

A PROOF OF THE RIEMANN HYPOTHESIS

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ABSTRACT. We prove the Riemann hypothesis.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ is defined on $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$. It is analytically continued to a meromorphic function on the whole plane with a pole at $s = 1$. It is well-known that negative even integers are zeros of $\zeta(s)$. Other zeros of $\zeta(s)$ are called complex zeros or nontrivial zeros.

Riemann stated the following statement, the so-called Riemann hypothesis, in [6] in 1859.

The Riemann hypothesis. All nontrivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$.

Hilbert listed it as the eighth problem of his 23 problems in his 1900 address to the Paris International Congress of Mathematicians. This is one of the most important unsolved problems in the twenty-first century.

Nobody has succeeded to prove it up to the present, but many computational results are known. In the early part of the twentieth century, they were obtained by hand computation ([1], [4], [5] and [7]). Numerical computations by computers have permitted us to check the truth of the Riemann hypothesis to extremely large t . We refer to [3] for a history of numerical verifications.

We will obtain further results with the development of computers and numerical methods. However, we can never reach to the goal in this way. We have to analyze the behaviour of the zeta function or the

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xi function for larger t than a fixed t_0 . For this purpose, we expand the real part and the imaginary part of the integration in a representation of the xi function into sums of powers of $1/t$ in Section 3. The precise representations of these parts are given in Section 6. We derive an equation which is satisfied if s is a zero of the xi function. We give a new zero-free region of $\zeta(s)$ by the above expansions and equation in Section 9. Let B be the supremum of the real parts of nontrivial zeros. Combining our new zero-free region with the theorem of de la Vallée Poussin, we first show $\frac{1}{2} \leq B < 1$. In Section 10 we give a lemma on Dirichlet series. It plays an important role to show that if $\frac{1}{2} < B < 1$, then there is no zero of $\zeta(s)$ on the line $\sigma = B$. We conclude $B = \frac{1}{2}$ by this fact and our new zero-free region.

2. PRELIMINARIES

The xi function $\xi(s)$ of Riemann is defined by

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It is an entire function and satisfies the functional equation $\xi(s) = \xi(1-s)$. It also satisfies $\xi(\bar{s}) = \overline{\xi(s)}$. The set of nontrivial zeros of $\zeta(s)$ coincides with the set of zeros of $\xi(s)$. Therefore, if s is a nontrivial zero of $\zeta(s)$, then \bar{s} , $1-s$ and $1-\bar{s}$ are also zeros of $\zeta(s)$.

The function $\xi(s)$ has the following integral representation

$$(2.1) \quad \xi(s) = \frac{1}{2} + \frac{s}{2}(s-1) \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx,$$

where $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ (for example, see p.22 in [8]).

Let

$$\Phi(s) = \int_1^\infty (x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1})\psi(x)dx.$$

We denote by $R(s) = \operatorname{Re}\Phi(s)$ and $I(s) = \operatorname{Im}\Phi(s)$ the real part and the imaginary part of $\Phi(s)$, respectively. Since

$$\begin{cases} \operatorname{Re}\xi(s) = \frac{1}{2} + \frac{1}{2}\operatorname{Re}(s(s-1)\Phi(s)), \\ \operatorname{Im}\xi(s) = \frac{1}{2}\operatorname{Im}(s(s-1)\Phi(s)), \end{cases}$$

we have that $\xi(s) = 0$ if and only if

$$(2.2) \quad \begin{cases} 1 + (\sigma(\sigma-1) - t^2)R(s) - t(2\sigma-1)I(s) = 0, \\ t(2\sigma-1)R(s) + (\sigma(\sigma-1) - t^2)I(s) = 0. \end{cases}$$

Solving (2.2) for $R(s)$ and $I(s)$, we obtain

$$(2.3) \quad \begin{cases} R(s) = \frac{t^2 - \sigma(\sigma - 1)}{(\sigma(\sigma - 1) - t^2)^2 + t^2(2\sigma - 1)^2}, \\ I(s) = \frac{t(2\sigma - 1)}{(\sigma(\sigma - 1) - t^2)^2 + t^2(2\sigma - 1)^2}. \end{cases}$$

Therefore, $\xi(s) = 0$ if and only if (2.3) holds.

3. EXPANSIONS OF $R(s)$ AND $I(s)$

We have

$$R(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} + x^{-\frac{\sigma}{2}-\frac{1}{2}}) \cos\left(\frac{t}{2} \log x\right) \psi(x) dx$$

and

$$I(s) = \int_1^\infty (x^{\frac{\sigma}{2}-1} - x^{-\frac{\sigma}{2}-\frac{1}{2}}) \sin\left(\frac{t}{2} \log x\right) \psi(x) dx.$$

First we expand $R(s)$. We set

$$f_0(x) := (x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}}) \psi(x).$$

Using integral by parts and

$$\frac{d}{dx} \left(\frac{2}{t} \sin\left(\frac{t}{2} \log x\right) \right) = x^{-1} \cos\left(\frac{t}{2} \log x\right),$$

we obtain

$$R(s) = -\frac{2}{t} \int_1^\infty \sin\left(\frac{t}{2} \log x\right) f_0'(x) dx.$$

Let $f_1(x) := x f_0'(x)$. Then we have

$$\begin{aligned} \int_1^\infty \sin\left(\frac{t}{2} \log x\right) f_0'(x) dx &= \int_1^\infty x^{-1} \sin\left(\frac{t}{2} \log x\right) f_1(x) dx \\ &= \frac{2}{t} f_1(1) + \frac{2}{t} \int_1^\infty \cos\left(\frac{t}{2} \log x\right) f_1'(x) dx \end{aligned}$$

by integral by parts. We inductively define $f_k(x) := x f_{k-1}'(x)$ for $k = 1, 2, \dots$. By the same way as above, we obtain

$$(3.1) \quad \begin{aligned} R(s) &= -\left(\frac{2}{t}\right)^2 f_1(1) + \left(\frac{2}{t}\right)^4 f_3(1) - \left(\frac{2}{t}\right)^6 f_5(1) \\ &\quad + \left(\frac{2}{t}\right)^7 \int_1^\infty \sin\left(\frac{t}{2} \log x\right) f_6'(x) dx. \end{aligned}$$

We use the same argument for $I(s)$. We define $g_0(x) := (x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}})\psi(x)$ and $g_k(x) := xg'_{k-1}(x)$ for $k = 1, 2, \dots$. Then, we similarly obtain

$$(3.2) \quad \begin{aligned} I(s) = & -\left(\frac{2}{t}\right)^3 g_2(1) + \left(\frac{2}{t}\right)^5 g_4(1) - \left(\frac{2}{t}\right)^7 g_6(1) \\ & + \left(\frac{2}{t}\right)^8 \int_1^\infty \sin\left(\frac{t}{2} \log x\right) g'_7(x) dx \end{aligned}$$

by integral by parts.

4. FUNCTIONS $f_k(x)$, $f'_k(x)$, $g_k(x)$ AND $g'_k(x)$

We collect precise forms of $f_k(x)$, $f'_k(x)$, $g_k(x)$ and $g'_k(x)$ in this section. By a direct calculation we obtain the following formulas

$$\begin{aligned} f'_0(x) = & \left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right) x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\ & + \left(x^{\frac{\sigma}{2}} + x^{-\frac{\sigma}{2}+\frac{1}{2}} \right) \psi^{(1)}(x), \end{aligned}$$

$$\begin{aligned} f_1(x) = & \left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\ & + \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}} \right) \psi^{(1)}(x), \end{aligned}$$

$$\begin{aligned} f'_1(x) = & \left\{ \left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\ & + \left\{ (\sigma+1)x^{\frac{\sigma}{2}} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\ & + \left(x^{\frac{\sigma}{2}+1} + x^{-\frac{\sigma}{2}+\frac{3}{2}} \right) \psi^{(2)}(x), \end{aligned}$$

$$\begin{aligned} f_2(x) = & \left\{ \left(\frac{\sigma}{2}\right)^2 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2}\right)^2 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\ & + \left\{ (\sigma+1)x^{\frac{\sigma}{2}+1} + (-\sigma+2)x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\ & + \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}} \right) \psi^{(2)}(x), \end{aligned}$$

$$\begin{aligned}
f_2'(x) = & \left\{ \left(\frac{\sigma}{2} \right)^3 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^3 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma + 3 \right) x^{\frac{\sigma}{2}+1} + \left(-\frac{3}{2}\sigma + \frac{9}{2} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left(x^{\frac{\sigma}{2}+2} + x^{-\frac{\sigma}{2}+\frac{5}{2}} \right) \psi^{(3)}(x),
\end{aligned}$$

$$\begin{aligned}
f_3(x) = & \left\{ \left(\frac{\sigma}{2} \right)^3 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^3 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{4}\sigma^2 + \frac{3}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}+1} + \left(\frac{3}{4}\sigma^2 - 3\sigma + \frac{13}{4} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma + 3 \right) x^{\frac{\sigma}{2}+2} + \left(-\frac{3}{2}\sigma + \frac{9}{2} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left(x^{\frac{\sigma}{2}+3} + x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(3)}(x),
\end{aligned}$$

$$\begin{aligned}
f_3'(x) = & \left\{ \left(\frac{\sigma}{2} \right)^4 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^4 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. + \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5 \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7 \right) x^{\frac{\sigma}{2}+1} + \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ (2\sigma + 6)x^{\frac{\sigma}{2}+2} + (-2\sigma + 8)x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left(x^{\frac{\sigma}{2}+3} + x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x),
\end{aligned}$$

$$\begin{aligned}
f_4(x) = & \left\{ \left(\frac{\sigma}{2} \right)^4 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^4 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. + \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5 \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7 \right) x^{\frac{\sigma}{2}+2} + \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ (2\sigma + 6)x^{\frac{\sigma}{2}+3} + (-2\sigma + 8)x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left(x^{\frac{\sigma}{2}+4} + x^{-\frac{\sigma}{2} + \frac{9}{2}} \right) \psi^{(4)}(x),
\end{aligned}$$

$$\begin{aligned}
f'_4(x) = & \left\{ \left(\frac{\sigma}{2} \right)^5 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^5 x^{-\frac{\sigma}{2} - \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. + \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16} \right) x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. + \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma^2 + 15\sigma + 25 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. + \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma + 10 \right) x^{\frac{\sigma}{2}+3} + \left(-\frac{5}{2}\sigma + \frac{25}{2} \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left(x^{\frac{\sigma}{2}+4} + x^{-\frac{\sigma}{2} + \frac{9}{2}} \right) \psi^{(5)}(x),
\end{aligned}$$

$$\begin{aligned}
f_5(x) = & \left\{ \left(\frac{\sigma}{2} \right)^5 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^5 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{5}{16} \sigma^4 + \frac{5}{4} \sigma^3 + \frac{5}{2} \sigma^2 + \frac{5}{2} \sigma + 1 \right) x^{\frac{\sigma}{2} + 1} \right. \\
& \quad \left. + \left(\frac{5}{16} \sigma^4 - \frac{5}{2} \sigma^3 + \frac{65}{8} \sigma^2 - \frac{25}{2} \sigma + \frac{121}{16} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{5}{4} \sigma^3 + \frac{15}{2} \sigma^2 + \frac{35}{2} \sigma + 15 \right) x^{\frac{\sigma}{2} + 2} \right. \\
& \quad \left. + \left(-\frac{5}{4} \sigma^3 + \frac{45}{4} \sigma^2 - \frac{145}{4} \sigma + \frac{165}{4} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma^2 + 15 \sigma + 25 \right) x^{\frac{\sigma}{2} + 3} \right. \\
& \quad \left. + \left(\frac{5}{2} \sigma^2 - 20 \sigma + \frac{85}{2} \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma + 10 \right) x^{\frac{\sigma}{2} + 4} + \left(-\frac{5}{2} \sigma + \frac{25}{2} \right) x^{-\frac{\sigma}{2} + \frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left(x^{\frac{\sigma}{2} + 5} + x^{-\frac{\sigma}{2} + \frac{11}{2}} \right) \psi^{(5)}(x),
\end{aligned}$$

$$\begin{aligned}
f'_5(x) = & \left\{ \left(\frac{\sigma}{2} \right)^6 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^6 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. + \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. + \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. + \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190 \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. + \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (3\sigma + 15)x^{\frac{\sigma}{2}+4} + (-3\sigma + 18)x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left(x^{\frac{\sigma}{2}+5} + x^{-\frac{\sigma}{2}+\frac{11}{2}} \right) \psi^{(6)}(x),
\end{aligned}$$

$$\begin{aligned}
f_6(x) = & \left\{ \left(\frac{\sigma}{2} \right)^6 x^{\frac{\sigma}{2}} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^6 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{16} \sigma^5 + \frac{15}{16} \sigma^4 + \frac{5}{2} \sigma^3 + \frac{15}{4} \sigma^2 + 3\sigma + 1 \right) x^{\frac{\sigma}{2} + 1} \right. \\
& \quad \left. + \left(-\frac{3}{16} \sigma^5 + \frac{15}{8} \sigma^4 - \frac{65}{8} \sigma^3 + \frac{75}{4} \sigma^2 - \frac{363}{16} \sigma + \frac{91}{8} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{15}{16} \sigma^4 + \frac{15}{2} \sigma^3 + \frac{105}{4} \sigma^2 + 45\sigma + 31 \right) x^{\frac{\sigma}{2} + 2} \right. \\
& \quad \left. + \left(\frac{15}{16} \sigma^4 - \frac{45}{4} \sigma^3 + \frac{435}{8} \sigma^2 - \frac{495}{4} \sigma + \frac{1771}{16} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma^3 + \frac{45}{2} \sigma^2 + 75\sigma + 90 \right) x^{\frac{\sigma}{2} + 3} \right. \\
& \quad \left. + \left(-\frac{5}{2} \sigma^3 + 30\sigma^2 - \frac{255}{2} \sigma + 190 \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{15}{4} \sigma^2 + 30\sigma + 65 \right) x^{\frac{\sigma}{2} + 4} \right. \\
& \quad \left. + \left(\frac{15}{4} \sigma^2 - \frac{75}{2} \sigma + \frac{395}{4} \right) x^{-\frac{\sigma}{2} + \frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (3\sigma + 15) x^{\frac{\sigma}{2} + 5} + (-3\sigma + 18) x^{-\frac{\sigma}{2} + \frac{11}{2}} \right\} \psi^{(5)}(x) \\
& + \left(x^{\frac{\sigma}{2} + 6} + x^{-\frac{\sigma}{2} + \frac{13}{2}} \right) \psi^{(6)}(x),
\end{aligned}$$

$$\begin{aligned}
f'_6(x) = & \left\{ \left(\frac{\sigma}{2} \right)^7 x^{\frac{\sigma}{2}-1} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^7 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64}\sigma^6 + \frac{21}{32}\sigma^5 + \frac{35}{16}\sigma^4 + \frac{35}{8}\sigma^3 + \frac{21}{4}\sigma^2 \right. \right. \\
& \quad \left. \left. + \frac{7}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}} + \left(\frac{7}{64}\sigma^6 - \frac{21}{16}\sigma^5 + \frac{455}{64}\sigma^4 - \frac{175}{8}\sigma^3 \right. \right. \\
& \quad \left. \left. + \frac{2541}{64}\sigma^2 - \frac{637}{16}\sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32}\sigma^5 + \frac{105}{16}\sigma^4 + \frac{245}{8}\sigma^3 + \frac{315}{4}\sigma^2 + \frac{217}{2}\sigma \right. \right. \\
& \quad \left. \left. + 63 \right) x^{\frac{\sigma}{2}+1} + \left(-\frac{21}{32}\sigma^5 + \frac{315}{32}\sigma^4 - \frac{1015}{16}\sigma^3 + \frac{3465}{16}\sigma^2 \right. \right. \\
& \quad \left. \left. - \frac{12397}{32}\sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16}\sigma^4 + \frac{105}{4}\sigma^3 + \frac{525}{4}\sigma^2 + 315\sigma + 301 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. + \left(\frac{35}{16}\sigma^4 - 35\sigma^3 + \frac{1785}{8}\sigma^2 - 665\sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^3 + \frac{105}{2}\sigma^2 + \frac{455}{2}\sigma + 350 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. + \left(-\frac{35}{8}\sigma^3 + \frac{525}{8}\sigma^2 - \frac{2765}{8}\sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4}\sigma^2 + \frac{105}{2}\sigma + 140 \right) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. + \left(\frac{21}{4}\sigma^2 - 63\sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2}\sigma + 21 \right) x^{\frac{\sigma}{2}+5} + \left(-\frac{7}{2}\sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2}+6} + x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(7)}(x),
\end{aligned}$$

$$\begin{aligned}
g'_0(x) = & \left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right) x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left(x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}+\frac{1}{2}} \right) \psi^{(1)}(x),
\end{aligned}$$

$$g_1(x) = \left\{ \frac{\sigma}{2} x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right) x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\ + \left(x^{\frac{\sigma}{2} + 1} - x^{-\frac{\sigma}{2} + \frac{3}{2}} \right) \psi^{(1)}(x),$$

$$g_1'(x) = \left\{ \left(\frac{\sigma}{2} \right)^2 x^{\frac{\sigma}{2} - 1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^2 x^{-\frac{\sigma}{2} - \frac{1}{2}} \right\} \psi(x) \\ + \left\{ (\sigma + 1) x^{\frac{\sigma}{2}} - (-\sigma + 2) x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi^{(1)}(x) \\ + \left(x^{\frac{\sigma}{2} + 1} - x^{-\frac{\sigma}{2} + \frac{3}{2}} \right) \psi^{(2)}(x),$$

$$g_2(x) = \left\{ \left(\frac{\sigma}{2} \right)^2 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^2 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\ + \left\{ (\sigma + 1) x^{\frac{\sigma}{2} + 1} - (-\sigma + 2) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\ + \left(x^{\frac{\sigma}{2} + 2} - x^{-\frac{\sigma}{2} + \frac{5}{2}} \right) \psi^{(2)}(x),$$

$$g_2'(x) = \left\{ \left(\frac{\sigma}{2} \right)^3 x^{\frac{\sigma}{2} - 1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^3 x^{-\frac{\sigma}{2} - \frac{1}{2}} \right\} \psi(x) \\ + \left\{ \left(\frac{3}{4} \sigma^2 + \frac{3}{2} \sigma + 1 \right) x^{\frac{\sigma}{2}} - \left(\frac{3}{4} \sigma^2 - 3\sigma + \frac{13}{4} \right) x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi^{(1)}(x) \\ + \left\{ \left(\frac{3}{2} \sigma + 3 \right) x^{\frac{\sigma}{2} + 1} - \left(-\frac{3}{2} \sigma + \frac{9}{2} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(2)}(x) \\ + \left(x^{\frac{\sigma}{2} + 2} - x^{-\frac{\sigma}{2} + \frac{5}{2}} \right) \psi^{(3)}(x),$$

$$g_3(x) = \left\{ \left(\frac{\sigma}{2} \right)^3 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^3 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\ + \left\{ \left(\frac{3}{4} \sigma^2 + \frac{3}{2} \sigma + 1 \right) x^{\frac{\sigma}{2} + 1} - \left(\frac{3}{4} \sigma^2 - 3\sigma + \frac{13}{4} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\ + \left\{ \left(\frac{3}{2} \sigma + 3 \right) x^{\frac{\sigma}{2} + 2} - \left(-\frac{3}{2} \sigma + \frac{9}{2} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\ + \left(x^{\frac{\sigma}{2} + 3} - x^{-\frac{\sigma}{2} + \frac{7}{2}} \right) \psi^{(3)}(x),$$

$$\begin{aligned}
g'_3(x) = & \left\{ \left(\frac{\sigma}{2} \right)^4 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^4 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. - \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5 \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7 \right) x^{\frac{\sigma}{2}+1} - \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ (2\sigma + 6)x^{\frac{\sigma}{2}+2} - (-2\sigma + 8)x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left(x^{\frac{\sigma}{2}+3} - x^{-\frac{\sigma}{2}+\frac{7}{2}} \right) \psi^{(4)}(x),
\end{aligned}$$

$$\begin{aligned}
g_4(x) = & \left\{ \left(\frac{\sigma}{2} \right)^4 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^4 x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{1}{2}\sigma^3 + \frac{3}{2}\sigma^2 + 2\sigma + 1 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. - \left(-\frac{1}{2}\sigma^3 + 3\sigma^2 - \frac{13}{2}\sigma + 5 \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{3}{2}\sigma^2 + 6\sigma + 7 \right) x^{\frac{\sigma}{2}+2} - \left(\frac{3}{2}\sigma^2 - 9\sigma + \frac{29}{2} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ (2\sigma + 6)x^{\frac{\sigma}{2}+3} - (-2\sigma + 8)x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left(x^{\frac{\sigma}{2}+4} - x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(4)}(x),
\end{aligned}$$

$$\begin{aligned}
g'_4(x) = & \left\{ \left(\frac{\sigma}{2} \right)^5 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^5 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{5}{16}\sigma^4 + \frac{5}{4}\sigma^3 + \frac{5}{2}\sigma^2 + \frac{5}{2}\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. - \left(\frac{5}{16}\sigma^4 - \frac{5}{2}\sigma^3 + \frac{65}{8}\sigma^2 - \frac{25}{2}\sigma + \frac{121}{16} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{5}{4}\sigma^3 + \frac{15}{2}\sigma^2 + \frac{35}{2}\sigma + 15 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. - \left(-\frac{5}{4}\sigma^3 + \frac{45}{4}\sigma^2 - \frac{145}{4}\sigma + \frac{165}{4} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma^2 + 15\sigma + 25 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. - \left(\frac{5}{2}\sigma^2 - 20\sigma + \frac{85}{2} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma + 10 \right) x^{\frac{\sigma}{2}+3} - \left(-\frac{5}{2}\sigma + \frac{25}{2} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left(x^{\frac{\sigma}{2}+4} - x^{-\frac{\sigma}{2}+\frac{9}{2}} \right) \psi^{(5)}(x),
\end{aligned}$$

$$\begin{aligned}
g_5(x) = & \left\{ \left(\frac{\sigma}{2} \right)^5 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^5 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{5}{16} \sigma^4 + \frac{5}{4} \sigma^3 + \frac{5}{2} \sigma^2 + \frac{5}{2} \sigma + 1 \right) x^{\frac{\sigma}{2} + 1} \right. \\
& \quad \left. - \left(\frac{5}{16} \sigma^4 - \frac{5}{2} \sigma^3 + \frac{65}{8} \sigma^2 - \frac{25}{2} \sigma + \frac{121}{16} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{5}{4} \sigma^3 + \frac{15}{2} \sigma^2 + \frac{35}{2} \sigma + 15 \right) x^{\frac{\sigma}{2} + 2} \right. \\
& \quad \left. - \left(-\frac{5}{4} \sigma^3 + \frac{45}{4} \sigma^2 - \frac{145}{4} \sigma + \frac{165}{4} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma^2 + 15 \sigma + 25 \right) x^{\frac{\sigma}{2} + 3} \right. \\
& \quad \left. - \left(\frac{5}{2} \sigma^2 - 20 \sigma + \frac{85}{2} \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma + 10 \right) x^{\frac{\sigma}{2} + 4} - \left(-\frac{5}{2} \sigma + \frac{25}{2} \right) x^{-\frac{\sigma}{2} + \frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left(x^{\frac{\sigma}{2} + 5} - x^{-\frac{\sigma}{2} + \frac{11}{2}} \right) \psi^{(5)}(x),
\end{aligned}$$

$$\begin{aligned}
g'_5(x) = & \left\{ \left(\frac{\sigma}{2} \right)^6 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^6 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{16}\sigma^5 + \frac{15}{16}\sigma^4 + \frac{5}{2}\sigma^3 + \frac{15}{4}\sigma^2 + 3\sigma + 1 \right) x^{\frac{\sigma}{2}} \right. \\
& \quad \left. - \left(-\frac{3}{16}\sigma^5 + \frac{15}{8}\sigma^4 - \frac{65}{8}\sigma^3 + \frac{75}{4}\sigma^2 - \frac{363}{16}\sigma + \frac{91}{8} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{15}{16}\sigma^4 + \frac{15}{2}\sigma^3 + \frac{105}{4}\sigma^2 + 45\sigma + 31 \right) x^{\frac{\sigma}{2}+1} \right. \\
& \quad \left. - \left(\frac{15}{16}\sigma^4 - \frac{45}{4}\sigma^3 + \frac{435}{8}\sigma^2 - \frac{495}{4}\sigma + \frac{1771}{16} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2}\sigma^3 + \frac{45}{2}\sigma^2 + 75\sigma + 90 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. - \left(-\frac{5}{2}\sigma^3 + 30\sigma^2 - \frac{255}{2}\sigma + 190 \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{15}{4}\sigma^2 + 30\sigma + 65 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. - \left(\frac{15}{4}\sigma^2 - \frac{75}{2}\sigma + \frac{395}{4} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (3\sigma + 15)x^{\frac{\sigma}{2}+4} - (-3\sigma + 18)x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left(x^{\frac{\sigma}{2}+5} - x^{-\frac{\sigma}{2}+\frac{11}{2}} \right) \psi^{(6)}(x),
\end{aligned}$$

$$\begin{aligned}
g_6(x) = & \left\{ \left(\frac{\sigma}{2} \right)^6 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^6 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{3}{16} \sigma^5 + \frac{15}{16} \sigma^4 + \frac{5}{2} \sigma^3 + \frac{15}{4} \sigma^2 + 3\sigma + 1 \right) x^{\frac{\sigma}{2} + 1} \right. \\
& \quad \left. - \left(-\frac{3}{16} \sigma^5 + \frac{15}{8} \sigma^4 - \frac{65}{8} \sigma^3 + \frac{75}{4} \sigma^2 - \frac{363}{16} \sigma + \frac{91}{8} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{15}{16} \sigma^4 + \frac{15}{2} \sigma^3 + \frac{105}{4} \sigma^2 + 45\sigma + 31 \right) x^{\frac{\sigma}{2} + 2} \right. \\
& \quad \left. - \left(\frac{15}{16} \sigma^4 - \frac{45}{4} \sigma^3 + \frac{435}{8} \sigma^2 - \frac{495}{4} \sigma + \frac{1771}{16} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{5}{2} \sigma^3 + \frac{45}{2} \sigma^2 + 75\sigma + 90 \right) x^{\frac{\sigma}{2} + 3} \right. \\
& \quad \left. - \left(-\frac{5}{2} \sigma^3 + 30\sigma^2 - \frac{255}{2} \sigma + 190 \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{15}{4} \sigma^2 + 30\sigma + 65 \right) x^{\frac{\sigma}{2} + 4} \right. \\
& \quad \left. - \left(\frac{15}{4} \sigma^2 - \frac{75}{2} \sigma + \frac{395}{4} \right) x^{-\frac{\sigma}{2} + \frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (3\sigma + 15) x^{\frac{\sigma}{2} + 5} - (-3\sigma + 18) x^{-\frac{\sigma}{2} + \frac{11}{2}} \right\} \psi^{(5)}(x) \\
& + \left(x^{\frac{\sigma}{2} + 6} - x^{-\frac{\sigma}{2} + \frac{13}{2}} \right) \psi^{(6)}(x),
\end{aligned}$$

$$\begin{aligned}
g'_6(x) = & \left\{ \left(\frac{\sigma}{2} \right)^7 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^7 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64}\sigma^6 + \frac{21}{32}\sigma^5 + \frac{35}{16}\sigma^4 + \frac{35}{8}\sigma^3 + \frac{21}{4}\sigma^2 + \frac{7}{2}\sigma \right. \right. \\
& \quad \left. \left. + 1 \right) x^{\frac{\sigma}{2}} - \left(\frac{7}{64}\sigma^6 - \frac{21}{16}\sigma^5 + \frac{455}{64}\sigma^4 - \frac{175}{8}\sigma^3 \right. \right. \\
& \quad \left. \left. + \frac{2541}{64}\sigma^2 - \frac{637}{16}\sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32}\sigma^5 + \frac{105}{16}\sigma^4 + \frac{245}{8}\sigma^3 + \frac{315}{4}\sigma^2 + \frac{217}{2}\sigma \right. \right. \\
& \quad \left. \left. + 63 \right) x^{\frac{\sigma}{2}+1} - \left(-\frac{21}{32}\sigma^5 + \frac{315}{32}\sigma^4 - \frac{1015}{16}\sigma^3 + \frac{3465}{16}\sigma^2 \right. \right. \\
& \quad \left. \left. - \frac{12397}{32}\sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16}\sigma^4 + \frac{105}{4}\sigma^3 + \frac{525}{4}\sigma^2 + 315\sigma + 301 \right) x^{\frac{\sigma}{2}+2} \right. \\
& \quad \left. - \left(\frac{35}{16}\sigma^4 - 35\sigma^3 + \frac{1785}{8}\sigma^2 - 665\sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^3 + \frac{105}{2}\sigma^2 + \frac{455}{2}\sigma + 350 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. - \left(-\frac{35}{8}\sigma^3 + \frac{525}{8}\sigma^2 - \frac{2765}{8}\sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4}\sigma^2 + \frac{105}{2}\sigma + 140 \right) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. - \left(\frac{21}{4}\sigma^2 - 63\sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2}\sigma + 21 \right) x^{\frac{\sigma}{2}+5} - \left(-\frac{7}{2}\sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2}+6} - x^{-\frac{\sigma}{2}+\frac{13}{2}} \right) \psi^{(7)}(x),
\end{aligned}$$

$$\begin{aligned}
g_7(x) = & \left\{ \left(\frac{\sigma}{2} \right)^7 x^{\frac{\sigma}{2}} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^7 x^{-\frac{\sigma}{2} + \frac{1}{2}} \right\} \psi(x) \\
& + \left\{ \left(\frac{7}{64} \sigma^6 + \frac{21}{32} \sigma^5 + \frac{35}{16} \sigma^4 + \frac{35}{8} \sigma^3 + \frac{21}{4} \sigma^2 + \frac{7}{2} \sigma \right. \right. \\
& \quad \left. \left. + 1 \right) x^{\frac{\sigma}{2} + 1} - \left(\frac{7}{64} \sigma^6 - \frac{21}{16} \sigma^5 + \frac{455}{64} \sigma^4 - \frac{175}{8} \sigma^3 \right. \right. \\
& \quad \left. \left. + \frac{2541}{64} \sigma^2 - \frac{637}{16} \sigma + \frac{1093}{64} \right) x^{-\frac{\sigma}{2} + \frac{3}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{21}{32} \sigma^5 + \frac{105}{16} \sigma^4 + \frac{245}{8} \sigma^3 + \frac{315}{4} \sigma^2 + \frac{217}{2} \sigma \right. \right. \\
& \quad \left. \left. + 63 \right) x^{\frac{\sigma}{2} + 2} - \left(-\frac{21}{32} \sigma^5 + \frac{315}{32} \sigma^4 - \frac{1015}{16} \sigma^3 + \frac{3465}{16} \sigma^2 \right. \right. \\
& \quad \left. \left. - \frac{12397}{32} \sigma + \frac{9219}{32} \right) x^{-\frac{\sigma}{2} + \frac{5}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{35}{16} \sigma^4 + \frac{105}{4} \sigma^3 + \frac{525}{4} \sigma^2 + 315 \sigma + 301 \right) x^{\frac{\sigma}{2} + 3} \right. \\
& \quad \left. - \left(\frac{35}{16} \sigma^4 - 35 \sigma^3 + \frac{1785}{8} \sigma^2 - 665 \sigma + \frac{12411}{16} \right) x^{-\frac{\sigma}{2} + \frac{7}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8} \sigma^3 + \frac{105}{2} \sigma^2 + \frac{455}{2} \sigma + 350 \right) x^{\frac{\sigma}{2} + 4} \right. \\
& \quad \left. - \left(-\frac{35}{8} \sigma^3 + \frac{525}{8} \sigma^2 - \frac{2765}{8} \sigma + \frac{5075}{8} \right) x^{-\frac{\sigma}{2} + \frac{9}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ \left(\frac{21}{4} \sigma^2 + \frac{105}{2} \sigma + 140 \right) x^{\frac{\sigma}{2} + 5} \right. \\
& \quad \left. - \left(\frac{21}{4} \sigma^2 - 63 \sigma + \frac{791}{4} \right) x^{-\frac{\sigma}{2} + \frac{11}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ \left(\frac{7}{2} \sigma + 21 \right) x^{\frac{\sigma}{2} + 6} - \left(-\frac{7}{2} \sigma + \frac{49}{2} \right) x^{-\frac{\sigma}{2} + \frac{13}{2}} \right\} \psi^{(6)}(x) \\
& + \left(x^{\frac{\sigma}{2} + 7} - x^{-\frac{\sigma}{2} + \frac{15}{2}} \right) \psi^{(7)}(x)
\end{aligned}$$

and

$$\begin{aligned}
g'_7(x) = & \left\{ \left(\frac{\sigma}{2} \right)^8 x^{\frac{\sigma}{2}-1} - \left(-\frac{\sigma}{2} + \frac{1}{2} \right)^8 x^{-\frac{\sigma}{2}-\frac{1}{2}} \right\} \psi(x) \\
& \left\{ \left(\frac{1}{16}\sigma^7 + \frac{7}{16}\sigma^6 + \frac{7}{4}\sigma^5 + \frac{35}{8}\sigma^4 + 7\sigma^3 + 7\sigma^2 + 4\sigma \right. \right. \\
& \quad \left. \left. + 1 \right) x^{\frac{\sigma}{2}} - \left(-\frac{1}{16}\sigma^7 + \frac{7}{8}\sigma^6 - \frac{91}{16}\sigma^5 + \frac{175}{8}\sigma^4 \right. \right. \\
& \quad \left. \left. - \frac{847}{16}\sigma^3 + \frac{637}{8}\sigma^2 - \frac{1093}{16}\sigma + \frac{205}{8} \right) x^{-\frac{\sigma}{2}+\frac{1}{2}} \right\} \psi^{(1)}(x) \\
& + \left\{ \left(\frac{7}{16}\sigma^6 + \frac{21}{4}\sigma^5 + \frac{245}{8}\sigma^4 + 105\sigma^3 + 217\sigma^2 + 252\sigma \right. \right. \\
& \quad \left. \left. + 127 \right) x^{\frac{\sigma}{2}+1} - \left(\frac{7}{16}\sigma^6 - \frac{63}{8}\sigma^5 + \frac{1015}{16}\sigma^4 - \frac{1155}{4}\sigma^3 \right. \right. \\
& \quad \left. \left. + \frac{12397}{16}\sigma^2 - \frac{9219}{8}\sigma + \frac{11797}{16} \right) x^{-\frac{\sigma}{2}+\frac{3}{2}} \right\} \psi^{(2)}(x) \\
& + \left\{ \left(\frac{7}{4}\sigma^5 + \frac{105}{4}\sigma^4 + 175\sigma^3 + 630\sigma^2 + 1204\sigma \right. \right. \\
& \quad \left. \left. + 966 \right) x^{\frac{\sigma}{2}+2} - \left(-\frac{7}{4}\sigma^5 + 35\sigma^4 - \frac{595}{2}\sigma^3 + 1330\sigma^2 \right. \right. \\
& \quad \left. \left. - \frac{12411}{4}\sigma + 3003 \right) x^{-\frac{\sigma}{2}+\frac{5}{2}} \right\} \psi^{(3)}(x) \\
& + \left\{ \left(\frac{35}{8}\sigma^4 + 70\sigma^3 + 455\sigma^2 + 1400\sigma + 1701 \right) x^{\frac{\sigma}{2}+3} \right. \\
& \quad \left. - \left(\frac{35}{8}\sigma^4 - \frac{175}{2}\sigma^3 + \frac{2765}{4}\sigma^2 - \frac{5075}{2}\sigma + \frac{29043}{8} \right) x^{-\frac{\sigma}{2}+\frac{7}{2}} \right\} \psi^{(4)}(x) \\
& + \left\{ (7\sigma^3 + 105\sigma^2 + 560\sigma + 1050) x^{\frac{\sigma}{2}+4} \right. \\
& \quad \left. - (-7\sigma^3 + 126\sigma^2 - 791\sigma + 1722) x^{-\frac{\sigma}{2}+\frac{9}{2}} \right\} \psi^{(5)}(x) \\
& + \left\{ (7\sigma^2 + 84\sigma + 266) x^{\frac{\sigma}{2}+5} \right. \\
& \quad \left. - (7\sigma^2 - 98\sigma + 357) x^{-\frac{\sigma}{2}+\frac{11}{2}} \right\} \psi^{(6)}(x) \\
& + \left\{ (4\sigma + 28) x^{\frac{\sigma}{2}+6} - (-4\sigma + 32) x^{-\frac{\sigma}{2}+\frac{13}{2}} \right\} \psi^{(7)}(x) \\
& + \left(x^{\frac{\sigma}{2}+7} - x^{-\frac{\sigma}{2}+\frac{15}{2}} \right) \psi^{(8)}(x).
\end{aligned}$$

5. FORMULAS OF $\psi^{(k)}(x)$

An important property of the series $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ is the reciprocal formula

$$(5.1) \quad 2\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1 \right)$$

for $x > 0$ (for example, see (2.6.3) in [8]). The integral representation (2.1) of $\xi(s)$ is derived from the above reciprocal formula.

It is rewritten as

$$(5.2) \quad \psi(x) = x^{-\frac{1}{2}} \psi\left(\frac{1}{x}\right) + \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2}.$$

Differentiating (5.2) one after another, we obtain the following equalities

$$(5.3) \quad \psi^{(1)}(x) = -\frac{1}{2} x^{-\frac{3}{2}} \psi\left(\frac{1}{x}\right) - x^{-\frac{5}{2}} \psi^{(1)}\left(\frac{1}{x}\right) - \frac{1}{4} x^{-\frac{3}{2}},$$

$$(5.4) \quad \begin{aligned} \psi^{(2)}(x) &= \frac{3}{4} x^{-\frac{5}{2}} \psi\left(\frac{1}{x}\right) + 3x^{-\frac{7}{2}} \psi^{(1)}\left(\frac{1}{x}\right) \\ &\quad + x^{-\frac{9}{2}} \psi^{(2)}\left(\frac{1}{x}\right) + \frac{3}{8} x^{-\frac{5}{2}}, \end{aligned}$$

$$(5.5) \quad \begin{aligned} \psi^{(3)}(x) &= -\frac{15}{8} x^{-\frac{7}{2}} \psi\left(\frac{1}{x}\right) - \frac{45}{4} x^{-\frac{9}{2}} \psi^{(1)}\left(\frac{1}{x}\right) \\ &\quad - \frac{15}{2} x^{-\frac{11}{2}} \psi^{(2)}\left(\frac{1}{x}\right) - x^{-\frac{13}{2}} \psi^{(3)}\left(\frac{1}{x}\right) \\ &\quad - \frac{15}{16} x^{-\frac{7}{2}}, \end{aligned}$$

$$(5.6) \quad \begin{aligned} \psi^{(4)}(x) &= \frac{105}{16} x^{-\frac{9}{2}} \psi\left(\frac{1}{x}\right) + \frac{105}{2} x^{-\frac{11}{2}} \psi^{(1)}\left(\frac{1}{x}\right) \\ &\quad + \frac{105}{2} x^{-\frac{13}{2}} \psi^{(2)}\left(\frac{1}{x}\right) + 14x^{-\frac{15}{2}} \psi^{(3)}\left(\frac{1}{x}\right) \\ &\quad + x^{-\frac{17}{2}} \psi^{(4)}\left(\frac{1}{x}\right) + \frac{105}{32} x^{-\frac{9}{2}}, \end{aligned}$$

$$\begin{aligned}
(5.7) \quad \psi^{(5)}(x) = & -\frac{945}{32}x^{-\frac{11}{2}}\psi\left(\frac{1}{x}\right) - \frac{4725}{16}x^{-\frac{13}{2}}\psi^{(1)}\left(\frac{1}{x}\right) \\
& - \frac{1575}{4}x^{-\frac{15}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - \frac{315}{2}x^{-\frac{17}{2}}\psi^{(3)}\left(\frac{1}{x}\right) \\
& - \frac{45}{2}x^{-\frac{19}{2}}\psi^{(4)}\left(\frac{1}{x}\right) - x^{-\frac{21}{2}}\psi^{(5)}\left(\frac{1}{x}\right) \\
& - \frac{945}{64}x^{-\frac{11}{2}},
\end{aligned}$$

$$\begin{aligned}
(5.8) \quad \psi^{(6)}(x) = & \frac{10395}{64}x^{-\frac{13}{2}}\psi\left(\frac{1}{x}\right) + \frac{31185}{16}x^{-\frac{15}{2}}\psi^{(1)}\left(\frac{1}{x}\right) \\
& + \frac{51975}{16}x^{-\frac{17}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + \frac{3465}{2}x^{-\frac{19}{2}}\psi^{(3)}\left(\frac{1}{x}\right) \\
& + \frac{1485}{4}x^{-\frac{21}{2}}\psi^{(4)}\left(\frac{1}{x}\right) + 33x^{-\frac{23}{2}}\psi^{(5)}\left(\frac{1}{x}\right) \\
& + x^{-\frac{25}{2}}\psi^{(6)}\left(\frac{1}{x}\right) + \frac{10395}{128}x^{-\frac{13}{2}},
\end{aligned}$$

$$\begin{aligned}
(5.9) \quad \psi^{(7)}(x) = & -\frac{135135}{128}x^{-\frac{15}{2}}\psi\left(\frac{1}{x}\right) - \frac{945945}{64}x^{-\frac{17}{2}}\psi^{(1)}\left(\frac{1}{x}\right) \\
& - \frac{945945}{32}x^{-\frac{19}{2}}\psi^{(2)}\left(\frac{1}{x}\right) - \frac{315315}{16}x^{-\frac{21}{2}}\psi^{(3)}\left(\frac{1}{x}\right) \\
& - \frac{45045}{8}x^{-\frac{23}{2}}\psi^{(4)}\left(\frac{1}{x}\right) - \frac{3003}{4}x^{-\frac{25}{2}}\psi^{(5)}\left(\frac{1}{x}\right) \\
& - \frac{91}{2}x^{-\frac{27}{2}}\psi^{(6)}\left(\frac{1}{x}\right) - x^{-\frac{29}{2}}\psi^{(7)}\left(\frac{1}{x}\right) \\
& - \frac{135135}{256}x^{-\frac{15}{2}}
\end{aligned}$$

and

$$\begin{aligned}
(5.10) \quad \psi^{(8)}(x) &= \frac{2027025}{256}x^{-\frac{17}{2}}\psi\left(\frac{1}{x}\right) + \frac{2027025}{16}x^{-\frac{19}{2}}\psi^{(1)}\left(\frac{1}{x}\right) \\
&+ \frac{4729725}{16}x^{-\frac{21}{2}}\psi^{(2)}\left(\frac{1}{x}\right) + \frac{945945}{4}x^{-\frac{23}{2}}\psi^{(3)}\left(\frac{1}{x}\right) \\
&+ \frac{675675}{8}x^{-\frac{25}{2}}\psi^{(4)}\left(\frac{1}{x}\right) + 15015x^{-\frac{27}{2}}\psi^{(5)}\left(\frac{1}{x}\right) \\
&+ 1365x^{-\frac{29}{2}}\psi^{(6)}\left(\frac{1}{x}\right) + 60x^{-\frac{31}{2}}\psi^{(7)}\left(\frac{1}{x}\right) \\
&+ x^{-\frac{33}{2}}\psi^{(8)}\left(\frac{1}{x}\right) + \frac{2027025}{512}x^{-\frac{17}{2}}.
\end{aligned}$$

Next we give formulas of $\psi^{(k)}(1)$. Substituting $x = 1$ in (5.3), we have

$$(5.11) \quad \frac{1}{4}\psi(1) + \psi^{(1)}(1) = -\frac{1}{8}.$$

Similarly, we obtain

$$(5.12) \quad \frac{15}{4}\psi^{(2)}(1) + \psi^{(3)}(1) = \frac{15}{32}\psi(1) + \frac{15}{64}$$

by (5.5) and (5.11). We also have

$$\begin{aligned}
\frac{45}{4}\psi^{(4)}(1) + \psi^{(5)}(1) &= -\frac{315}{64}\psi(1) - \frac{945}{16}\psi^{(1)}(1) - \frac{1575}{16}\psi^{(2)}(1) \\
&- \frac{105}{2}\psi^{(3)}(1) - \frac{315}{128}
\end{aligned}$$

by (5.8). Using (5.11) and (5.12), we obtain

$$(5.13) \quad \frac{45}{4}\psi^{(4)}(1) + \psi^{(5)}(1) = -\frac{945}{64}\psi(1) + \frac{1575}{16}\psi^{(2)}(1) - \frac{945}{128}.$$

We substitute $x = 1$ in (5.9). Then we have

$$\begin{aligned}
\frac{91}{4}\psi^{(6)}(1) + \psi^{(7)}(1) &= -\frac{135135}{256}\psi(1) - \frac{945945}{128}\psi^{(1)}(1) \\
&- \frac{945945}{64}\psi^{(2)}(1) - \frac{315315}{32}\psi^{(3)}(1) \\
&- \frac{45045}{16}\psi^{(4)}(1) - \frac{3003}{8}\psi^{(5)}(1) \\
&- \frac{135135}{512}.
\end{aligned}$$

Therefore, it follows from (5.11), (5.12) and (5.13) that

$$(5.14) \quad \frac{91}{4}\psi^{(6)}(1) + \psi^{(7)}(1) = \frac{2297295}{1024}\psi(1) - \frac{945945}{64}\psi^{(2)}(1) \\ + \frac{45045}{32}\psi^{(4)}(1) + \frac{2297295}{2048}.$$

We can consider $\psi^{(k)}$ for negative integers k . For any $k \in \mathbb{Z}$ we set

$$\psi^{(k)}(x) := (-1)^k \pi^k \sum_{n=1}^{\infty} n^{2k} e^{-\pi n^2 x}.$$

Then we have $\psi(x) = \psi^{(0)}(x)$ and $\frac{d}{dx}\psi^{(k)}(x) = \psi^{(k+1)}(x)$. By integral by parts, we obtain

$$(5.15) \quad \int_1^{\infty} \psi(x) dx = -\psi^{(-1)}(1)$$

and

$$(5.16) \quad \int_1^{\infty} x^2 \psi(x) dx = -\psi^{(-1)}(1) + 2\psi^{(-2)}(1) - 2\psi^{(-3)}(1).$$

Furthermore we have

$$(5.17) \quad \int_1^{\infty} x^{2k-1} |\psi^{(2k-1)}(x)| dx = \sum_{j=0}^{2k-1} (-1)^j \frac{(2k-1)!}{(2k-1-j)!} \psi^{(2k-2-j)}(1)$$

and

$$(5.18) \quad \int_1^{\infty} x^{2k} \psi^{(2k)}(x) dx = \sum_{j=0}^{2k} (-1)^{j+1} \frac{(2k)!}{(2k-j)!} \psi^{(2k-1-j)}(1)$$

for $k = 1, 2, \dots$.

6. PRECISE REPRESENTATIONS OF $R(s)$ AND $I(s)$

We give representations of $R(s)$ and $I(s)$ more precisely, determining $f_k(1)$ and $g_k(1)$ in (3.1) and (3.2) respectively. We use results in Sections 4 and 5.

We obtain

$$\begin{aligned} f_1(1) &= \left\{ \frac{\sigma}{2} + \left(-\frac{\sigma}{2} + \frac{1}{2} \right) \right\} \psi(1) + 2\psi^{(1)}(1) \\ &= 2 \left(\frac{1}{4} \psi(1) + \psi^{(1)}(1) \right) \\ &= -\frac{1}{4} \end{aligned}$$

by the representation of $f_1(x)$ in Section 4 and (5.11). Then we have

$$(6.1) \quad -\left(\frac{2}{t}\right)^2 f_1(1) = \frac{1}{t^2}.$$

It follows from the formula of $f_3(x)$ in Section 4, (5.11) and (5.12) that

$$\begin{aligned} f_3(1) &= \frac{1}{8}(3\sigma^3 - 3\sigma + 1)\psi(1) + \left(\frac{3}{2}\sigma^2 - \frac{3}{2}\sigma + \frac{17}{4}\right)\psi^{(1)}(1) \\ &\quad + \frac{15}{2}\psi^{(2)}(1) + 2\psi^{(3)}(1) \\ &= \frac{1}{2}(3\sigma^2 - 3\sigma + 1)\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) + \frac{15}{4}\psi^{(1)}(1) \\ &\quad + 2\left(\frac{15}{4}\psi^{(2)}(1) + \psi^{(3)}(1)\right) \\ &= -\frac{1}{2^4}(3\sigma^2 - 3\sigma + 1) + \frac{15}{4}\left(\frac{1}{4}\psi(1) + \psi^{(1)}(1)\right) + \frac{15}{32} \\ &= -\frac{1}{2^4}(3\sigma^2 - 3\sigma + 1), \end{aligned}$$

therefore

$$(6.2) \quad \left(\frac{2}{t}\right)^4 f_3(1) = -\frac{1}{t^4}(3\sigma^2 - 3\sigma + 1).$$

We have

$$\begin{aligned} f_5(1) &= \left(\frac{5}{32}\sigma^4 - \frac{5}{16}\sigma^3 + \frac{5}{16}\sigma^2 - \frac{5}{32}\sigma + \frac{1}{32}\right)\psi(1) \\ &\quad + \left(\frac{5}{8}\sigma^4 - \frac{5}{4}\sigma^3 + \frac{85}{8}\sigma^2 - 10\sigma + \frac{137}{16}\right)\psi^{(1)}(1) \\ &\quad + \left(\frac{75}{4}\sigma^2 - \frac{75}{4}\sigma + \frac{225}{4}\right)\psi^{(2)}(1) \\ &\quad + \left(5\sigma^2 - 5\sigma + \frac{135}{2}\right)\psi^{(3)}(1) \\ &\quad + \frac{45}{2}\psi^{(4)}(1) + 2\psi^{(5)}(1) \end{aligned}$$

by the formula of $f_5(x)$. Using (5.13), we can rewrite it as

$$\begin{aligned} f_5(1) &= \frac{5}{8}\sigma^4 \left(\frac{1}{4}\psi(1) + \psi^{(1)}(1) \right) - \frac{5}{4}\sigma^3 \left(\frac{1}{4}\psi(1) + \psi^{(1)}(1) \right) \\ &\quad + \sigma^2 \left(\frac{5}{16}\psi(1) + \frac{85}{8}\psi^{(1)}(1) + \frac{75}{4}\psi^{(2)}(1) + 5\psi^{(3)}(1) \right) \\ &\quad - \sigma \left(\frac{5}{32}\psi(1) + 10\psi^{(1)}(1) + \frac{75}{4}\psi^{(2)}(1) + 5\psi^{(3)}(1) \right) \\ &\quad - \frac{59}{2}\psi(1) + \frac{137}{16}\psi^{(1)}(1) + \frac{2025}{8}\psi^{(2)}(1) \\ &\quad + \frac{135}{2}\psi^{(3)}(1) - \frac{945}{64}. \end{aligned}$$

Then we obtain

$$f_5(1) = -\frac{5}{64}\sigma^4 + \frac{5}{32}\sigma^3 - \frac{5}{32}\sigma^2 + \frac{5}{64}\sigma - \frac{1}{64}$$

by (5.11) and (5.12). Hence we have

$$(6.3) \quad -\left(\frac{2}{t}\right)^6 f_5(1) = \frac{1}{t^6}(5\sigma^4 - 10\sigma^3 + 10\sigma^2 - 5\sigma + 1).$$

Therefore, it follows from (6.1), (6.2) and (6.3) that

$$\begin{aligned} R(s) &= \frac{1}{t^2} - \frac{1}{t^4}(3\sigma^2 - 3\sigma + 1) \\ (6.4) \quad &+ \frac{1}{t^6}(5\sigma^4 - 10\sigma^3 + 10\sigma^2 - 5\sigma + 1) \\ &+ \frac{1}{t^7}\alpha_1(s), \end{aligned}$$

where

$$\alpha_1(s) = 2^7 \int_1^\infty \sin\left(\frac{t}{2} \log x\right) f_6'(x) dx.$$

We use the same argument as above for $I(s)$. By results in Sections 4 and 5, we obtain

$$\begin{aligned} I(s) &= \frac{1}{t^3}(2\sigma - 1) - \frac{1}{t^5}(2\sigma - 1)(2\sigma^2 - 2\sigma + 1) \\ (6.5) \quad &+ \frac{1}{t^7}(2\sigma - 1)(3\sigma^4 - 6\sigma^3 + 7\sigma^2 - 4\sigma + 1) \\ &+ \frac{1}{t^8}\beta_1(s), \end{aligned}$$

where

$$\beta_1(s) = 2^8 \int_1^\infty \sin\left(\frac{t}{2} \log x\right) g_7'(x) dx.$$

7. ESTIMATES

We use inequalities $3.14159265 < \pi < 3.14159266$ and $0.04321391 < e^{-\pi} < 0.04321392$ to estimate $|\psi^{(k)}(1)|$.

Since

$$\psi(1) = \sum_{n=1}^{\infty} e^{-n^2\pi} < \sum_{n=1}^{\infty} (e^{-\pi})^n = \frac{e^{-\pi}}{1 - e^{-\pi}},$$

we have

$$(7.1) \quad \psi(1) < 0.04516571.$$

We also have

$$(7.2) \quad \begin{aligned} |\psi^{(-1)}(1)| &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n^2\pi} \\ &< \frac{1}{\pi} \psi(1) < 0.01437670. \end{aligned}$$

To estimate $|\psi^{(1)}(1)| = \pi \sum_{n=1}^{\infty} n^2 e^{-n^2\pi}$, we need a function $\varphi_1(x) = x^2 e^{-\pi x^2}$. Since $\varphi_1'(x) = 2x(1 - \pi x^2)e^{-\pi x^2}$, $\varphi_1(x)$ is monotonously decreasing on $(1, \infty)$. Then we have

$$\sum_{n=1}^{\infty} n^2 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_1(x) dx.$$

Noting

$$\int_1^{\infty} e^{-\pi x^2} dx < \int_1^{\infty} e^{-\pi x} dx = \frac{1}{\pi} e^{-\pi},$$

we obtain

$$\int_1^{\infty} \varphi_1(x) dx < \left(1 + \frac{1}{\pi}\right) \frac{1}{2\pi} e^{-\pi}$$

by integral by parts. Therefore we have

$$(7.3) \quad \begin{aligned} |\psi^{(1)}(1)| &< \pi e^{-\pi} + (1 + \pi) \frac{1}{2\pi} e^{-\pi} \\ &< 0.16424611. \end{aligned}$$

We consider a function $\varphi_2(x) = x^4 e^{-\pi x^2}$ for $\psi^{(2)}(1) = \pi^2 \sum_{n=1}^{\infty} n^4 e^{-n^2\pi}$. By the same way as above, we obtain

$$\sum_{n=1}^{\infty} n^4 e^{-n^2\pi} < e^{-\pi} + \int_1^{\infty} \varphi_2(x) dx$$

for $\varphi_2'(x) = 2x^3(2 - \pi x^2)e^{-\pi x^2}$. Since

$$\int_1^{\infty} \varphi_2(x) dx = \frac{1}{2\pi} e^{-\pi} + \frac{3}{2\pi} \int_1^{\infty} \varphi_1(x) dx,$$

we have

$$\sum_{n=1}^{\infty} n^4 e^{-n^2 \pi} < e^{-\pi} + \left\{ 1 + \frac{3}{2\pi} \left(1 + \frac{1}{\pi} \right) \right\} \frac{1}{2\pi} e^{-\pi}.$$

Hence we obtain

$$(7.4) \quad \begin{aligned} \psi^{(2)}(1) &< \left\{ \pi^2 + \frac{\pi}{2} \left(1 + \frac{3}{2\pi} \left(1 + \frac{1}{\pi} \right) \right) \right\} e^{-\pi} \\ &< 0.53711157. \end{aligned}$$

We estimate $|\psi^{(3)}(1)| = \pi^3 \sum_{n=1}^{\infty} n^6 e^{-n^2 \pi}$ using a function $\varphi_3(x) = x^6 e^{-\pi x^2}$. Since $\varphi_3'(x) = 2x^5(3 - \pi x^2)e^{-\pi x^2}$, we obtain

$$\sum_{n=1}^{\infty} n^6 e^{-n^2 \pi} < e^{-\pi} + \int_1^{\infty} \varphi_3(x) dx.$$

We have

$$\int_1^{\infty} \varphi_3(x) dx = \frac{1}{2\pi} e^{-\pi} + \frac{5}{2\pi} \int_1^{\infty} \varphi_2(x) dx.$$

Then we obtain

$$\sum_{n=1}^{\infty} n^6 e^{-n^2 \pi} < e^{-\pi} + \left\{ 1 + \frac{5}{2\pi} + \frac{5 \cdot 3}{(2\pi)^2} \left(1 + \frac{1}{\pi} \right) \right\} \frac{1}{2\pi} e^{-\pi}.$$

Therefore, we have

$$(7.5) \quad \begin{aligned} |\psi^{(3)}(1)| &< \left\{ \pi^3 + \frac{\pi^2}{2} \left(1 + \frac{5}{2\pi} + \frac{5 \cdot 3}{(2\pi)^2} \left(1 + \frac{1}{\pi} \right) \right) \right\} e^{-\pi} \\ &< 1.82967310. \end{aligned}$$

We can estimate other $|\psi^{(k)}(1)|$ by the same way. We do not give their estimation furthermore, because we do not need their real values in our argument.

We show the following lemma using the above estimates.

Lemma 1. *If $t > 41.232345$, then we have*

$$0 < R(s) < 0.001176398$$

for $\frac{1}{2} \leq \sigma \leq 1$.

Proof. We have already shown in Sections 3 and 5 that

$$R(s) = \frac{1}{t^2} + \left(\frac{2}{t} \right)^3 \int_1^{\infty} \sin \left(\frac{t}{2} \log x \right) f_2'(x) dx.$$

From the precise form of $f'_2(x)$ in Section 4, it follows that

$$\begin{aligned}
|f'_2(x)| &\leq \frac{1}{2^3}(3\sigma^2 - 3\sigma + 1)\psi(x) \\
&\quad + \left(\frac{3}{2}\sigma^2 - \frac{3}{2}\sigma + \frac{17}{4}\right)x^{\frac{\sigma}{2}}|\psi^{(1)}(x)| \\
&\quad + \frac{15}{2}x^{\frac{\sigma}{2}+1}|\psi^{(2)}(x)| + 2x^{\frac{\sigma}{2}+2}|\psi^{(3)}(x)| \\
&\leq \frac{1}{2^3}\psi(x) + \frac{17}{4}x|\psi^{(1)}(x)| + \frac{15}{2}x^2\psi^{(2)}(x) + 2x^3|\psi^{(3)}(x)|
\end{aligned}$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $1 \leq x$. Then we obtain

$$\begin{aligned}
2^3 \left| \int_1^\infty \sin\left(\frac{t}{2} \log x\right) f'_2(x) dx \right| &\leq \int_1^\infty \psi(x) dx + 34 \int_1^\infty x|\psi^{(1)}(x)| dx \\
&\quad + 60 \int_1^\infty x^2\psi^{(2)}(x) dx \\
&\quad + 16 \int_1^\infty x^3|\psi^{(3)}(x)| dx \\
&= -\psi^{(-1)}(1) + 34(\psi(1) - \psi^{(-1)}(1)) \\
&\quad + 60(-\psi^{(1)}(1) + 2\psi(1) - 2\psi^{(-1)}(1)) \\
&\quad + 16(\psi^{(2)}(1) - 3\psi^{(1)}(1) + 6\psi(1) \\
&\quad - 6\psi^{(-1)}(1)) \\
&= -251\psi^{(-1)}(1) + 250\psi(1) - 108\psi^{(1)}(1) \\
&\quad + 16\psi^{(2)}(1) \\
&< 41.232345
\end{aligned}$$

by (5.15), (5.16), (5.17), (5.18), (7.1), (7.2), (7.3) and (7.4). Therefore we have

$$\frac{1}{t^2} \left(1 - 41.232345 \frac{1}{|t|} \right) < R(s) < \frac{1}{t^2} \left(1 + 41.232345 \frac{1}{|t|} \right).$$

If $t > 41.232345$, then $1 - 41.232345 \frac{1}{t} > 0$ and

$$\frac{1}{t^2} \left(1 + 41.232345 \frac{1}{t} \right) < \frac{2}{(41.232345)^2} < 0.001176398.$$

□

We estimate $\alpha_1(s)$ in (6.4) by the same argument as the proof of Lemma 1.

Lemma 2. *There exists a constant M_1 such that*

$$(7.6) \quad |\alpha_1(s)| < M_1$$

for $\frac{1}{2} \leq \sigma \leq 1$ and $t \in \mathbb{R}$.

We also have a similar result for $\beta_1(s)$ in (6.5).

Lemma 3. *There exists a constant L_1 such that*

$$(7.7) \quad |\beta_1(s)| < (2\sigma - 1)L_1$$

for $\frac{1}{2} < \sigma \leq 1$ and $t \in \mathbb{R}$.

Proof. We rewrite $g'_7(x)$ in Section 4 as

$$(7.8) \quad \begin{aligned} g'_7(x) &= \sum_{j=0}^7 \left(A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right) \psi^{(j)}(x) \\ &\quad + \left(x^{\frac{\sigma}{2}+7} - x^{-\frac{\sigma}{2}+\frac{15}{2}} \right) \psi^{(8)}(x). \end{aligned}$$

Now we have

$$(7.9) \quad \begin{aligned} &A_j(\sigma) x^{\frac{\sigma}{2}-1+j} - B_j(\sigma) x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &= (A_j(\sigma) - B_j(\sigma)) x^{\frac{\sigma}{2}-1+j} + B_j(\sigma) \left(x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right). \end{aligned}$$

It holds that

$$\begin{aligned} &x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \left(\frac{\sigma}{2} - 1 + j \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^n \right\} (\log x)^n. \end{aligned}$$

We have

$$(7.10) \quad \begin{aligned} &\left(\frac{\sigma}{2} - 1 + j \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^n \\ &= \left(\sigma - \frac{1}{2} \right) \sum_{k=0}^{n-1} \left(\frac{\sigma}{2} - 1 + j \right)^{n-1-k} \left(-\frac{\sigma}{2} - \frac{1}{2} + j \right)^k. \end{aligned}$$

Since

$$\left| \frac{\sigma}{2} - 1 \right| < \frac{3}{4} < 1 \quad \text{and} \quad \left| -\frac{\sigma}{2} - \frac{1}{2} \right| \leq 1$$

for σ with $\frac{1}{2} < \sigma \leq 1$, we obtain

$$\left| \left(\frac{\sigma}{2} - 1 \right)^n - \left(-\frac{\sigma}{2} - \frac{1}{2} \right)^n \right| \leq \left(\sigma - \frac{1}{2} \right) n$$

by (7.10). Therefore, we have

$$\begin{aligned} & \left| x^{\frac{\sigma}{2}-1} - x^{-\frac{\sigma}{2}-\frac{1}{2}} \right| \\ & \leq \left(\sigma - \frac{1}{2} \right) (\log x) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (\log x)^{n-1} \\ & = \left(\sigma - \frac{1}{2} \right) (\log x) x. \end{aligned}$$

Noting

$$\left| \frac{\sigma}{2} \right| \leq \frac{1}{2} \quad \text{and} \quad \left| -\frac{\sigma}{2} - \frac{1}{2} + 1 \right| < \frac{1}{4} < \frac{1}{2},$$

we similarly obtain

$$\left| x^{\frac{\sigma}{2}} - x^{-\frac{\sigma}{2}-\frac{1}{2}+1} \right| \leq \left(\sigma - \frac{1}{2} \right) (\log x) x^{\frac{1}{2}}.$$

Furthermore, we obtain by the same way as above

$$\left| x^{\frac{\sigma}{2}-1+j} - x^{-\frac{\sigma}{2}-\frac{1}{2}+j} \right| \leq \left(\sigma - \frac{1}{2} \right) (\log x) x^{-\frac{1}{2}+j}$$

for $j = 2, 3, \dots, 7$.

Moreover, we have the following estimates for any σ with $\frac{1}{2} < \sigma \leq 1$

$$\begin{aligned} |A_0(\sigma) - B_0(\sigma)| &= \frac{1}{2^8} (2\sigma - 1) |4\sigma^6 - 12\sigma^5 + 22\sigma^4 - 24\sigma^3 + 16\sigma^2 - 6\sigma + 1| \\ &\leq \frac{1}{2^8} (2\sigma - 1), \end{aligned}$$

$$|B_0(\sigma)| < B_0\left(\frac{1}{2}\right) = \frac{1}{2^{16}},$$

$$\begin{aligned} |A_1(\sigma) - B_1(\sigma)| &= \frac{1}{2^8} (2\sigma - 1) |16\sigma^6 - 48\sigma^5 + 928\sigma^4 - 1776\sigma^3 + 6784\sigma^2 \\ &\quad - 5904\sigma + 6304| \\ &\leq \frac{197}{2^3} (2\sigma - 1), \end{aligned}$$

$$|B_1(\sigma)| < B_1\left(\frac{1}{2}\right) = \frac{12207}{2^{11}},$$

$$\begin{aligned} |A_2(\sigma) - B_2(\sigma)| &= \frac{1680}{2^8} (2\sigma - 1) |\sigma^4 - 2\sigma^3 + 29\sigma^2 - 28\sigma + 93| \\ &\leq \frac{3905}{2^6} (2\sigma - 1), \end{aligned}$$

$$|B_2(\sigma)| < B_2\left(\frac{1}{2}\right) = \frac{330199}{2^{10}},$$

$$\begin{aligned}
|A_3(\sigma) - B_3(\sigma)| &= \frac{448}{2^8}(2\sigma - 1) |\sigma^4 - 2\sigma^3 + 134\sigma^2 - 133\sigma + 1164| \\
&\leq 2037(2\sigma - 1), \\
|B_3(\sigma)| &< B_3\left(\frac{1}{2}\right) = \frac{223881}{2^7}, \\
|A_4(\sigma) - B_4(\sigma)| &= \frac{1}{2^3}(2\sigma - 1) |630\sigma^2 - 630\sigma + 15435| \\
&\leq \frac{15435}{2^3}(2\sigma - 1), \\
|B_4(\sigma)| &< B_4\left(\frac{1}{2}\right) = \frac{323043}{2^7}, \\
|A_5(\sigma) - B_5(\sigma)| &= (2\sigma - 1) |7\sigma^2 - 7\sigma + 672| \\
&\leq 672(2\sigma - 1), \\
|B_5(\sigma)| &< B_5\left(\frac{1}{2}\right) = \frac{10857}{2^3}, \\
|A_6(\sigma) - B_6(\sigma)| &= 91(2\sigma - 1), \quad |B_6(\sigma)| < B_6\left(\frac{1}{2}\right) = \frac{1239}{2^2}, \\
|A_7(\sigma) - B_7(\sigma)| &= 4(2\sigma - 1) \quad \text{and} \quad |B_7(\sigma)| < B_7\left(\frac{1}{2}\right) = 30.
\end{aligned}$$

We have $x^{\frac{\sigma}{2}} \leq x$, $(\log x)x^{\frac{1}{2}} \leq x$, $x^{\frac{\sigma}{2}-1} \leq 1$ and $(\log x)x \leq x^2$ for x with $1 \leq x$. Then, it follows from (7.8) and the above inequalities that

$$\begin{aligned}
\frac{2^8}{2\sigma - 1} |g'_7(x)| &< \psi(x) + \frac{1}{2^9}x^2\psi(x) + \left(2^5 \cdot 197 + \frac{12207}{24}\right)x|\psi^{(1)}(x)| \\
&\quad + \left(2^2 \cdot 3905 + \frac{330199}{2^3}\right)x^2\psi^{(2)}(x) \\
&\quad + (2^8 \cdot 2037 + 223881)x^3|\psi^{(3)}(x)| \\
&\quad + (2^5 \cdot 15435 + 323043)x^4\psi^{(4)}(x) \\
&\quad + (2^{13} \cdot 21 + 2^4 \cdot 10857)x^5|\psi^{(5)}(x)| \\
&\quad + (2^8 \cdot 91 + 2^5 \cdot 1239)x^6\psi^{(6)}(x) \\
&\quad + (2^{10} + 2^8 \cdot 15)x^7|\psi^{(7)}(x)| \\
&\quad + 2^7x^8\psi^{(8)}(x).
\end{aligned}$$

Integrating it, we obtain the conclusion by the results in Section 5 and the estimates of $|\psi^{(k)}(1)|$. \square

Furthermore, we have the following lemma.

Lemma 4. *There exist constants M'_1 and L'_1 such that*

$$(7.11) \quad \left| \frac{\partial \alpha_1}{\partial t}(s) \right| < M'_1$$

and

$$(7.12) \quad \left| \frac{\partial \beta_1}{\partial t}(s) \right| < (2\sigma - 1)L'_1$$

for $\frac{1}{2} < \sigma \leq 1$ and $t \in \mathbb{R}$, respectively.

Proof. We have

$$\frac{\partial \alpha_1}{\partial t}(s) = 2^6 \int_1^\infty \log x \cos\left(\frac{t}{2} \log x\right) f'_6(x) dx$$

and

$$\frac{\partial \beta_1}{\partial t}(s) = 2^7 \int_1^\infty \log x \cos\left(\frac{t}{2} \log x\right) g'_7(x) dx.$$

Since $\log x < x^{\frac{1}{2}}$ for $1 \leq x$, we obtain the statement by the same argument as the proofs of Lemmas 2 and 3. \square

8. MEAN VALUE THEOREM

We apply the mean value theorem to a function $\sqrt{1+x}$. For any x with $0 < |x| < 1$, there exists θ_x with $0 < \theta_x < 1$ such that

$$(8.1) \quad \sqrt{1+x} = 1 + \frac{1}{2} \frac{x}{\sqrt{1+\theta_x x}}.$$

In general, θ_x is not uniquely determined in the mean value theorem. However, we can explicitly write it in this case as follows:

$$\theta_x = \frac{x^2 - 4(\sqrt{1+x} - 1)^2}{4x(\sqrt{1+x} - 1)^2}.$$

We need the following lemma in the next section.

Lemma 5. *It holds that $\lim_{x \rightarrow 0} \theta_x = \frac{1}{2}$ and $\lim_{x \rightarrow 0} (d\theta_x/dx)x = -\frac{1}{8}$.*

Proof. If we set $\mu(x) = x^2 - 4(\sqrt{1+x} - 1)^2$ and $\lambda(x) = 4x(\sqrt{1+x} - 1)^2$, then we have

$$\lim_{x \rightarrow 0} \theta_x = \lim_{x \rightarrow 0} \frac{\mu(x)}{\lambda(x)} = \frac{\mu'''(0)}{\lambda'''(0)} = \frac{1}{2}$$

by l'Hôpital's rule.

We consider the next limit. We have

$$\begin{aligned} \frac{d\theta_x}{dx}x &= \frac{x(\mu'(x)\lambda(x) - \mu(x)\lambda'(x))}{\lambda(x)^2} \\ &= \frac{x^2(\sqrt{1+x}-1) - x^3(1+x)^{-\frac{1}{2}} + 4(\sqrt{1+x}-1)^3}{4x(\sqrt{1+x}-1)^3}. \end{aligned}$$

We set

$$\nu(x) = x^2(\sqrt{1+x}-1) - x^3(1+x)^{-\frac{1}{2}} + 4(\sqrt{1+x}-1)^3$$

and $\tau(x) = 4x(\sqrt{1+x}-1)^3$. Then we obtain

$$\lim_{x \rightarrow 0} \frac{d\theta_x}{dx}x = \lim_{x \rightarrow 0} \frac{\nu(x)}{\tau(x)} = \frac{\nu^{(4)}(0)}{\tau^{(4)}(0)} = -\frac{1}{8}$$

using l'Hôpital's rule again. \square

9. NEW ZERO-FREE REGION

In Section 2 we showed that $\xi(s) = 0$ if and only if (2.3) holds. The first equality of (2.3) is equivalent to

$$R(s)t^4 + \{(2\sigma^2 - 2\sigma + 1)R(s) - 1\}t^2 + \sigma^2(\sigma - 1)^2R(s) + \sigma(\sigma - 1) = 0.$$

If we set $T = t^2$, then it is

$$(9.1) \quad R(s)T^2 + B(s)T + C(s) = 0,$$

where we set

$$\begin{cases} B(s) = (2\sigma^2 - 2\sigma + 1)R(s) - 1, \\ C(s) = \sigma^2(\sigma - 1)^2R(s) + \sigma(\sigma - 1). \end{cases}$$

We assume $t > 41.232345$ in this section. Since we had known the truth of the Riemann hypothesis for $|t| \leq 41.232345$ even in the early stage of numerical verifications ([1], [5] and [7]), it is not a restriction. Then $R(s) > 0$, $B(s) < 0$ and $C(s) < 0$ by Lemma 1. The equation (9.1) means

$$T = \frac{-B(s) + \sqrt{B(s)^2 - 4R(s)C(s)}}{2R(s)}$$

for $T > 0$. We set $D(s) = B(s)^2 - 4R(s)C(s)$ and $A(s) = -B(s) + \sqrt{D(s)}$. Then, we have

$$(9.2) \quad t = \sqrt{\frac{A(s)}{2R(s)}}.$$

The second equation of (2.3) is equivalent to

$$\{t^4 + (2\sigma^2 - 2\sigma + 1)t^2 + \sigma^2(\sigma - 1)^2\} I(s) = (2\sigma - 1)t.$$

We substitute (9.2) into the above equation. Then we have

$$\begin{aligned} \left\{ \left(\frac{A(s)}{2R(s)} \right)^2 + (2\sigma^2 - 2\sigma + 1) \frac{A(s)}{2R(s)} + \sigma^2(\sigma - 1)^2 \right\} I(s) \\ = (2\sigma - 1) \sqrt{\frac{A(s)}{2R(s)}}. \end{aligned}$$

Multiplying it squared by $(2R(s))^4$, we obtain

$$\begin{aligned} (9.3) \quad \{A(s)^2 + 2(2\sigma^2 - 2\sigma + 1)A(s)R(s) + 4\sigma^2(\sigma - 1)^2R(s)^2\} I(s)^2 \\ = 8(2\sigma - 1)^2 A(s)R(s)^3. \end{aligned}$$

For the sake of simplicity, we set $\alpha(\sigma) = 4\sigma^2 - 4\sigma + 1$, $\beta(\sigma) = 2\sigma^2 - 2\sigma + 1$ and $\gamma(\sigma) = 3\sigma^2 - 3\sigma + 1$. Since

$$\begin{aligned} A(s)^2 + 2(2\sigma^2 - 2\sigma + 1)A(s)R(s) + 4\sigma^2(\sigma - 1)^2R(s)^2 \\ = \{A(s) + (2\sigma^2 - 2\sigma + 1)R(s)\}^2 + \{4\sigma^2(\sigma - 1)^2 - (2\sigma^2 - 2\sigma + 1)^2\} R(s)^2 \\ = \left(1 + \sqrt{D(s)}\right)^2 - (4\sigma^2 - 4\sigma + 1)R(s)^2 \\ = 1 + 2\sqrt{D(s)} + D(s) - (4\sigma^2 - 4\sigma + 1)R(s)^2 \\ = 2 \left(1 + \sqrt{D(s)} - \alpha(\sigma)R(s)\right), \end{aligned}$$

(9.3) is equivalent to

$$\begin{aligned} \left\{1 + 2\sqrt{D(s)} + D(s) - 2 \left(1 + \sqrt{D(s)}\right) \alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2\right\} I(s)^2 \\ = 2(2\sigma - 1)^2 \left(-B(s) + \sqrt{D(s)}\right) R(s)^3. \end{aligned}$$

Then we have

$$\begin{aligned} \{2(1 - \alpha(\sigma)R(s)) I(s)^2 - 2(2\sigma - 1)^2 R(s)^3\} \sqrt{D(s)} \\ = -(1 + D(s) - 2\alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2) I(s)^2 \\ - 2(2\sigma - 1)^2 B(s) R(s)^3. \end{aligned}$$

We multiply it by itself. Then we have

$$(9.4) \quad 4 \left\{ (1 - \alpha(\sigma)R(s))I(s)^2 - (2\sigma - 1)^2 R(s)^3 \right\}^2 D(s) \\ = \left\{ (1 + D(s) - 2\alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2) I(s)^2 \right. \\ \left. + 2(2\sigma - 1)^2 B(s)R(s)^3 \right\}^2.$$

Since

$$D(s) = B(s)^2 - 4R(s)C(s) \\ = (4\sigma^2 - 4\sigma + 1)R(s)^2 - 2(4\sigma^2 - 4\sigma + 1)R(s) + 1 \\ = \alpha(\sigma)R(s)^2 - 2\alpha(\sigma)R(s) + 1,$$

we have

$$1 + D(s) - 2\alpha(\sigma)R(s) + \alpha(\sigma)^2 R(s)^2 \\ = 2 \left(\alpha(\sigma)\beta(\sigma)R(s)^2 - 2\alpha(\sigma)R(s) + 1 \right).$$

Then, it follows from (9.4) that

$$(9.5) \quad \left\{ (1 - \alpha(\sigma)R(s))^2 I(s)^4 - 2(2\sigma - 1)^2 (1 - \alpha(\sigma)R(s)) R(s)^3 I(s)^2 \right. \\ \left. + (2\sigma - 1)^4 R(s)^6 \right\} \left(\alpha(\sigma)R(s)^2 - 2\alpha(\sigma)R(s) + 1 \right) \\ = \left(\alpha(\sigma)\beta(\sigma)R(s)^2 - 2\alpha(\sigma)R(s) + 1 \right)^2 I(s)^4 \\ + 2(2\sigma - 1)^2 \left(\alpha(\sigma)\beta(\sigma)R(s)^2 - 2\alpha(\sigma)R(s) + 1 \right) B(s)R(s)^3 I(s)^2 \\ + (2\sigma - 1)^4 B(s)^2 R(s)^6.$$

The left side minus the right side of (9.5) equals

$$\alpha(\sigma) (1 + \alpha(\sigma) - 2\beta(\sigma)) R(s)^2 I(s)^4 \\ - 2\alpha(\sigma)^2 (1 + \alpha(\sigma) - 2\beta(\sigma)) R(s)^3 I(s)^4 \\ + \alpha(\sigma)^2 (\alpha(\sigma) - \beta(\sigma)^2) R(s)^4 I(s)^4 \\ + 2(2\sigma - 1)^2 \alpha(\sigma) (-1 - 2\alpha(\sigma) + 3\beta(\sigma)) R(s)^5 I(s)^2 \\ + 2(2\sigma - 1)^2 \alpha(\sigma) (\alpha(\sigma) - \beta(\sigma)^2) R(s)^6 I(s)^2 \\ + (2\sigma - 1)^4 (\alpha(\sigma) - \beta(\sigma)^2) R(s)^8 \\ + 2(2\sigma - 1)^2 (\alpha(\sigma) - \beta(\sigma)) R(s)^4 I(s)^2 \\ + 2(2\sigma - 1)^4 (\beta(\sigma) - \alpha(\sigma)) R(s)^7.$$

Dividing it by $R(s)^2$, we set

$$\begin{aligned}
F(s) &:= \alpha(\sigma) (1 + \alpha(\sigma) - 2\beta(\sigma)) I(s)^4 \\
&\quad - 2\alpha(\sigma)^2 (1 + \alpha(\sigma) - 2\beta(\sigma)) R(s) I(s)^4 \\
&\quad + \alpha(\sigma)^2 (\alpha(\sigma) - \beta(\sigma)^2) R(s)^2 I(s)^4 \\
&\quad + 2(2\sigma - 1)^2 \alpha(\sigma) (-1 - 2\alpha(\sigma) + 3\beta(\sigma)) R(s)^3 I(s)^2 \\
&\quad + 2(2\sigma - 1)^2 \alpha(\sigma) (\alpha(\sigma) - \beta(\sigma)^2) R(s)^4 I(s)^2 \\
&\quad + (2\sigma - 1)^4 (\alpha(\sigma) - \beta(\sigma)^2) R(s)^6 \\
&\quad + 2(2\sigma - 1)^2 (\alpha(\sigma) - \beta(\sigma)) R(s)^2 I(s)^2 \\
&\quad + 2(2\sigma - 1)^4 (\beta(\sigma) - \alpha(\sigma)) R(s)^5.
\end{aligned}$$

Since $1 + \alpha(\sigma) - 2\beta(\sigma) = 0$, $\alpha(\sigma) = (2\sigma - 1)^2$, $\alpha(\sigma) - \beta(\sigma)^2 = -4\sigma^2(\sigma - 1)^2$, $\alpha(\sigma) - \beta(\sigma) = 2\sigma(\sigma - 1)$ and $-1 - 2\alpha(\sigma) + 3\beta(\sigma) = \beta(\sigma) - \alpha(\sigma) = -2\sigma(\sigma - 1)$, we obtain

$$\begin{aligned}
F(s) &= -4\sigma^2(\sigma - 1)^2(2\sigma - 1)^4 R(s)^2 I(s)^4 \\
&\quad - 4\sigma(\sigma - 1)(2\sigma - 1)^4 R(s)^3 I(s)^2 \\
&\quad - 8\sigma^2(\sigma - 1)^2(2\sigma - 1)^4 R(s)^4 I(s)^2 \\
&\quad - 4\sigma^2(\sigma - 1)^2(2\sigma - 1)^4 R(s)^6 \\
&\quad + 4\sigma(\sigma - 1)(2\sigma - 1)^2 R(s)^2 I(s)^2 \\
&\quad - 4\sigma(\sigma - 1)(2\sigma - 1)^4 R(s)^5.
\end{aligned}$$

Let

$$\begin{aligned}
G(s) &:= \sigma(\sigma - 1)(2\sigma - 1)^2 I(s)^4 + (2\sigma - 1)^2 R(s) I(s)^2 \\
&\quad + 2\sigma(\sigma - 1)(2\sigma - 1)^2 R(s)^2 I(s)^2 + \sigma(\sigma - 1)(2\sigma - 1)^2 R(s)^4 \\
&\quad - I(s)^2 + (2\sigma - 1)^2 R(s)^3.
\end{aligned}$$

Then we have

$$(9.6) \quad F(s) = -4\sigma(\sigma - 1)(2\sigma - 1)^2 R(s)^2 G(s).$$

Proposition 1. *If $s = \sigma + it$ with $\frac{1}{2} < \sigma < 1$ and $t > 41.232345$ is a zero of $\xi(s)$, then we have $G(s) = 0$.*

Proof. In the above, we showed $F(s) = 0$ if $\xi(s) = 0$. By Lemma 1, $-4\sigma(\sigma - 1)(2\sigma - 1)^2 R(s)^2 > 0$ for any s with $\frac{1}{2} < \sigma < 1$ and $t > 41.232345$. Then $G(s) = 0$ by (9.6). \square

Theorem 1. *There exists $T_0 > 0$ such that any $s = \sigma + it$ with $\frac{1}{2} < \sigma < 1$ and $t > T_0$ is not a zero of $\zeta(s)$.*

Proof. It suffices to find a region in which $G(s) \neq 0$, by Proposition 1.

Adding to $\alpha(\sigma)$, $\beta(\sigma)$ and $\gamma(\sigma)$, we set

$$\begin{cases} C_1(\sigma) = 5\sigma^4 - 10\sigma^3 + 10\sigma^2 - 5\sigma + 1, \\ C_2(\sigma) = 3\sigma^4 - 6\sigma^3 + 7\sigma^2 - 4\sigma + 1. \end{cases}$$

From (6.4) and (6.5), it follows that

$$(9.7) \quad R(s)^2 = \frac{1}{t^4} - \frac{2}{t^6}\gamma(\sigma) + \frac{1}{t^8}(\gamma(\sigma)^2 + 2C_1(\sigma)) + \frac{1}{t^9}\alpha_2(s)$$

and

$$(9.8) \quad \begin{aligned} I(s)^2 &= \frac{1}{t^6}(2\sigma - 1)^2 - \frac{2}{t^8}(2\sigma - 1)^2\beta(\sigma) \\ &\quad + \frac{1}{t^{10}}(2\sigma - 1)^2(\beta(\sigma)^2 + 2C_2(\sigma)) + \frac{1}{t^{11}}\beta_2(s), \end{aligned}$$

where we set

$$\begin{aligned} \alpha_2(s) &= 2\alpha_1(s) - \frac{2}{t}\gamma(\sigma)C_1(\sigma) - \frac{2}{t^2}\gamma(\sigma)\alpha_1(s) + \frac{1}{t^3}C_1(\sigma)^2 \\ &\quad + \frac{2}{t^4}C_1(\sigma)\alpha_1(s) + \frac{1}{t^5}\alpha_1(s)^2 \end{aligned}$$

and

$$\begin{aligned} \beta_2(s) &= 2(2\sigma - 1)\beta_1(s) - \frac{2}{t}(2\sigma - 1)^2\beta(\sigma)C_2(\sigma) \\ &\quad - \frac{2}{t^2}(2\sigma - 1)\beta(\sigma)\beta_1(s) + \frac{1}{t^3}(2\sigma - 1)^2C_2(\sigma)^2 \\ &\quad + \frac{2}{t^4}(2\sigma - 1)C_2(\sigma)\beta_1(s) + \frac{1}{t^5}\beta_1(s)^2. \end{aligned}$$

We rewrite $G(s)$ as

$$\begin{aligned} G(s) &= \sigma(\sigma - 1)(2\sigma - 1)^2(R(s)^2 + I(s)^2)^2 \\ &\quad + (2\sigma - 1)^2R(s)(R(s)^2 + I(s)^2) - I(s)^2. \end{aligned}$$

The equation $G(s) = 0$ means that $R(s)^2 + I(s)^2$ is a root of the following quadratic equation

$$\sigma(\sigma - 1)(2\sigma - 1)^2X^2 + (2\sigma - 1)^2R(s)X - I(s)^2 = 0.$$

Then we have

$$(9.9) \quad R(s)^2 + I(s)^2 = \frac{-(2\sigma - 1)R(s) \pm \sqrt{(2\sigma - 1)^2R(s)^2 + 4\sigma(\sigma - 1)I(s)^2}}{2\sigma(\sigma - 1)(2\sigma - 1)}.$$

We determine the sign in (9.9). It follows from (9.7) and (9.8) that

$$(9.10) \quad R(s)^2 + I(s)^2 = \frac{1}{t^4} - \frac{1}{t^6}\beta(\sigma) + \frac{1}{t^8}C_3(\sigma) + \frac{1}{t^9}\gamma_1(s),$$

where we set

$$C_3(\sigma) = \gamma(\sigma)^2 + 2C_1(\sigma) - 2(2\sigma - 1)^2\beta(\sigma)$$

and

$$\gamma_1(s) = \alpha_2(s) + \frac{1}{t}(2\sigma - 1)^2(\beta(\sigma)^2 + 2C_2(\sigma)) + \frac{1}{t^2}\beta_2(s).$$

The functions $\beta(\sigma)$ and $C_3(\sigma)$ are uniformly bounded on $\frac{1}{2} < \sigma < 1$. By Lemmas 2 and 3, we also see that $\gamma_1(s)$ is uniformly bounded on $\frac{1}{2} < \sigma < 1$ and $t > t_0$, where t_0 is some constant with $t_0 > 41.232345$. Then $t^2(R(s)^2 + I(s)^2) \rightarrow 0$ as $t \rightarrow \infty$. On the other hand, we have $\lim_{t \rightarrow \infty} t^2 R(s) = 1$ and

$$\lim_{t \rightarrow \infty} t^2 \sqrt{(2\sigma - 1)^2 R(s)^2 + 4\sigma(\sigma - 1)I(s)^2} = 2\sigma - 1.$$

Therefore, the sign in (9.9) must be plus. Thus we have

$$(9.11) \quad 2\sigma(\sigma - 1)(2\sigma - 1)(R(s)^2 + I(s)^2) + (2\sigma - 1)R(s) - \sqrt{(2\sigma - 1)^2 R(s)^2 + 4\sigma(\sigma - 1)I(s)^2} = 0$$

if s is a zero of $\xi(s)$ with $\frac{1}{2} < \sigma < 1$ and $t > t_0$.

By the same way as (9.10), we obtain

$$\begin{aligned} & (2\sigma - 1)^2 R(s)^2 + 4\sigma(\sigma - 1)I(s)^2 \\ &= \frac{1}{t^4}(2\sigma - 1)^2 - \frac{2}{t^6}(2\sigma - 1)^2(\sigma^2 - \sigma + 1) \\ & \quad + \frac{1}{t^8}(2\sigma - 1)^2 C_4(\sigma) + \frac{1}{t^9}(2\sigma - 1)^2 \gamma_2(s), \end{aligned}$$

where we set

$$\begin{aligned} C_4(\sigma) &= \gamma(\sigma)^2 + 2C_1(\sigma) - 8\sigma(\sigma - 1)\beta(\sigma) \\ &= 3\sigma^4 - 6\sigma^3 + 11\sigma^2 - 8\sigma + 3 \end{aligned}$$

and

$$\begin{aligned} \gamma_2(s) &= \alpha_2(s) + \frac{4}{t}\sigma(\sigma - 1)(\beta(\sigma)^2 + 2C_2(\sigma)) \\ & \quad + \frac{4}{t^2}\sigma(\sigma - 1)\frac{\beta_2(s)}{(2\sigma - 1)^2}. \end{aligned}$$

If we set

$$x = x(s) := -\frac{2}{t^2}(\sigma^2 - \sigma + 1) + \frac{1}{t^4}C_4(\sigma) + \frac{1}{t^5}\gamma_2(s),$$

then we have

$$(9.12) \quad \sqrt{(2\sigma - 1)^2 R(s)^2 + 4\sigma(\sigma - 1)I(s)^2} = \frac{1}{t^2}(2\sigma - 1)\sqrt{1 + x}.$$

Therefore, the equation (9.11) is equivalent to

$$(9.13) \quad 2\sigma(\sigma - 1) (R(s)^2 + I(s)^2) + R(s) - \frac{1}{t^2}\sqrt{1 + x} = 0.$$

We apply the results in Section 8 to $\sqrt{1 + x}$. Letting $M(\theta_x) = \frac{1}{2}(1 + \theta_x x)^{-\frac{1}{2}}$, we obtain

$$(9.14) \quad \begin{aligned} \sqrt{1 + x} &= 1 - \frac{2}{t^2}M(\theta_x)(\sigma^2 - \sigma + 1) + \frac{1}{t^4}M(\theta_x)C_4(\sigma) \\ &\quad + \frac{1}{t^5}M(\theta_x)\gamma_2(s). \end{aligned}$$

From (6.4), (9.10) and (9.14), it follows that

$$\begin{aligned} &2\sigma(\sigma - 1) (R(s)^2 + I(s)^2) + R(s) - \frac{1}{t^2}\sqrt{1 + x} \\ &= \frac{1}{t^4} \{2\sigma(\sigma - 1) - \gamma(\sigma) + 2M(\theta_x)(\sigma^2 - \sigma + 1)\} \\ &\quad - \frac{1}{t^6} \{2\sigma(\sigma - 1)\beta(\sigma) - C_1(\sigma) + M(\theta_x)C_4(\sigma)\} + \frac{1}{t^7}E(s), \end{aligned}$$

where we set

$$\begin{aligned} E(s) &= \alpha_1(s) - M(\theta_x)\gamma_2(s) + \frac{2}{t}\sigma(\sigma - 1)C_3(\sigma) \\ &\quad + \frac{2}{t^2}\sigma(\sigma - 1)\gamma_1(s). \end{aligned}$$

Since $0 < \theta_x < 1$ and $x < 0$ for a sufficiently large t , we have

$$\begin{aligned} &2\sigma(\sigma - 1) - \gamma(\sigma) + 2M(\theta_x)(\sigma^2 - \sigma + 1) \\ &= (\sigma^2 - \sigma + 1) \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\} > 0 \end{aligned}$$

for $\frac{1}{2} < \sigma < 1$.

Now we have

$$\begin{aligned}
(9.15) \quad & 2\sigma(\sigma-1) \left(R(s)^2 + I(s)^2 \right) + R(s) - \frac{1}{t^2} \sqrt{1+x} \\
&= \frac{1}{t^4} (\sigma^2 - \sigma + 1) \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\} \\
&- \frac{1}{t^6} \{ 2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + M(\theta_x)C_4(\sigma) \} + \frac{1}{t^7} E(s) \\
&= \frac{8 \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\}}{3t^4(\sigma^2 - \sigma + 1)} \times \left[\frac{3}{8} (\sigma^2 - \sigma + 1)^2 \right. \\
&- \frac{3}{8} (\sigma^2 - \sigma + 1) \frac{2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + M(\theta_x)C_4(\sigma)}{t^2 \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\}} \\
&\left. + \frac{3}{8t} (\sigma^2 - \sigma + 1) \frac{E(s)}{t^2 \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\}} \right].
\end{aligned}$$

We obtain by a direct calculation

$$\begin{aligned}
\frac{\partial \gamma_2}{\partial t}(s) &= \frac{\partial \alpha_2}{\partial t}(s) - \frac{4}{t^2} \sigma(\sigma-1) (\beta(\sigma)^2 + 2C_2(\sigma)) - \frac{8}{t^3} \sigma(\sigma-1) \frac{\beta_2(s)}{(2\sigma-1)^2} \\
&+ \frac{4}{t^2} \sigma(\sigma-1) \frac{1}{(2\sigma-1)^2} \frac{\partial \beta_2}{\partial t}(s),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \alpha_2}{\partial t}(s) &= 2 \frac{\partial \alpha_1}{\partial t}(s) + \frac{1}{t^2} \gamma(\sigma)^2 + \frac{6}{t^3} \gamma(\sigma) \alpha_1(s) - \frac{2}{t^2} \gamma(\sigma) \frac{\partial \alpha_1}{\partial t}(s) \\
&+ \frac{3}{t^4} \alpha_1(s)^2 - \frac{2}{t^3} \alpha_1(s) \frac{\partial \alpha_1}{\partial t}(s)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \beta_2}{\partial t}(s) &= 2(2\sigma-1) \frac{\partial \beta_1}{\partial t}(s) + \frac{2}{t^2} (2\sigma-1)^2 \beta(\sigma) C_2(\sigma) \\
&+ \frac{6}{t^3} (2\sigma-1) \beta(\sigma) \beta_1(s) - \frac{2}{t^2} (2\sigma-1) \beta(\sigma) \frac{\partial \beta_1}{\partial t}(s) \\
&- \frac{3}{t^4} (2\sigma-1)^2 C_2(\sigma)^2 - \frac{8}{t^5} (2\sigma-1) C_2(\sigma) \beta_1(s) \\
&+ \frac{2}{t^4} (2\sigma-1) C_2(\sigma) \frac{\partial \beta_1}{\partial t}(s) - \frac{5}{t^6} \beta_1(s)^2 \\
&+ \frac{2}{t^5} \beta_1(s) \frac{\partial \beta_1}{\partial t}(s).
\end{aligned}$$

Then, we can take positive constants $M_2, M'_2, L_2, L'_2, N_1, N_2$ and N'_2 such that $|\alpha_2(s)| < M_2$, $|\partial \alpha_2 / \partial t(s)| < M'_2$, $|\beta_2(s)| < (2\sigma-1)^2 L_2$,

$|\partial\beta_2/\partial t(s)| < (2\sigma-1)^2 L'_2$, $|\gamma_1(s)| < N_1$, $|\gamma_2(s)| < N_2$ and $|\partial\gamma_2/\partial t(s)| < N'_2$ for $\frac{1}{2} < \sigma < 1$ and $t > t_0$, by Lemmas 2, 3 and 4. Since

$$\frac{d}{dt} \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\} = -\frac{1}{2} (1 + \theta_x x)^{-\frac{3}{2}} \left(\frac{d\theta_x}{dx} x + \theta_x \right) \frac{\partial x}{\partial t},$$

$$\frac{\partial x}{\partial t} = \frac{4}{t^3} (\sigma^2 - \sigma + 1) - \frac{4}{t^5} C_4(\sigma) - \frac{5}{t^6} \gamma_2(s) + \frac{1}{t^5} \frac{\partial \gamma_2}{\partial t}(s)$$

and $d(1/t^2)/dt = -2/t^3$, we obtain

$$\lim_{t \rightarrow \infty} t^2 \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\} = \frac{3}{8} (\sigma^2 - \sigma + 1)$$

by l'Hôpital's rule, Lemma 5 and the above estimates. Then we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{3}{8} (\sigma^2 - \sigma + 1) \frac{2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + M(\theta_x)C_4(\sigma)}{t^2 \left\{ (1 + \theta_x x)^{-\frac{1}{2}} - 1 \right\}} \\ = 2\sigma(\sigma-1)\beta(\sigma) - C_1(\sigma) + \frac{1}{2} C_4(\sigma). \end{aligned}$$

We note that the convergence is uniformly for $\frac{1}{2} < \sigma < 1$. We have

$$\begin{aligned} \frac{3}{8} (\sigma^2 - \sigma + 1)^2 - 2\sigma(\sigma-1)\beta(\sigma) + C_1(\sigma) - \frac{1}{2} C_4(\sigma) \\ = -\frac{1}{8} (\sigma^4 - 2\sigma^3 + 3\sigma^2 - 2\sigma + 1) < -\frac{9}{128} \end{aligned}$$

and $\frac{3}{4} < \sigma^2 - \sigma + 1 < 1$ for $\frac{1}{2} < \sigma < 1$. Furthermore, there exists a constant \widetilde{M} such that $|E(s)| < \widetilde{M}$ for $\frac{1}{2} < \sigma < 1$ and $t > t_0$. Therefore, there exists $T_0 > 0$ such that if $t > T_0$, then the right side of (9.15) is negative for $\frac{1}{2} < \sigma < 1$. Thus, the proof is complete. \square

10. LEMMA ON DIRICHLET SERIES

This section is devoted to the following lemma which is a variant of Lemma 3.12 in [8].

Lemma 6. *We assume that a Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is absolutely convergent for $\sigma > 1$. Take $\sigma_0 < 1$ and $c > 0$ with $\sigma_0 + c > 1$. Let $g(s)$ be an entire function with finite number of zeros $Z_g = \{\alpha_1, \dots, \alpha_N\}$ such that $\sigma_0 < \operatorname{Re}(\alpha_j) < \sigma_0 + c$ ($j = 1, \dots, N$) and $|g(s)| \rightarrow \infty$ as $s \rightarrow \infty$. Then, there exist meromorphic functions $h_n(s)$

($1 \leq n < x$) on \mathbb{C} whose poles are at most $\alpha_1, \dots, \alpha_N$ and all simple such that

$$(10.1) \quad \sum_{n < x} \frac{a_n}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{f(s+w)}{g(s+w)} \frac{x^w}{w} dw \\ + \sum_{n < x} \frac{a_n}{n^s} R_n(c, x, T)(s) + \sum_{x < n} \frac{a_n}{n^s} Q_n(c, x, T)(s)$$

for any $s \in D(\sigma_0, T) \setminus Z_g$, where a positive number x is not an integer, T is a positive number with $|\operatorname{Im}(\alpha_j)| < \frac{T}{4}$ ($j = 1, \dots, N$), $D(\sigma_0, T) = \{\sigma + it; \sigma_0 \leq \sigma, |t| < \frac{T}{4}\}$, and $R_n(c, x, T)(s)$ and $Q_n(c, x, T)(s)$ are holomorphic functions depending only on $g(s), n, c, x, T$ and satisfying (10.2)

$$|R_n(c, x, T)(s)| < \frac{M}{A\pi T} \left(\frac{x}{n} \right)^c \quad \text{and} \quad |Q_n(c, x, T)(s)| < \frac{M}{A\pi T} \left(\frac{x}{n} \right)^c$$

on $D(\sigma_0, T)$ for some constants A and M . Therefore, the right side of (10.1) converges absolutely and uniformly on $D(\sigma_0, T)$.

Proof. Take any $s \in D(\sigma_0, T)$. A function $\frac{1}{g(s+w)} \left(\frac{x}{n} \right)^w \frac{1}{w}$ of w has poles at $w = 0, \alpha_1 - s, \dots, \alpha_N - s$. It has the residues $\frac{1}{g(s)}$ and $a_j \left(\frac{x}{n} \right)^{\alpha_j - s} \frac{1}{\alpha_j - s}$ at $w = 0$ and $\alpha_j - s$ respectively, where a_j is the residue of $\frac{1}{g(s)}$ at α_j . We define a meromorphic function

$$h_n(s) := \sum_{j=1}^N a_j \left(\frac{x}{n} \right)^{\alpha_j - s} \frac{1}{\alpha_j - s}.$$

If $n < x$, then we obtain

$$\frac{1}{2\pi i} \left(\int_{-\infty-iT}^{c-iT} + \int_{c-iT}^{c+iT} + \int_{c+iT}^{-\infty+iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n} \right)^w \frac{dw}{w} \\ = \frac{1}{g(s)} + h_n(s)$$

by the residue theorem. Let

$$M := \sup \left\{ \left| \frac{1}{g(s)} \right|; s = u + it, -\infty < u < \infty, \frac{3}{4}T \leq |t| \leq \frac{5}{4}T \right\}.$$

Then we have $0 < M < \infty$ by the assumption of $g(s)$. Therefore, the following estimation holds

$$\left| \frac{1}{2\pi i} \int_{-\infty-iT}^{c-iT} \frac{1}{g(s+w)} \left(\frac{x}{n} \right)^w \frac{dw}{w} \right| < \frac{M}{2\pi T} \frac{(x/n)^c}{\log(x/n)}.$$

Similarly we have

$$\left| \frac{1}{2\pi i} \int_{c+iT}^{-\infty+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} \right| < \frac{M}{2\pi T} \frac{(x/n)^c}{\log(x/n)}.$$

We define

$$R_n(c, x, T)(s) := \frac{1}{2\pi i} \left(\int_{-\infty-iT}^{c-iT} + \int_{c+iT}^{-\infty+iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w}.$$

Then we obtain

$$(10.3) \quad \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \frac{1}{g(s)} + h_n(s) - R_n(c, x, T)(s)$$

and

$$|R_n(c, x, T)(s)| < \frac{M}{\pi T} \frac{(x/n)^c}{\log(x/n)}$$

for $s \in D(\sigma_0, T)$.

For $x < n$, we similarly obtain

$$\frac{1}{2\pi i} \left(\int_{\infty+iT}^{c+iT} + \int_{c+iT}^{c-iT} + \int_{c-iT}^{\infty-iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = 0,$$

because there is no residue term. If we set

$$Q_n(c, x, T)(s) := \frac{1}{2\pi i} \left(\int_{c+iT}^{\infty+iT} + \int_{\infty-iT}^{c-iT} \right) \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w},$$

then we have

$$(10.4) \quad \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} = -Q_n(c, x, T)(s).$$

We also obtain

$$|Q_n(c, x, T)(s)| < \frac{M}{\pi T} \frac{(x/n)^c}{|\log(x/n)|}$$

for $s \in D(\sigma_0, T)$ by the same way as above.

From (10.3) and (10.4), it follows that

$$\begin{aligned}
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{f(s+w)}{g(s+w)} \frac{x^w}{w} dw &= \sum_{n < x} \frac{a_n}{n^s} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} \\
&\quad + \sum_{x < n} \frac{a_n}{n^s} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{g(s+w)} \left(\frac{x}{n}\right)^w \frac{dw}{w} \\
&= \sum_{n < x} \frac{a_n}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) \\
&\quad - \sum_{n < x} \frac{a_n}{n^s} R_n(c, x, T)(s) \\
&\quad - \sum_{x < n} \frac{a_n}{n^s} Q_n(c, x, T)(s).
\end{aligned}$$

Then we obtain (10.1). Furthermore, we can take $A > 0$ such that $|\log(x/n)| > A$ for any $n \in \mathbb{N}$, by the assumption of x . Hence, we also obtain (10.2). \square

11. PROOF OF THE RIEMANN HYPOTHESIS

Let

$$B := \sup\{\beta; \zeta(\beta + i\gamma) = 0, \gamma \neq 0\}.$$

The Riemann hypothesis states $B = \frac{1}{2}$. We recall the theorem of de la Vallée Poussin ([2]) which says that there is a constant $A > 0$ such that $\zeta(s)$ is not zero for

$$\sigma \geq 1 - \frac{A}{\log t} \quad (t > t_0),$$

where t_0 is some positive constant. We may restate it as follows: if $s = \sigma + it$ ($t > t_0$) satisfies

$$(11.1) \quad t \leq \exp\left(\frac{A}{1-\sigma}\right),$$

then $\zeta(s) \neq 0$.

Proposition 2. *It holds that $\frac{1}{2} \leq B < 1$.*

Proof. Since

$$\exp\left(\frac{A}{1-\sigma}\right) \longrightarrow \infty \quad \text{as} \quad \sigma \longrightarrow 1-0,$$

there exists $\sigma_0 < 1$ such that $T_0 < \exp\left(\frac{A}{1-\sigma}\right)$ for $\sigma_0 < \sigma < 1$, where T_0 is the constant in Theorem 1. Then, there is no zero of $\zeta(s)$ in a region $\sigma_0 < \sigma$ by Theorem 1 and the theorem of de la Vallée Poussin. \square

Proposition 3. *If $\frac{1}{2} < B < 1$, then there is no zero of $\zeta(s)$ on the line $\sigma = B$.*

Proof. If the function $\zeta(s)$ has a zero on $\sigma = B$, then the number of zeros of $\zeta(s)$ on $\sigma = B$ is finite by Theorem 1. Let $Z_B = \{\rho_1, \bar{\rho}_1, \dots, \rho_N, \bar{\rho}_N\}$ be the set of zeros of $\zeta(s)$ on the line $\sigma = B$. By Theorem 1, there exists $\delta_0 > 0$ such that there are no zeros of $\zeta(s)$ except Z_B in the set $\{\sigma + it; B - \delta_0 \leq \sigma, -\infty < t < \infty\}$.

It is well-known that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

for $\sigma > 1$. We define an entire function $g(s)$ by

$$g(s) := \prod_{j=1}^N (s - \rho_j)(s - \bar{\rho}_j).$$

Then, its zeros are Z_B , and $|g(s)| \rightarrow \infty$ as $s \rightarrow \infty$. We take $T > 0$ such that $|\operatorname{Im}(\rho_j)| < \frac{T}{4}$ for $j = 1, \dots, N$. Putting $\sigma_0 = B - \delta_0$ and $c = 2$, we apply Lemma 6. If $x > 1$ is not an integer, then we have

(11.2)

$$\begin{aligned} \sum_{n < x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \\ &+ \sum_{n < x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) \\ &+ \sum_{x < n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s) \end{aligned}$$

for $s \in D(\sigma_0, T) \setminus Z_B$, where $h_n(s)$ is a meromorphic function on \mathbb{C} whose poles are at most Z_B and all simple.

We consider the integral of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ along $C = C_0 + C_1 + C_2 + C_3$, where C_0, C_1, C_2 and C_3 are segments from $2 - iT$ to $2 + iT$, from $2 + iT$ to $-\frac{\delta_0}{2} + iT$, from $-\frac{\delta_0}{2} + iT$ to $-\frac{\delta_0}{2} - iT$ and from $-\frac{\delta_0}{2} - iT$ to $2 - iT$, respectively. We set

$$D_0 := \left\{ \sigma + it; B - \frac{1}{4}\delta_0 < \sigma < B + \frac{1}{4}\delta_0, |t| < \frac{T}{4} \right\}.$$

Then $D_0 \subset D(\sigma_0, T)$ and $Z_B \subset D_0$. For any $s \in D_0 \setminus Z_B$, the poles of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ in a domain surrounded by C are $w = 0, \rho_1 - s, \bar{\rho}_1 - s, \dots, \rho_N - s$ and $\bar{\rho}_N - s$. The residue of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ at $w = 0$ is $\frac{1}{\zeta(s)g(s)}$. Let a_j and b_j be the residues of $\frac{1}{\zeta(s)g(s)}$ at ρ_j and $\bar{\rho}_j$ respectively.

Then, the residues of $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ at $\rho_j - s$ and $\bar{\rho}_j - s$ are $a_j \frac{x^{\rho_j - s}}{\rho_j - s}$ and $b_j \frac{x^{\bar{\rho}_j - s}}{\bar{\rho}_j - s}$ respectively. We define

$$Q(s) := \sum_{j=1}^N \left(a_j \frac{x^{\rho_j - s}}{\rho_j - s} + b_j \frac{x^{\bar{\rho}_j - s}}{\bar{\rho}_j - s} \right).$$

Then, $Q(s)$ is a meromorphic function on \mathbb{C} whose poles are at most Z_B and all simple. We note that $\operatorname{Re}(s+w) > B - \frac{3}{4}\delta_0$ if $s \in D_0$ and w is on C . If $s \in D_0$ and w is on C_2 , then $\operatorname{Re}(s+w) < B - \frac{1}{4}\delta_0$. Then, $\frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w}$ is holomorphic on C as a function of w for any $s \in D_0$. By the residue theorem, we obtain

$$\frac{1}{2\pi i} \int_C \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw = \frac{1}{\zeta(s)g(s)} + Q(s)$$

for $s \in D_0 \setminus Z_B$. Therefore we have

$$(11.3) \quad \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw = \frac{1}{\zeta(s)g(s)} + Q(s) + P(s)$$

for $s \in D_0 \setminus Z_B$, where

$$P(s) = -\frac{1}{2\pi i} \int_{C_1+C_2+C_3} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw$$

is a holomorphic function on D_0 . We set

$$E := \left\{ \sigma + it; B - \frac{3}{4}\delta_0 \leq \sigma \leq B + 2 + \frac{1}{4}\delta_0, \frac{3}{4}T \leq |t| \leq \frac{5}{4}T \right\} \\ \cup \left\{ \sigma + it; B - \frac{3}{4}\delta_0 \leq \sigma \leq B - \frac{1}{4}\delta_0, |t| \leq \frac{5}{4}T \right\}.$$

Then we have $\{s+w; s \in D_0, w \in C_1 \cup C_2 \cup C_3\} \subset E$. Since there is no pole of $\frac{1}{\zeta(s)g(s)}$ on E , we can take $M_0 > 0$ such that

$$\left| \frac{1}{\zeta(s)g(s)} \right| < M_0$$

on E . Then we obtain

$$\left| \int_{C_1} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \right| < \frac{M_0 x^2}{T \log x}$$

and

$$\left| \int_{C_3} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \right| < \frac{M_0 x^2}{T \log x}$$

for $s \in D_0$. Since we have $w = -\frac{\delta_0}{2} + it$ and $x^w = e^{it \log x} x^{-\frac{\delta_0}{2}}$ on C_2 , we obtain

$$\left| \int_{C_2} \frac{1}{\zeta(s+w)g(s+w)} \frac{x^w}{w} dw \right| < \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}}$$

for $s \in D_0$. It follows from the above estimates that

$$(11.4) \quad |P(s)| < \frac{1}{2\pi} \left(\frac{2M_0 x^2}{T \log x} + \frac{4}{\delta_0} M_0 T x^{-\frac{\delta_0}{2}} \right)$$

on D_0 .

By (11.2) and (11.3), we obtain

$$(11.5) \quad \begin{aligned} \sum_{n < x} \frac{\mu(n)}{n^s} \left(\frac{1}{g(s)} + h_n(s) \right) &= \frac{1}{\zeta(s)g(s)} + Q(s) + P(s) \\ &+ \sum_{n < x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) \\ &+ \sum_{x < n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s) \end{aligned}$$

for $s \in D_0 \setminus Z_B$. We see that a function

$$P(s) + \sum_{n < x} \frac{\mu(n)}{n^s} R_n(2, x, T)(s) + \sum_{x < n} \frac{\mu(n)}{n^s} Q_n(2, x, T)(s)$$

is bounded on D_0 by the properties of $R_n(2, x, T)(s)$ and $Q_n(2, x, T)(s)$, and (11.4). The functions $\frac{1}{g(s)}$, $h_n(s)$ and $Q(s)$ are meromorphic functions on \mathbb{C} whose poles are at most Z_B and all simple. On the other hand, the function $\frac{1}{\zeta(s)g(s)}$ has poles of order at least 2 at every point in Z_B . This contradicts to the equation (11.5). Hence, there is no zero of $\zeta(s)$ on the line $\sigma = B$. \square

Proof of the Riemann hypothesis.

We may assume $\frac{1}{2} \leq B < 1$ by Proposition 2. Suppose that $\frac{1}{2} < B < 1$. Then there is no zero of $\zeta(s)$ on the line $\sigma = B$ by Proposition 3. Hence, we can take B' with $\frac{1}{2} < B' < B$ such that $\zeta(s) \neq 0$ for $B' < \sigma$ by Theorem 1. This contradicts to the definition of B . Thus we conclude $B = \frac{1}{2}$.

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