ON THE ZEROS OF THE RIEMANN ZETA FUNCTION ON THE CRITICAL LINE

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Abstract. This is the continuation of the previous paper [1]. By the same method as in the proof of the Riemann hypothesis, we prove that every zero of the Riemann zeta function on the critical line is simple for $t > T_0$, where T_0 is a constant.

1. Introduction

We proved the Riemann hypothesis in the previous paper [1]. The method used in it is also valid to investigate the simpleness of zeros of the Riemann zeta function $\zeta(s)$. We use the same notation as in [1], and frequently quote from it. Since nontrivial zeros of $\zeta(s)$ are zeros of the xi function $\xi(s)$, we consider $\xi(s)$.

It has the following representation

$$\xi(s) = \frac{1}{2} + \frac{s}{2}(s-1)\int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx, \tag{1.1}$$

where $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ and $s = \sigma + it$ ((2.1) in [1]). Let

$$\Phi(s) = \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx.$$

We denote by $R(s) = \mathrm{Re}\Phi(s)$ and $I(s) = \mathrm{Im}\Phi(s)$ the real part and the imaginary part of $\Phi(s)$, respectively. Then we have that $\xi(s) = 0$ if and only if

$$\begin{cases} R(s) = \dfrac{t^2 - \sigma(\sigma-1)}{(\sigma(\sigma-1)-t^2)^2 + t^2(2\sigma-1)^2}, \\ I(s) = \dfrac{t(2\sigma-1)}{(\sigma(\sigma-1)-t^2)^2 + t^2(2\sigma-1)^2} \end{cases} \tag{1.2}$$

(see Section 2 in [1]).

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We also consider $\xi'(s) = 0$. Differentiating (1.1), we obtain

$$\xi'(s) = \left(s - \frac{1}{2}\right)\Phi(s) + \frac{s}{2}(s-1)\Phi'(s), \tag{1.3}$$

where

$$\Phi'(s) = \frac{1}{2}\int_1^\infty \log x \left(x^{\frac{s}{2}-1} - x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)dx. \tag{1.4}$$

If we set $U(s) = \mathrm{Re}\Phi'(s)$ and $V(s) = \mathrm{Im}\Phi'(s)$, then we have

$$\begin{cases} \mathrm{Re}\xi'(s) = \left(\sigma - \frac{1}{2}\right)R(s) - tI(s) + \frac{1}{2}\left(\sigma(\sigma-1) - t^2\right)U(s) \\ \qquad\qquad - \frac{1}{2}t(2\sigma-1)V(s), \\ \mathrm{Im}\xi'(s) = tR(s) + \left(\sigma - \frac{1}{2}\right)I(s) + \frac{1}{2}t(2\sigma-1)U(s) \\ \qquad\qquad + \frac{1}{2}\left(\sigma(\sigma-1) - t^2\right)V(s) \end{cases}$$

by (1.3). Therefore, $\xi'(s) = 0$ if and only if

$$\begin{cases} \left(\sigma - \frac{1}{2}\right)R(s) - tI(s) + \frac{1}{2}\left(\sigma(\sigma-1) - t^2\right)U(s) \\ \qquad - \frac{1}{2}t(2\sigma-1)V(s) = 0, \\ tR(s) + \left(\sigma - \frac{1}{2}\right)I(s) + \frac{1}{2}t(2\sigma-1)U(s) \\ \qquad + \frac{1}{2}\left(\sigma(\sigma-1) - t^2\right)V(s) = 0. \end{cases} \tag{1.5}$$

Since the Riemann hypothesis is true, all nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. It is well-known that every trivial zero of $\zeta(s)$ is simple. Therefore, it suffices to consider zeros of $\xi(s)$ on $\sigma = \frac{1}{2}$.

We note $I(\frac{1}{2} + it) = 0$ and $U(\frac{1}{2} + it) = 0$. From (1.2), it follows that $\xi(\frac{1}{2} + it) = 0$ if and only if

$$R\left(\frac{1}{2} + it\right) = \frac{1}{t^2 + \frac{1}{4}}. \tag{1.6}$$

Similarly, we have that $\xi'(\frac{1}{2} + it) = 0$ if and only if

$$tR\left(\frac{1}{2} + it\right) - \frac{1}{2}\left(t^2 + \frac{1}{4}\right)V\left(\frac{1}{2} + it\right) = 0. \tag{1.7}$$

By showing that the equalities (1.6) and (1.7) do not hold simultaneously for sufficiently large t, we obtain our result that there exists T_0 such that every zero of $\zeta(\frac{1}{2}+it)$ with $t > T_0$ is simple (Theorem 1).

2. Expansion of $V(\frac{1}{2}+it)$

We have

$$V\left(\frac{1}{2}+it\right) = \int_1^\infty \sin\left(\frac{t}{2}\log x\right) x^{-\frac{3}{4}} \log x\psi(x)dx.$$

We define $h_0(x) := x^{\frac{1}{4}} \log x\psi(x)$ and $h_k(x) := xh'_{k-1}(x)$ for $k = 1, 2, \dots, 7$. Then we have

$$V\left(\frac{1}{2}+it\right) = -\left(\frac{2}{t}\right)^3 h_2(1) + \left(\frac{2}{t}\right)^4 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h'_3(x)dx \tag{2.1}$$

by integration by parts. Furthermore, we obtain

$$\begin{aligned} V\left(\frac{1}{2}+it\right) &= -\left(\frac{2}{t}\right)^3 h_2(1) + \left(\frac{2}{t}\right)^5 h_4(1) - \left(\frac{2}{t}\right)^7 h_6(1) \\ &\quad + \left(\frac{2}{t}\right)^8 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h'_7(x)dx \end{aligned} \tag{2.2}$$

by the successive use of integration by parts. We have the following formulas

$$h_1(x) = x^{\frac{1}{4}}\left(\frac{1}{4}\log x + 1\right)\psi(x) + x^{\frac{5}{4}}\log x\psi^{(1)}(x),$$

$$\begin{aligned} h_2(x) &= \frac{1}{4}x^{\frac{1}{4}}\left(\frac{1}{4}\log x + 2\right)\psi(x) + x^{\frac{5}{4}}\left(\frac{3}{2}\log x + 2\right)\psi^{(1)}(x) \\ &\quad + x^{\frac{9}{4}}\log x\psi^{(2)}(x), \end{aligned}$$

$$\begin{aligned} h_3(x) &= \frac{1}{16}x^{\frac{1}{4}}\left(\frac{1}{4}\log x + 3\right)\psi(x) + x^{\frac{5}{4}}\left(\frac{31}{16}\log x + \frac{9}{2}\right)\psi^{(1)}(x) \\ &\quad + x^{\frac{9}{4}}\left(\frac{15}{4}\log x + 3\right)\psi^{(2)}(x) + x^{\frac{13}{4}}\log x\psi^{(3)}(x), \end{aligned}$$

$$\begin{aligned} h'_3(x) &= \frac{1}{64}x^{-\frac{3}{4}}\left(\frac{1}{4}\log x + 4\right)\psi(x) + x^{\frac{1}{4}}\left(\frac{39}{16}\log x + \frac{31}{4}\right)\psi^{(1)}(x) \\ &\quad + x^{\frac{5}{4}}\left(\frac{83}{8}\log x + 15\right)\psi^{(2)}(x) + x^{\frac{9}{4}}(7\log x + 4)\psi^{(3)}(x) \\ &\quad + x^{\frac{13}{4}}\log x\psi^{(4)}(x), \end{aligned}$$

$$\begin{aligned}h_4(x) &= \frac{1}{64}x^{\frac{1}{4}}\left(\frac{1}{4}\log x+4\right)\psi(x)+x^{\frac{5}{4}}\left(\frac{39}{16}\log x+\frac{31}{4}\right)\psi^{(1)}(x)\\ &\quad+x^{\frac{9}{4}}\left(\frac{83}{8}\log x+15\right)\psi^{(2)}(x)+x^{\frac{13}{4}}(7\log x+4)\psi^{(3)}(x)\\ &\quad+x^{\frac{17}{4}}\log x\psi^{(4)}(x),\end{aligned}$$

$$\begin{aligned}h_5(x) &= \frac{1}{256}x^{\frac{1}{4}}\left(\frac{1}{4}\log x+5\right)\psi(x)+x^{\frac{5}{4}}\left(\frac{781}{256}\log x+\frac{195}{16}\right)\psi^{(1)}(x)\\ &\quad+x^{\frac{9}{4}}\left(\frac{825}{32}\log x+\frac{415}{8}\right)\psi^{(2)}(x)+x^{\frac{13}{4}}\left(\frac{265}{8}\log x+35\right)\psi^{(3)}(x)\\ &\quad+x^{\frac{17}{4}}\left(\frac{45}{4}\log x+5\right)\psi^{(4)}(x)+x^{\frac{21}{4}}\log x\psi^{(5)}(x),\end{aligned}$$

$$\begin{aligned}h_6(x) &= \frac{1}{1024}x^{\frac{1}{4}}\left(\frac{1}{4}\log x+6\right)\psi(x)+x^{\frac{5}{4}}\left(\frac{1953}{512}\log x+\frac{2343}{128}\right)\psi^{(1)}(x)\\ &\quad+x^{\frac{9}{4}}\left(\frac{15631}{256}\log x+\frac{2475}{16}\right)\psi^{(2)}(x)\\ &\quad+x^{\frac{13}{4}}\left(\frac{2135}{16}\log x+\frac{795}{4}\right)\psi^{(3)}(x)\\ &\quad+x^{\frac{17}{4}}\left(\frac{1295}{16}\log x+\frac{135}{2}\right)\psi^{(4)}(x)\\ &\quad+x^{\frac{21}{4}}\left(\frac{33}{2}\log x+6\right)\psi^{(5)}(x)+x^{\frac{25}{4}}\log x\psi^{(6)}(x),\end{aligned}$$

$$\begin{aligned}h_7(x) &= \frac{1}{4096}x^{\frac{1}{4}}\left(\frac{1}{4}\log x+7\right)\psi(x)\\ &\quad+x^{\frac{5}{4}}\left(\frac{19531}{4096}\log x+\frac{13671}{512}\right)\psi^{(1)}(x)\\ &\quad+x^{\frac{9}{4}}\left(\frac{144585}{1024}\log x+\frac{109417}{256}\right)\psi^{(2)}(x)\\ &\quad+x^{\frac{13}{4}}\left(\frac{126651}{256}\log x+\frac{14945}{16}\right)\psi^{(3)}(x)\\ &\quad+x^{\frac{17}{4}}\left(\frac{30555}{64}\log x+\frac{9065}{16}\right)\psi^{(4)}(x)\\ &\quad+x^{\frac{21}{4}}\left(\frac{2681}{16}\log x+\frac{231}{2}\right)\psi^{(5)}(x)\\ &\quad+x^{\frac{25}{4}}\left(\frac{91}{4}\log x+7\right)\psi^{(6)}(x)+x^{\frac{29}{4}}\log x\psi^{(7)}(x)\end{aligned}$$

and

$$\begin{aligned} h_7'(x) = & \frac{1}{16384}x^{-\frac{3}{4}}\left(\frac{1}{4}\log x + 8\right)\psi(x) \\ & + x^{\frac{1}{4}}\left(\frac{12207}{2048}\log x + \frac{19531}{512}\right)\psi^{(1)}(x) \\ & + x^{\frac{5}{4}}\left(\frac{330199}{1024}\log x + \frac{144585}{128}\right)\psi^{(2)}(x) \\ & + x^{\frac{9}{4}}\left(\frac{223881}{128}\log x + \frac{126651}{32}\right)\psi^{(3)}(x) \\ & + x^{\frac{13}{4}}\left(\frac{323043}{128}\log x + \frac{30555}{8}\right)\psi^{(4)}(x) \\ & + x^{\frac{17}{4}}\left(\frac{10857}{8}\log x + \frac{2681}{2}\right)\psi^{(5)}(x) \\ & + x^{\frac{21}{4}}\left(\frac{1239}{4}\log x + 182\right)\psi^{(6)}(x) \\ & + x^{\frac{25}{4}}(30\log x + 8)\psi^{(7)}(x) + x^{\frac{29}{4}}\log x\psi^{(8)}(x). \end{aligned}$$

Using (5.11) in [1], we obtain

$$h_2(1) = \frac{1}{2}\psi(1) + 2\psi^{(1)}(1) = 2\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) = -\frac{1}{4}. \tag{2.3}$$

Similarly, we have

$$\begin{aligned} h_4(1) &= \frac{1}{16}\psi(1) + \frac{31}{4}\psi^{(1)}(1) + 15\psi^{(2)}(1) + 4\psi^{(3)}(1) \\ &= -\frac{1}{32} \end{aligned} \tag{2.4}$$

by (5.11) and (5.12) in [1]. Furthermore, we have

$$\begin{aligned} h_6(1) &= \frac{6}{1024}\psi(1) + \frac{2343}{128}\psi^{(1)}(1) + \frac{2475}{16}\psi^{(2)}(1) \\ &\quad + \frac{795}{4}\psi^{(3)}(1) + \frac{135}{2}\psi^{(4)}(1) + 6\psi^{(5)}(1) \\ &= -\frac{3}{2^{10}} \end{aligned} \tag{2.5}$$

by (5.11), (5.12) and (5.13) in [1]. Therefore, we finally obtain

$$V\left(\frac{1}{2} + it\right) = \frac{2}{t^3} - \frac{1}{t^5} + \frac{3}{8t^7} + \frac{\varepsilon_V(t)}{t^8} \tag{2.6}$$

by (2.2), where we set

$$\varepsilon_V(t) := 2^8 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_7'(x)dx. \tag{2.7}$$

3. Estimates

We use the argument and results in Section 7 in [1].

Lemma 1. *If* $t > 243.4664312$*, then*

$$0 < V\left(\frac{1}{2}+it\right) < 0.00000028. \tag{3.1}$$

Proof. We have

$$V\left(\frac{1}{2}+it\right) = \frac{2}{t^3} + \left(\frac{2}{t}\right)^4 \int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx$$

by (2.1) and (2.3). Noting $x^{-\frac{3}{4}}\log x < 1$ and $x^{-\frac{3}{4}} \le 1$ for $x \ge 1$, we obtain

$$\begin{aligned}|h_3'(x)| \le \frac{17}{256}\psi(x) + \frac{163}{16}x|\psi^{(1)}(x)| + \frac{203}{8}x^2\psi^{(2)}(x) \\ + 11x^3|\psi^{(3)}(x)| + x^4\psi^{(4)}(x).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|\int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx\right| \\ &\quad\le \frac{17}{256}\int_1^\infty \psi(x)dx + \frac{163}{16}\int_1^\infty x\left|\psi^{(1)}(x)\right|dx + \frac{203}{8}\int_1^\infty x^2\psi^{(2)}(x)dx \\ &\qquad + 11\int_1^\infty x^3\left|\psi^{(3)}(x)\right|dx + \int_1^\infty x^4\psi^{(4)}(x)dx \\ &\quad= -\frac{38657}{256}\psi^{(-1)}(1) + \frac{2415}{16}\psi(1) - \frac{563}{8}\psi^{(1)}(1) + 15\psi^{(2)}(1) - \psi^{(3)}(1)\end{aligned}$$

by (5.15), (5.16), (5.17) and (5.18) in [1]. Hence, it follows that

$$2^4\left|\int_1^\infty \sin\left(\frac{t}{2}\log x\right) h_3'(x)dx\right| < 486.9328622$$

from (7.1), (7.2), (7.3), (7.4) and (7.5) in [1]. Thus we obtain

$$\frac{1}{t^3}\left(2 - 486.9328622\frac{1}{t}\right) < V\left(\frac{1}{2}+it\right) < \frac{1}{t^3}\left(2 + 486.9328622\frac{1}{t}\right)$$

for $t > 0$. If $t > 243.4664312$, then $2 - 486.9328622\frac{1}{t} > 0$ and

$$\frac{1}{t^3}\left(2 + 486.9328622\frac{1}{t}\right) < \frac{4}{t^3} < 0.00000028.$$

□

We also obtain the following lemma. We omit its proof because the argument is the same as above.

Lemma 2. *There exist positive constants M and M' such that*

$$|\varepsilon_V(t)| < M \quad \text{and} \quad |\varepsilon'_V(t)| < M' \tag{3.2}$$

for $t > 0$.

4. Necessary condition

Suppose that the equalities (1.6) and (1.7) hold at t simultaneously. In this case, the following equation must be satisfied

$$\left(t^2 + \frac{1}{4}\right)^2 V\left(\frac{1}{2} + it\right) = 2t. \tag{4.1}$$

If we set

$$X(t) := \left(t^2 + \frac{1}{4}\right) V\left(\frac{1}{2} + it\right),$$

then we have

$$X(t)t^2 - 2t + \frac{1}{4}X(t) = 0 \tag{4.2}$$

by (4.1). It holds that $0 < X(t) < 0.016597323$ for $t > 243.4664312$ by Lemma 1. By (4.2) we obtain

$$t = \frac{1 \pm \sqrt{1 - \frac{1}{4}X(t)^2}}{X(t)}$$

or

$$tX(t) - 1 = \pm\sqrt{1 - \frac{1}{4}X(t)^2}.$$

Since

$$\begin{aligned} tX(t) &= \left(t^3 + \frac{1}{4}t\right) V\left(\frac{1}{2} + it\right) \\ &= 2 - \frac{1}{2t^2} + \frac{1}{8t^4} + \frac{\varepsilon_V(t)}{t^5} + \frac{3}{32t^6} + \frac{\varepsilon_V(t)}{4t^7} \end{aligned}$$

by (2.6), we can take $t_0 > 0$ such that $\frac{3}{2} < tX(t)$ for $t > t_0$ by Lemma 2. Here we may assume $t_0 > 243.4664312$. Then $tX(t) - 1 > 0$. Therefore we have

$$tX(t) - 1 = \sqrt{1 - \frac{1}{4}X(t)^2} \tag{4.3}$$

if $t > t_0$.

We summarize the above result in the following proposition.

Proposition 1. *Let $t > t_0$. If the equalities (1.6) and (1.7) hold at t simultaneously, then the equation (4.3) is satisfied.*

5. Result

Theorem 1. *There exists $T_0 > 0$ such that every zero of $\zeta(\frac{1}{2} + it)$ with $t > T_0$ is simple.*

Proof. By Proposition 1, it suffices to show that the equation (4.3) does not hold if t is sufficiently large.

Consider a function

$$tX(t) - 1 - \sqrt{1 - \frac{1}{4}X(t)^2}. \tag{5.1}$$

We set $x = -\frac{1}{4}X(t)^2$, and apply the mean value theorem to a function $\sqrt{1+x}$. Then we obtain

$$\sqrt{1 - \frac{1}{4}X(t)^2} = 1 + \frac{1}{2}\frac{-\frac{1}{4}X(t)^2}{\sqrt{1 - \frac{\theta_x}{4}X(t)^2}},$$

where

$$\theta_x = \frac{x^2 - 4(\sqrt{1+x} - 1)^2}{4x(\sqrt{1+x} - 1)}$$

(see Section 8 in [1]). Therefore we have

$$\begin{aligned} tX(t) - 1 - \sqrt{1 - \frac{1}{4}X(t)^2} \\ = tX(t) - 2 + \frac{1}{8}\frac{X(t)^2}{\sqrt{1 - \frac{\theta_x}{4}X(t)^2}}. \end{aligned} \tag{5.2}$$

From (2.6) it follows that

$$X(t) = \frac{2}{t} - \frac{1}{2t^3} + \frac{1}{8t^5} + \frac{\varepsilon_X(t)}{t^6}, \tag{5.3}$$

where we set

$$\varepsilon_X(t) = \varepsilon_V(t) + \frac{3}{32t} + \frac{\varepsilon_V(t)}{4t^2}. \tag{5.4}$$

Then we have

$$X(t)^2 = \frac{4}{t^2} - \frac{2}{t^4} + \frac{3}{4t^6} + \frac{\varepsilon_{X^2}(t)}{t^7}, \tag{5.5}$$

where we set

$$\varepsilon_{X^2}(t) = 4\varepsilon_X(t) - \frac{1}{8t} - \frac{\varepsilon_X(t)}{t^2} + \frac{1}{64t^3} + \frac{\varepsilon_X(t)}{4t^4} + \frac{\varepsilon_X(t)^2}{t^5}. \tag{5.6}$$

Let

$$Y(t) := \sqrt{1 - \frac{\theta_x}{4}X(t)^2}.$$

Then we have

$$\begin{aligned}
&tX(t) - 1 - \sqrt{1 - \frac{1}{4}X(t)^2} \\
&= tX(t) - 2 + \frac{X(t)^2}{8Y(t)} \\
&= -\frac{1}{2t^2} + \frac{1}{8t^4} + \frac{\varepsilon_X(t)}{t^5} \\
&\quad + \frac{1}{2Y(t)t^2} - \frac{1}{4Y(t)t^4} + \frac{3}{32Y(t)t^6} + \frac{\varepsilon_{X^2}(t)}{8Y(t)t^7} \\
&= \frac{1}{2}\left(\frac{1}{Y(t)} - 1\right)\frac{1}{t^2} - \frac{1}{4}\left(\frac{1}{Y(t)} - \frac{1}{2}\right)\frac{1}{t^4} \\
&\quad + \frac{\varepsilon_X(t)}{t^5} + \frac{3}{32Y(t)t^6} + \frac{\varepsilon_{X^2}(t)}{8Y(t)t^7} \\
&= \frac{1 - Y(t)}{2Y(t)t^2}A(t)
\end{aligned} \tag{5.7}$$

by (5.2), (5.3) and (5.5), where we set

$$\begin{aligned}
A(t) := 1 &- \frac{1}{4}\frac{2 - Y(t)}{t^2(1 - Y(t))} + \frac{2Y(t)}{t^2(1 - Y(t))}\frac{\varepsilon_X(t)}{t} \\
&+ \frac{3}{16}\frac{1}{t^2(1 - Y(t))}\frac{1}{t^2} + \frac{1}{4}\frac{1}{t^2(1 - Y(t))}\frac{\varepsilon_{X^2}(t)}{t^3}.
\end{aligned} \tag{5.8}$$

Since

$$\frac{1 - Y(t)}{2Y(t)t^2} > 0$$

for $t > t_0$, it is sufficient to show $A(t) \neq 0$ if t is large enough.

We consider $\lim_{t\to\infty} t^2(1-Y(t))$. We have

$$\begin{aligned}
&\frac{d}{dt}(1-Y(t))\\
&= -\frac{1}{2}\left(1-\frac{\theta_x}{4}X(t)^2\right)^{-\frac{1}{2}}\frac{d}{dt}\left(-\frac{\theta_x}{4}X(t)^2\right)\\
&= -\frac{1}{2}\left(1-\frac{\theta_x}{4}X(t)^2\right)^{-\frac{1}{2}}\left\{-\frac{1}{4}\frac{d\theta_x}{dx}\frac{d}{dt}\left(-\frac{1}{4}X(t)^2\right)X(t)^2\right.\\
&\qquad\left.-\frac{1}{4}\theta_x\cdot 2X(t)\frac{dX}{dt}(t)\right\}\\
&= -\frac{1}{2}\left(1-\frac{\theta_x}{4}X(t)^2\right)^{-\frac{1}{2}}\left(-\frac{1}{2}\frac{d\theta_x}{dx}x-\frac{1}{2}\theta_x\right)\frac{dX}{dt}(t)X(t).
\end{aligned}$$

From (5.3) it follows that

$$\frac{dX}{dt}(t) = -\frac{2}{t^2}+\frac{3}{2t^4}-\frac{5}{8t^6}+\frac{d\varepsilon_X}{dt}(t)\frac{1}{t^6}-\frac{6\varepsilon_X(t)}{t^7}. \tag{5.9}$$

Then we obtain

$$\begin{aligned}
\frac{dX}{dt}(t)X(t) = &-\frac{4}{t^3}+\frac{4}{t^5}-\left(\frac{9}{4}-2\frac{d\varepsilon_X}{dt}(t)\right)\frac{1}{t^7}\\
&-\frac{14\varepsilon_X(t)}{t^8}+\frac{1}{2}\left(1-\frac{d\varepsilon_X}{dt}(t)\right)\frac{1}{t^9}\\
&+\frac{9\varepsilon_X(t)}{2t^{10}}-\frac{1}{8}\left(\frac{5}{8}-\frac{d\varepsilon_X}{dt}(t)\right)\frac{1}{t^{11}}\\
&-\left(\frac{11}{8}\varepsilon_X(t)-\frac{d\varepsilon_X}{dt}(t)\varepsilon_X(t)\right)\frac{1}{t^{12}}\\
&-\frac{6\varepsilon_X(t)^2}{t^{13}}
\end{aligned}$$

by (5.3) and (5.9). We have $\lim_{x\to 0}\theta_x = \frac{1}{2}$ and $\lim_{x\to 0}\frac{d\theta_x}{dx}x = -\frac{1}{8}$ (Lemma 5 in [1]). There exist constants M_X, M'_X and M_{X^2} such that $|\varepsilon_X(t)| < M_X$, $\left|\frac{d\varepsilon_X}{dt}(t)\right| < M'_X$ and $|\varepsilon_{X^2}(t)| < M_{X^2}$ for $t > t_0$ by Lemma 2. Then we obtain

$$\lim_{t\to\infty} t^3\frac{d}{dt}(1-Y(t)) = -\frac{3}{8}.$$

By l'Hôpital's rule, we have

$$\lim_{t\to\infty} t^2(1-Y(t)) = \lim_{t\to\infty}\frac{\frac{d}{dt}(1-Y(t))}{\frac{d}{dt}\left(\frac{1}{t^2}\right)} = \frac{3}{16}. \tag{5.10}$$

Since $\lim_{t\to\infty} Y(t) = 1$, we obtain $\lim_{t\to\infty} A(t) = -\frac{1}{3}$ by (5.8) and (5.10). Hence, there exists T_0 such that $A(t) < 0$ for $t > T_0$. This completes the proof. □

References

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