Sum of the Summations of Binomial Expansions with Geometric Series

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Abstract: This paper presents a theorem on binomial coefficients. This theorem states that sum of the summations of binomial expansions is equal to the sum of a geometric series with the exponents of two.

MSC Classification codes: 05A10, 40A05 (65B10)

Keywords: binomial coefficient, geometric series, summation

1. Introduction

Combinatorics consisting of both combination and permutation plays a vital role in computing, computational science, and management. The computational science is a rapidly growing interdisciplinary area where combinatorics, science, engineering, mathematics, computation, and collaboration use advance computing capabilities to meet today's challenges and solve the most complex real life problems.

The factorial or factorial function of a nonnegative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n.

Let $N = \{0, 1, 2, 3, \ldots .,\}$ be the set of natural umbers including zero element.

The binomial coefficient denotes that $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}, where\ n, r\ \in N.$

2. Binomial Theorem

Theorem: $\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 1.$

The binomial theorem states that sum of the summations of binomial expansions is equal to the sum of a geometric series with exponents of two [1-3].

Proof. $\binom{0}{0} = \frac{0!}{0!} = 1 \Longrightarrow \sum_{i=0}^{0}\binom{0}{i} = 2^0;\ \sum_{i=0}^{1}\binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1;$

$\sum_{i=0}^{2}\binom{2}{i} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 2^2;\ \sum_{i=0}^{3}\binom{3}{i} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3;\ \cdots;$

Similarly, we can continue this process upto n such that $\sum_{i=0}^{n}\binom{n}{i} = 2^n.$

When making the summation on both sides, we get

$$\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=0}^{n}2^i,$$

where $\sum_{i=0}^{n}2^i = \dfrac{2^{n+1}-1}{2-1} = 2^{n+1}-1$ is the geometric sereis with exponents of two [1,2].

$$\therefore \sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-1.$$

Hence, theorem is proved.

Lemma: $\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\sum_{i=0}^{k+3}\binom{k+3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}= 2^{n+1}-2^k,$ where $k \le n$ & $k, n \in N$.

Proof: The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n}2^i = 2^{n+1}-2^k.$$

$$\text{Then, } \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=k}^{n}2^i.$$

Therefore, $$\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^k.$$

3. Conclusion

In this article, theorem and lemma on binomial expansions and geometric series have been built using binomial coefficients. These combinatorial results can be useful for researchers who are working in science, engineering, management, and medicine [4].

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