Algorithmic and Numerical Techniques for Computation of Binomial and Geometric Series

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Abstract: Geometric series plays a vital role in the areas of combinatorics, science, economics, and medicine. This paper presents algorithmic and numerical techniques for computing the summation of multiple series of binomial coefficients and the multiple summations of geometric series in an innovative way and also the relation between the binomial expansions and geometric series. These are the methodological advances which are useful for researchers who are working in science, economics, engineering, computation, and management.

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1. Introduction

In the earlier days, geometric series served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. The geometric series and its summations and sums have significant applications in science, engineering, economics, queuing theory, computation, and management. In this article, innovative binomial expansions and geometric series [1-14] are introduced as methodological advances used in computational science. Computational science is a rapidly growing multi-and inter-disciplinary area where science, engineering, computation, mathematics, and collaboration use advance computing capabilities to understand and solve the most complex real life problems.

1.1 Computation of Geometric Series and its Sum

In this section, computation of geometric series and its sum [5-7] are developed without using the traditional computing method.

Let $N = \{0, 1, 2, 3, ...,\}$ be the set of natural umbers including zero element.

In general, if x is an integer, then $x^n = \overbrace{x^{n-1} + x^{n-1} + x^{n-1} + x^{n-1} + \cdots + x^{n-1}}^{x\ times}$

$= (\mathrm{x} - 1)x^{n-1} + x^{n-1} = (x-1)\,x^{n-1} + \overbrace{x^{n-2} + x^{n-2} + x^{n-2} + x^{n-2} + \cdots + x^{n-2}}^{x\ times}$

$= (\mathrm{x}-1)x^{n-1} + (\mathrm{x}-1)x^{n-2} + x^{n-2}$. Similarly, we can develop the algebraic expression,

$i.e., x^n = (\mathrm{x}-1)x^{n-1} + (\mathrm{x}-1)x^{n-2} + (\mathrm{x}-1)x^{n-3} + (\mathrm{x}-1)x^{n-3} + \cdots + (\mathrm{x}-1)x^k + x^k$

$$x^n = (x-1)\sum_{i=k}^{n-1} x^i + x^k \Rightarrow \sum_{i=k}^{n} x^i = \frac{x^{n+1} - x^k}{x-1} \Rightarrow \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1}, where\ k \le n.$$

For example,

$$3^n = 3^{n-1} + 3^{n-1} + 3^{n-1} = (3-1)3^{n-1} + 3^{n-1} = (3-1)3^{n-1} + (3-1)3^{n-2} + 3^{n-2}$$
$$\Rightarrow 3^n = (3-1)3^{n-1} + (3-1)3^{n-2} + (3-1)3^{n-3} + \cdots + (3-1)3^k + 3^k$$
$$\Rightarrow 3^n = (3-1)\sum_{i=k}^{n-1} 3^k + \ 3^k \Rightarrow \sum_{i=k}^{n-1} 3^k = \frac{3^n - 3^k}{3-1} \Rightarrow \sum_{i=0}^{n-1} 3^k = \frac{3^n - 1}{2}.$$

If x is any number, then we can develop the geometric series as follows:

$$x^n = (\mathrm{x} - 1)\, x^{n-1} + x^{n-1} \Rightarrow (\mathrm{x}-1)x^{n-1} + (\mathrm{x}-1)x^{n-2} + \cdots + (\mathrm{x}-1)x^k + x^k$$
$$\Rightarrow x^n = (x-1)\sum_{i=k}^{n-1} x^i + x^k \Rightarrow \sum_{i=k}^{n-1} x^i = \frac{x^n - x^k}{x-1} \Rightarrow \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x-1}.$$

For example,

$$(9.05)^n = (9.05-1)(9.05)^{n-1} + (9.05)^{n-1} \Rightarrow \sum_{i=k}^{n-1} (9.05)^i = \frac{(9.05)^n - (9.05)^k}{(9.05-1)}.$$

1.2 Geometric Series with exponents of 2

Let us develop the sum of geometric series [5] with exponents of 2 independently.

$$2^n = 2^{n-1} + 2^{n-1} = 2^{n-1} + 2^{n-2} + 2^{n-2} = \cdots = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots 2^k + 2^k$$
$$\Rightarrow 2^k + 2^{k+2} + 2^{k+3} + \cdots + 2^{n-1} = 2^n - 2^k \Rightarrow \sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k,$$

where $k \le n$ *and* $k, n \in N$. In the geometric series *if* $k = 0$, then $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Next, let us develop a geometric series using the arithmetic equation 2 = 2.

$$2 = 1 + 1 = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^2} = \cdots = 1 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n}$$
$$\Rightarrow \sum_{i=0}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n} \ \text{ and } \ \sum_{i=k}^{n} \frac{1}{2^i} = \frac{1}{2^{k+1}} - \frac{1}{2^n} = \frac{2^n - 2^{k+1}}{2^{n+k+1}}\ , (k \le n \ \&\ k, n \in N).$$

1.3 Traditional Binomial Coefficient

The factorial or factorial function [17-20] of a nonnegative integer *n*, denoted by *n*!, is the product of all positive integers less than or equal to *n*.

A binomial coefficient is always an integer that denotes $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, where $n, r \in N$.

Here, $\binom{n+r}{r} = \frac{(n+r)}{r!\,n!} \Rightarrow (n+r) = l \times r!\,n!$, where l is an integer.

2. Binomial and Geometric Series

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind for establishing a novel binomial series along with an innovative binomial coefficient 15, 16].

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \ \&\ \ V_r^n = \frac{(r+1)(r+2)(r+3)\cdots\cdots(r+n-1)(r+n)}{n!},$$

where $n \geq 1, r \geq 0$ *and* $n, r \in N$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ and V_r^n refer to the binomial sereis and binomial coefficient respectively.

Let us compare the binomial coefficient V_x^y with the traditional binomial coefficient as follows:

Let $z = x + y$. Then, $zC_x = \frac{z!}{x!\,y!}$. Here, $V_x^y = V_y^x \Longrightarrow zC_x = zC_y$, $\quad (x, y, z \in N)$.

For example,

$$V_3^5 = V_5^3 = (5+3)C_3 = (5+3)C_5 = 56.$$

Also, $V_n^0 = V_0^n = nC_0 = nC_n = \frac{n!}{n!\,0!} = 1$ *and* $V_0^0 = 0C_0 = \frac{0!}{0!} = 1 (\because 0! = 1)$.

2.1 Binomial Expansions equal to Multiple of 2

Let us develop some series of binomial coefficients or binomial expansions [15-16] which are equal to the multiple of 2 or exponents of 2 or both.

(1) $\sum_{i=0}^{n} V_i^{n-i} = 2^n$. (2) $\sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}$. (3) $\sum_{i=0}^{n} (i+1)V_i^{n-i} = (n+2)2^{n-1}$.

(4) $\sum_{i=0}^{n} (i-1)V_i^{n-i} = (n-2)2^{n-1}$, $V_r^n = \prod_{i=1}^{n} \frac{(r+i)}{n!}$, ($n \geq 1, r \geq 0$ & $n, r \in N$).

2.2 Relations between Binomial Expansions

Relation 1: $\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{i=0}^{n} (i-1)V_i^{n-i} = \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}$.

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 1) as follows:

$(i+1)V_i^{n-i} + (i-1)V_i^{n-i} = 2iV_i^{n-i}$. This idea can be applied to Relation 1.

$$\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{i=0}^{n} (i-1)V_i^{n-i} = 2\sum_{i=0}^{n} iV_i^{n-i} = (n+2)2^{n-1} + (n-2)2^{n-1} = 2n2^{n-1}.$$

$$\text{Then, } 2\sum_{i=0}^{n} iV_i^{n-i} = 2n2^{n-1} \Longrightarrow \sum_{i=0}^{n} iV_i^{n-i} = n2^{n-1}.$$

Relation 2: $\sum_{i=0}^{n} (i+1)V_i^{n-i} - \sum_{i=0}^{n} (i-1)V_i^{n-i} = \sum_{i=0}^{n} V_i^{n-i} = 2^n$.

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 2) as follows:

$(i+1)V_i^{n-i} - (i-1)V_i^{n-i} = 2V_i^{n-i}$. This idea can be applied to Relation 2.

$$\sum_{i=0}^{n} (i+1)V_i^{n-i} - \sum_{i=0}^{n} (i-1)V_i^{n-i} = 2\sum_{i=0}^{n} V_i^{n-i} = (n+2)2^{n-1} - (n-2)2^{n-1} = 4 \times 2^{n-1}.$$

$$\text{Then, } 2\sum_{i=0}^{n} V_i^{n-i} = 22^{\mathrm{n}} \Rightarrow \sum_{i=0}^{n} V_i^{n-i} = 2^{\mathrm{n}}.$$

Hence, two relations are proved.

2.3 Annamalai's Binomial Expansion

The following binomial expansions [15-18], named as Annamalai's binomial expansions, are derived from the Annamalai's (iii) binomial identity $\sum_{i=0}^{r} V_i^p = V_r^{p+1}$.

$$(1).\quad \sum_{i=0}^{n} \frac{(i+1)}{1!} = 1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2!}.$$

$$(2).\quad \sum_{i=0}^{n} \frac{(i+1)(i+2)}{2!} = 1+3+6+\cdots+\frac{(n+1)(n+2)}{2!} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

$$(3).\quad \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)}{3!} = \frac{(n+1)(n+2)(n+3)(n+4)}{4!}.$$

$$(4).\quad \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)(i+4)}{4!} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!}.$$

Similarly, this process continues up to *r* times. The *r*th binomial expansion is as follows:

$$(r).\quad \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)\cdots(i+r)}{r!} = \frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!}$$

$$i.e., \sum_{i=0}^{n} \prod_{j=1}^{r} \frac{i+j}{r!} = \prod_{i=1}^{r+1} \frac{n+i}{(r+1)!}.$$

This Annamalai's binomial expansion [15-17] is used to create the Annamalai's binomial series as follows.

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i=0}^{n} \prod_{j=1}^{r} \frac{i+j}{r!} x^i.$$

3. Binomial Expansion equal to the Sum of Geometric Series

Binomial expansion denotes a series of binomial coefficients. In this section, we focus on the summation of multiple binomial expansions or summation of multiple series of binomial coefficients.

$$\textbf{Theorem 3.1}: \sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 1.$$

This binomial theorem states that the sum of multiple summations of series of binomial coefficients [15-18] is equal to the sum of a geometric series with exponents of 2.

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step\ 0$: $\binom{0}{0} = \frac{0!}{0!} = 1 \Rightarrow \sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 2^0.$

$Step\ 1$: $\sum_{i=0}^{1} \binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1.$

$Step\ 2$: $\sum_{i=0}^{2} \binom{2}{i} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 = 2^2.$

$Step\ 4$: $\sum_{i=0}^{3} \binom{3}{i} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8.$

Similarly, we can continue the expressions up to "$step\ n$" such that $\sum_{i=0}^{n} \binom{n}{i} = 2^n.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} 2^i,$$

where $\sum_{i=0}^{n} 2^i = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1$ is the geometric sereis with exponents of two.

$$\therefore \sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 1.$$

Hence, theorem is proved.

Theorem 3.2: $\sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k+1}{i} + \sum_{i=0}^{k+2} \binom{k+2}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^k,$
where $k \leq n$ & $k, n \in N$.

Proof. The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k.$$

Then, $\sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k+1}{i} + \sum_{i=0}^{k+2} \binom{k+2}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=k}^{n} 2^i.$

Therefore, $\sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k+1}{i} + \sum_{i=0}^{k+2} \binom{k+2}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^k.$

Some results of Theorem 2.3 are given below:

(i) $\sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^n = 2^n.$ $(ii) \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n-1}(2^2 - 1) = 3(2^{n-1}).$

$$(iii) \sum_{i=0}^{n-2} \binom{n-2}{i} + \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^{n-2} = 2^{n-2}(2^3 - 1) = 7(2^{n-2}).$$

$$(iv) \sum_{i=0}^{n-3} \binom{n-3}{i} + \sum_{i=0}^{n-2} \binom{n-2}{i} + \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^{n-3} = 15(2^{n-3}).$$

These results can be generalized as follows:

$$\sum_{i=0}^{p} \binom{p}{i} + \sum_{i=0}^{p+1} \binom{p+1}{i} + \sum_{i=0}^{p+2} \binom{p+2}{i} + \cdots + \sum_{i=0}^{q-1} \binom{q-1}{i} + \sum_{i=0}^{q} \binom{q}{i} = 2^p(2^{q-p+1} - 1),$$

where $0 \le p \le q$ and $p, q \in N$.

Some results of Theorem 3.1 are given below:

$$(a) \sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{p-1} \binom{p-1}{i} = 2^p - 1, \text{where } 1 \le p \in N.$$

$$(b) \sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{p-1} \binom{q-1}{i} = 2^q - 1, \text{where } 1 \le q \in N.$$

By subtracting (a) from (b), we get

$$\left(\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \cdots + \sum_{i=0}^{q-1} \binom{q-1}{i} \right) - \left(\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \cdots + \sum_{i=0}^{p-1} \binom{p-1}{i} \right) = 2^q - 2^p,$$

$$i.e., \sum_{i=0}^{p} \binom{p}{i} + \sum_{i=0}^{p+1} \binom{p+1}{i} + \sum_{i=0}^{p+2} \binom{p+2}{i} + \cdots + \sum_{i=0}^{q-2} \binom{q-2}{i} + \sum_{i=0}^{q-1} \binom{q-1}{i} = 2^q - 2^p,$$

where p $< q$ & $p, q \in N$.

By adding (a) and (b), we get

$$\left(\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \cdots + \sum_{i=0}^{p-1} \binom{p-1}{i} \right) + \left(\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \cdots + \sum_{i=0}^{q-1} \binom{q-1}{i} \right) = 2^p + 2^q - 2,$$

$$If\ p = q, \qquad then\ 2\left(\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \cdots + \sum_{i=0}^{p-1} \binom{p-1}{i} \right) = 22^p - 2 = 2(2^p - 1),$$

$$i.e., \qquad \sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{p-1} \binom{p-1}{i} = 2^p - 1, \text{where } 1 \le q \in N.$$

4. Conclusion

This article presented the algorithmic and numerical techniques for computing the summation of multiple series of binomial coefficients and the multiple summations of geometric series in an innovative way and also introduced theorems and relations between the binomial expansions and geometric series. These techniques and its results can be useful for researchers who are working in science, economics, engineering, management, and medicine [21].

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