# Computation and combinatorial Techniques for Binomial Coefficients and Geometric Series

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**Abstract:** This paper presents a computing method for the sum of summation of geometric series and the summation of series of binomial expansions in an innovative way. Geometric Series plays a vital role in the field of combinatorics including binomial coefficients. The multiple summations of series of binomial coefficients or computation of multiple binomial expansions are equal to the exponents of two. These methodological advances are useful for the researchers who are working in science, engineering, economics, computation, and management.

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**Keywords:** binomial coefficient, geometric series, computation, summation

## 1. Introduction

In the earlier days, geometric series served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. The geometric series and its summations and sums have significant applications in science, engineering, economics, queuing theory, computation, and management. In this article, the sum of geometric series [1-11] whose terms are exponents of 2 is developed that is equal to the summation of series of binomial coefficients and binomial expansions. Also, this article discusses the general geometric series whose terms are multiple of any integer and/or any number.

Let $N$= {0, 1, 2, 3, . . . , } be the set of natural number including zero element. The elements of set $N$ are used in the geometric series.

### 1.1 Computation of Geometric Series and its Sum

In this section, computation of geometric series and its sum are developed without using the traditional computing method.

In general, if $x$ is an integer, then $x^n = \overbrace{x^{n-1} + x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1}}^{x\ times}$

$= (x-1)x^{n-1} + x^{n-1} = (x-1)\, x^{n-1} + \overbrace{x^{n-2} + x^{n-2} + x^{n-2} + x^{n-2} + \cdots + x^{n-2}}^{x\ times}$

$= (x-1)x^{n-1} + (x-1)x^{n-2} + x^{n-2}$. Similarly, we can develop the algebraic expression, $i.e., x^n = (x-1)x^{n-1} + (x-1)x^{n-2} + (x-1)x^{n-3} + (x-1)x^{n-3} + \cdots + (x-1)x^k + x^k$

$\Longrightarrow\ x^n = (x-1)\sum_{i=k}^{n-1} x^i + x^k \Longrightarrow \sum_{i=k}^{n} x^i = \frac{x^{n+1}-x^k}{x-1} \Longrightarrow \sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$,

where $k \le n$ and $k, n \in N$.

For example,

$3^n = 3^{n-1} + 3^{n-1} + 3^{n-1} = (3-1)3^{n-1} + 3^{n-1} = (3-1)3^{n-1} + (3-1)3^{n-2} + 3^{n-2}$

$$\Rightarrow 3^n = (3-1)3^{n-1} + (3-1)3^{n-2} + (3-1)3^{n-3} + \cdots + (3-1)3^k + 3^k$$
$$\Rightarrow 3^n = (3-1)\sum_{i=k}^{n-1} 3^k + \ 3^k \Rightarrow \sum_{i=k}^{n-1} 3^k = \frac{3^n - 3^k}{3-1} \Rightarrow \sum_{i=0}^{n-1} 3^k = \frac{3^n - 1}{2}.$$

If x is any number, then we can develop the geometric series as follows:
$$x^n = (\mathrm{x} - 1)\, x^{n-1} + x^{n-1} \Rightarrow (\mathrm{x}-1)x^{n-1} + (\mathrm{x}-1)x^{n-2} + \cdots + (\mathrm{x}-1)x^k + x^k$$
$$\Rightarrow x^n = (x-1)\sum_{i=k}^{n-1} x^i + x^k \Rightarrow \sum_{i=k}^{n-1} x^i = \frac{x^n - x^k}{x-1} \Rightarrow \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}.$$
For example,
$$(9.05)^n = (9.05-1)(9.05)^{n-1} + (9.05)^{n-1} \Rightarrow \sum_{i=k}^{n-1} (9.05)^i = \frac{(9.05)^n - (9.05)^k}{(9.05-1)}.$$

### 1.2 Geometric Series with exponents of 2

Let us develop the sum of geometric series [3] with exponents of 2 independently.
$$2^n = 2^{n-1} + 2^{n-1} = 2^{n-1} + 2^{n-2} + 2^{n-2} = \cdots = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots 2^k + 2^k$$
$$\Rightarrow 2^k + 2^{k+2} + 2^{k+3} + \cdots + 2^{n-1} = 2^n - 2^k \Rightarrow \sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k,$$

where $k \le n \ \ and \ \ k, n \in N$. In the geometric series $if \ \ k = 0$, then $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Next, let us develop a geometric series using the arithmetic equation 2 = 2.
$$2 = 1 + 1 = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^2} = \cdots = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n}$$
$$\Rightarrow \sum_{i=0}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n} \ \text{ and } \ \sum_{i=k}^{n} \frac{1}{2^i} = \frac{1}{2^{k+1}} - \frac{1}{2^n} = \frac{2^n - 2^{k+1}}{2^{n+k+1}}\ , (k \le n \ \& \ k, n \in N).$$

### 1.3 Binomial Coefficient

The factorial or factorial function [12-14] of a nonnegative integer *n*, denoted by *n*!, is the product of all positive integers less than or equal to *n*.

Let *N* = {0, 1, 2, 3, . . . .,} be the set of natural umbers including zero element.

A binomial coefficient is always an integer that denotes $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, where $n, r \ \in N$.

Here, $\binom{n+r}{r} = \frac{(n+r)}{r!\,n!} \Rightarrow (n+r) = l \times r!\,n!$, where $l$ is an integer.

## 2. Computation of Binomial Expansion

Here, a binomial expansion denotes a series of binomial coefficients. We know that a binomial coefficient has two independent variables. In the instructive section, binomial coefficient has been explained in more details.

For example, the following algebraic expression is a binomial expansion whose sum is equal to the exponent *n* of two [12].

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n.$$

**Theorem** : $\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-1.$

This binomial theorem states that the sum of multiple summations of series of binomial coefficients [12] is equal to the sum of a geometric series with exponents of 2 [1-5].

$Proof$ for this theorem: $\binom{0}{0}=\frac{0!}{0!}=1 \Rightarrow \sum_{i=0}^{0}\binom{0}{i}=2^0;\ \sum_{i=0}^{1}\binom{1}{i}=\binom{1}{0}+\binom{1}{1}=1+1=\ 2^1;$ $\sum_{i=0}^{2}\binom{2}{i}=\binom{2}{0}+\binom{2}{1}+\binom{2}{2}=1+2+1=\ 2^2;\ \sum_{i=0}^{3}\binom{3}{i}=\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=2^3;\ \cdots;$

$\text{Similarly, we can continue this process upto } n \text{ such that } \sum_{i=0}^{n}\binom{n}{i}=2^n.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=0}^{n}2^i,$$

$$\text{where } \sum_{i=0}^{n}2^i=\ \frac{2^{n+1}-1}{2-1}=2^{n+1}-1 \text{ is the geometric sereis with exponents of two.}$$

$$\therefore \sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-1.$$

Hence, theorem is proved.

**Lemma**: $\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\sum_{i=0}^{k+3}\binom{k+3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\ 2^{n+1}-2^k,$

where $k \leq n$ & $k, n\ \in N$.

*Proof* for this lemma: The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n}2^i\ =2^{n+1}-2^k.$$

$$\text{Then,} \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=k}^{n}2^i.$$

Therefore, $\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{k}.$

Some results of the lemma on binomial expansions are given below:

(i) $\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n}=2^{n}.$ $(ii)\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n-1}(2^{3}-1)=3(2^{n-1}).$

$(iii)\sum_{i=0}^{n-2}\binom{n-2}{i}+\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n-2}=\ 2^{n-2}(2^{3}-1)=7(2^{n-2}).$

$(iv)\sum_{i=0}^{n-3}\binom{n-3}{i}+\sum_{i=0}^{n-2}\binom{n-2}{i}+\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n-3}=15(2^{n-3}).$

These results can be generalized as follows:

$$\sum_{i=0}^{p}\binom{p}{i}+\sum_{i=0}^{p+1}\binom{p+1}{i}+\sum_{i=0}^{p+2}\binom{p+2}{i}+\cdots+\sum_{i=0}^{q-1}\binom{q-1}{i}+\sum_{i=0}^{q}\binom{q}{i}=2^{p}(2^{q-p+1}-1),$$

where $0\leq p\leq q$ *and* $p,q\in\ N$.

Some results of the theorem on binomial expansion are given below:

$(a)\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}=2^{p}-1,$ where $1\leq p\in N.$

$(b)\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{p-1}\binom{q-1}{i}=2^{q}-1,$ where $1\leq q\in N.$

By subtracting (a) from (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{q-1}\binom{q}{i}\right)-\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}\right)=2^{q}-2^{p},$$

$$i.e.,\sum_{i=0}^{p}\binom{p}{i}+\sum_{i=0}^{p+1}\binom{p+1}{i}+\sum_{i=0}^{p+2}\binom{p+2}{i}+\cdots+\sum_{i=0}^{q-2}\binom{q-2}{i}+\sum_{i=0}^{q-1}\binom{q-1}{i}=2^{q}-2^{p}.$$

## 3. Conclusion

In this article, theorem and lemma have been constituted based on the binomial expansions relating to the summation of geometric series where whose terms are exponents of 2. These combinatorial results can be useful for researchers working in science, engineering, management, and medicine [15].

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