Slide Presentation on the Riemann Hypothesis

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Patrick Solé and Michel Planat stated a new criterion for the Riemann Hypothesis. We prove the Riemann Hypothesis is true using this criterion.

Keywords

Riemann Hypothesis, Prime numbers, Chebyshev function, Riemann zeta function.

Objectives

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. We provide a proof for the Riemann Hypothesis using the properties of the Chebyshev function.

Methods

Say Dedekinds(q_n) holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, q_n is the nth prime number and $\zeta(x)$ is the Riemann zeta function.

Theorem 1

Dedekinds(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true (Solé and Planat, 2011).*

Theorem 2

If the Riemann Hypothesis is false, then there are infinitely many prime numbers q_n for which $\mathsf{Dedekinds}(q_n)$ *does not hold.*

The Riemann Hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ (Solé and Planat, 2011):

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound (Solé and Planat, 2011):

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f is introduced in the Nicolas paper (Nicolas, 1983):

$$f(x) = e^{\gamma} \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Remark

When the Riemann Hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ (Nicolas, 1983).

According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0 \colon \log f(y) \geq k \times y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \times y^{-b} \times \sqrt{y}\right) \times \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \times y^{-b} \times \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$.

In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0 \colon \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann Hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$ (Solé and Planat, 2011).

In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. ■

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant (Mertens, 1874).

Theorem 3

We have that (Choie et al., 2007):

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) = \gamma - B = H.$$

The Basel Problem

Theorem 4

It is known that (Edwards, 2001):

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

Theorem 5

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - \log(1+\frac{1}{q_k})\right) = \log(\zeta(2)) - H.$$

We obtain that

$$\begin{aligned}
\log(\zeta(2)) - H &= \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \times (q_k + 1)}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1}) \right) - H
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) \right) - \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) - \log(\frac{q_k}{q_k - 1}) + \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right)
\end{aligned}
$$

and the proof is done. ■

A New Criterion

Theorem 6

Dedekinds(q_n) *holds if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log \log \theta(q_n)$$

is satisfied for the prime number q_n, where the set $S = \{x :\ x > q_n\}$ contains all the real numbers greater than q_n and χ_S is the characteristic function of the set S (This is the function defined by $\chi_S(x) = 1$ when $x \in S$ and $\chi_S(x) = 0$ otherwise).

When Dedekinds(q_n) holds, we apply the logarithm to the both sides of the inequality:

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log\log\theta(q_n)$$

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n).$$

Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

when Dedekinds(q_n) holds. The same happens in the reverse implication. ■

The Main Insight

Theorem 7

The Riemann Hypothesis is true if the inequality

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n.

The inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is also satisfied, where the set $S = \{x :\ x \geq q_n\}$ contains all the real numbers greater than or equal to q_n.

In the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

only change the values of

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes q_n and q_{n+1}. It is enough to show that

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n.

Indeed, the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k}-(\chi_{\{x:\ x\geq q_n\}}(q_k))\times\log(1+\frac{1}{q_k})\right)>B+\log\log\theta(q_n)$$

is the same as

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k}-(\chi_{\{x:\ x\geq q_{n+1}\}}(q_k))\times\log(1+\frac{1}{q_k})\right)$$

$$>B+\log\log\theta(q_{n+1})+\log(1+\frac{1}{q_n})+\log\log\theta(q_n)-\log\log\theta(q_{n+1})$$

where q_n and q_{n+1} are consecutive primes.

If the Riemann Hypothesis is false, then

$$\log(1 + \frac{1}{q_n}) + \log \log \theta(q_n) \geq \log \log \theta(q_{n+1})$$

must be violated for infinitely many n's, since Dedekinds(q_{n+1}) will not hold for infinitely many q_{n+1}'s. By contraposition, the Riemann Hypothesis should be true when the previous inequality is satisfied for all sufficiently large prime numbers q_n.

This is

$$\log\left((1+\frac{1}{q_n})\times\log\theta(q_n)\right)\geq\log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}}\geq\log\log\theta(q_{n+1}).$$

To sum up, the Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}}\geq\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. ■

Theorem 8

For $n \geq 2$ (Ghosh, 2019):

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times \left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right).$$

Theorem 9

For $n \geq 8602$ (Massias and Robin, 1996):

$$q_n \leq n \times (\log n + \log \log n - 0.9385).$$

The Main Theorem

Theorem 10

The Riemann Hypothesis is true.

The Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. That is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

after dividing the both sides of the inequality by $\theta(q_n)$.

We would only need to prove that

$$1+\frac{\log\theta(q_n)}{q_n}\geq 1+\frac{1}{n\times(1-\frac{1}{\log n}+\frac{\log\log n}{4\times\log^2 n})}$$

because of

$$\frac{\theta(q_n)}{\log q_{n+1}}\geq n\times(1-\frac{1}{\log n}+\frac{\log\log n}{4\times\log^2 n})$$

$$\theta(q_n)^{\frac{1}{q_n}}=e^{\frac{\log\theta(q_n)}{q_n}}\geq 1+\frac{\log\theta(q_n)}{q_n}$$

using the known results. That is equivalent to

$$\left(n\times(1-\frac{1}{\log n}+\frac{\log\log n}{4\times\log^2 n})\right)\times\log\theta(q_n)\geq q_n.$$

Therefore,

$$\left(n \times (1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n})\right) \times \log\theta(q_n) \geq n \times (\log n + \log\log n - 0.9385)$$

which is

$$\left(1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}\right) \times \log\theta(q_n) + 0.9385 \geq \log n + \log\log n$$

$$\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}} \times e^{0.9385} \geq n \times \log n$$

$$e^{0.9385} \geq \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}}}$$

under the assumption of the known results.

However, we know that

$$\overline{\lim_{n\to\infty}} \frac{n \times \log n}{\theta(q_n)^{1-\frac{1}{\log n}+\frac{\log\log n}{4\times\log^2 n}}} = \lim_{n\to\infty} \frac{n \times \log n}{\theta(q_n)^{1-\frac{1}{\log n}+\frac{\log\log n}{4\times\log^2 n}}} = 1$$

since

$$\lim_{n\to\infty} \left(1 - \frac{1}{\log n} + \frac{\log\log n}{4 \times \log^2 n}\right) = 1$$

$$\theta(q_n) \sim q_n, \quad (n \to \infty)$$

$$q_n \sim n \times \log n, \quad (n \to \infty).$$

For any positive real number ε, there exists a natural number m such that

$$\frac{n \times \log n}{\theta(q_n)^{1-\frac{1}{\log n}+\frac{\log \log n}{4 \times \log^2 n}}} < 1 + \varepsilon$$

for all $n > m$, because of the definition of limit superior. Moreover, we can see that $e^{0.9385} > 2.5561$. Consequently, it is enough to take any positive real number $\varepsilon \leq 1.5561$. By the definition of the limit superior yields the proof of the Riemann Hypothesis. ■

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