Proof of the Riemann hypothesis.

We may assume $\frac{1}{2} \leq B < 1$ by Proposition 2. Suppose that $\frac{1}{2} < B < 1$. Then there is no zero of $\zeta(s)$ on the line $\sigma = B$ by Proposition 3. Hence, we can take B' with $\frac{1}{2} < B' < B$ such that $\zeta(s) \neq 0$ for $B' < \sigma$ by Theorem 1. This contradicts to the definition of B. Thus we conclude $B = \frac{1}{2}$.

12. Simpleness of zeros

We prove the simpleness of zeros in this section. We have already proved the Riemann hypothesis in the previous section. Then, it suffices to consider $\xi'(s)$ on the critical line $\sigma = \frac{1}{2}$.

Lemma 7. *We have the following inequalities*

$$2^3 \left| \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx \right| < 41.04293449$$

and

$$2^3 \left| \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx \right| < 509.9044819$$

when $\sigma = \frac{1}{2}$.

Proof. On $\sigma = \frac{1}{2}$, we have

$$\begin{aligned} |f_2'(x)| \leq \frac{1}{2^3} \cdot \frac{1}{4}\psi(x) + \frac{31}{2^3}x|\psi^{(1)}(x)| + \frac{15}{2}x^2\psi^{(2)}(x) \\ + 2x^3|\psi^{(3)}(x)|. \end{aligned}$$

Then we obtain

$$\begin{aligned} 2^3 \left| \int_1^\infty \sin\left(\frac{t}{2}\log x\right) f_2'(x)dx \right| &\leq -\frac{989}{4}\psi^{(-1)}(1) + 247\psi(1) \\ &\quad - 108\psi^{(1)}(1) + 16\psi^{(2)}(1) \\ &< 41.04293449 \end{aligned}$$

by the same way as the proof of Lemma 1.

If $\sigma = \frac{1}{2}$, then we obtain

$$f_0^{(1)}(x) = \frac{1}{2}x^{-\frac{3}{4}}\psi(x) + 2x^{\frac{1}{4}}\psi^{(1)}(x),$$

$$f_0^{(2)}(x) = -\frac{3}{8}x^{-\frac{7}{4}}\psi(x) + x^{-\frac{3}{4}}\psi^{(1)}(x) + 2x^{\frac{1}{4}}\psi^{(2)}(x),$$

$$\begin{aligned} f_0^{(3)}(x) = \frac{21}{32}x^{-\frac{11}{4}}\psi(x) - \frac{9}{8}x^{-\frac{7}{4}}\psi^{(1)}(x) + \frac{3}{2}x^{-\frac{3}{4}}\psi^{(2)}(x) \\ + 2x^{\frac{1}{4}}\psi^{(3)}(x) \end{aligned}$$

and

$$\begin{aligned} f_0^{(4)}(x) = -\frac{231}{128}x^{-\frac{15}{4}}\psi(x) + \frac{21}{8}x^{-\frac{11}{4}}\psi^{(1)}(x) - \frac{9}{4}x^{-\frac{7}{4}}\psi^{(2)}(x) \\ + 2x^{-\frac{3}{4}}\psi^{(3)}(x) + 2x^{\frac{1}{4}}\psi^{(4)}(x). \end{aligned}$$

Then we have

$$\begin{aligned}|h_3'(x)| \le &\left(\frac{1135}{128}x^{-\frac{3}{4}}\psi(x) + \frac{147}{8}x^{\frac{1}{4}}|\psi^{(1)}(x)| + \frac{101}{4}x^{\frac{5}{4}}\psi^{(2)}(x)\right.\\ &\left. +14x^{\frac{9}{4}}|\psi^{(3)}(x)| + 2x^{\frac{13}{4}}\psi^{(4)}(x)\right)\log x\\ &+\frac{73}{8}x^{-\frac{3}{4}}\psi(x) + \frac{49}{2}x^{\frac{1}{4}}|\psi^{(1)}(x)| + 30x^{\frac{5}{4}}\psi^{(2)}(x)\\ &+8x^{\frac{9}{4}}|\psi^{(3)}(x)|\end{aligned}$$

by (4.1). Since $x^{-\frac{3}{4}}\log x < 1$ for $x \ge 1$, we obtain

$$\begin{aligned}|h_3'(x)| \le &\frac{2303}{128}\psi(x) + \frac{343}{8}x|\psi^{(1)}(x)| + \frac{221}{4}x^2\psi^{(2)}(x)\\ &+22x^3|\psi^{(3)}(x)| + 2x^4\psi^{(4)}(x).\end{aligned}$$

Therefore, we obtain

$$2^3\left|\int_1^\infty \sin\left(\frac{t}{2}\log x\right)h_3'(x)dx\right| < 509.9044819$$

by the same way as above. □

Theorem 2. *Every zero of* $\zeta(s)$ *is simple.*

Proof. It suffices to show that any zero of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$ is simple.

By (2.5) we have

$$\begin{cases} U(s) = \dfrac{1}{2}\displaystyle\int_1^\infty \log x\left(x^{\frac{\sigma}{2}-1} - x^{-\frac{\sigma}{2}-\frac{1}{2}}\right)\cos\left(\frac{t}{2}\log x\right)\psi(x)dx,\\ V(s) = \dfrac{1}{2}\displaystyle\int_1^\infty \log x\left(x^{\frac{\sigma}{2}-1} + x^{-\frac{\sigma}{2}-\frac{1}{2}}\right)\sin\left(\frac{t}{2}\log x\right)\psi(x)dx.\end{cases}$$

Since $I(\frac{1}{2}+it) = U(\frac{1}{2}+it) = 0$, (2.6) becomes

$$R\left(\frac{1}{2}+it\right) - \left(\frac{1}{4}+t^2\right)V\left(\frac{1}{2}+it\right) = 0 \tag{12.1}$$

on $\sigma = \frac{1}{2}$. If we set

$$\varepsilon_R = 2^3\int_1^\infty \sin\left(\frac{t}{2}\log x\right)f_2'(x)dx,$$

then we have

$$R\left(\frac{1}{2}+it\right) = \frac{1}{t^2} + \frac{1}{t^3}\varepsilon_R$$

as shown in the proof of Lemma 1. Putting

$$\varepsilon_V = 2^3 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx,$$

we have

$$V\left(\frac{1}{2}+it\right) = \frac{2}{t^3} + \frac{1}{t^4}\varepsilon_V$$

by (3.3). Then we obtain

$$R\left(\frac{1}{2}+it\right) - \left(\frac{1}{4}+t^2\right)V\left(\frac{1}{2}+it\right)$$
$$= -\frac{1}{t^4}\left\{2t^3 - (1-\varepsilon_V)t^2 + \left(\frac{1}{2}-\varepsilon_R\right)t - \frac{\varepsilon_V}{4}\right\}.$$

It follows from Lemma 7 that

$$2t^3 - (1-\varepsilon_V)t^2 + \left(\frac{1}{2}-\varepsilon_R\right)t - \frac{\varepsilon_V}{4} > 2t^3 - 510.9044819t^2 - 41.54293449t - 127.4761206. \tag{12.2}$$

If $t > 256$, then the right side of (12.2) is positive. Hence, the equality (12.1) does not hold. This means that any zero $\frac{1}{2}+it$ of $\zeta(s)$ with $t > 256$ is simple. It is well-known that all zeros for $|t| \le 256$ are simple ([5]). Thus, we complete the proof. □

References

[1] R. Backlund, Sur les zéros de la fonction $\zeta(s)$ de Riemann, C. R. Acad. Sci. Paris, V., **158** (1914), 1979–1982.

[2] C. J. de la Vallée Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donée, Mém. Couronnés et Autres Mém. Publ. Acad. Roy. Sci. des lettres Beaux-Arts Belg., **59** (1899–1900), 1–74.

[3] X. Gourdon, The 10^{13} first zeros of the Riemann zeta function, and zeros computation at very large height, http://numbers.computation.free.fr/Constants/Miscellaneouszetazeros1e13-1e24.pdf.

[4] J. Gram, Note sur les zéros de la fonction $\zeta(s)$ de Riemann, Acta Math., **27** (1903), 289–304.

[5] J. I. Hutchinson, On the roots of the Riemann zeta-function, Trans. Amer. Math. Soc., **27** (1925), 49–60.

[6] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Monat. der Königl. Preuss. Akad. der Wissen. zu Berlin aus der Jahre 1859, (1860), 671–680.

[7] E. C. Titchmarsh, The zeros of the Riemann zeta-function, Proc. Roy. Soc. London, **151** (1935), 234–255; also ibid., **157** (1936), 261–263.

[8] E. C. Titchmarsh, The Theory of the Riemann Zeta Function, second edition, edited and with a preface by D. R. Heath-Brown, The Clarendon Press, Oxford University Press, New York, 1986.

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