Computation Method for the Summation of Series of Binomial Expansions and Geometric Series with its Derivatives

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Abstract: Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical, combinatorial, and numerical equations for solving today's scientific problems and challenges. Combinatorics involves integers, factorials, binomial coefficients, binomial series, and numerical computation with geometric series for finding solutions to the problems in computing and computational science. This paper presents computation method for the summation of series of binomial expansions and geometric series with its derivatives in an innovative way and also theorems and relationship between the binomial expansions and geometric series. These computational approaches refer to the methodological advances which are useful for researchers who are working in computational science. Computational science is a rapidly growing multi-and inter-disciplinary area where science, engineering, computation, mathematics, and collaboration use advance computing capabilities to understand and solve the most complex real life problems.

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1. Introduction

In the earlier days, geometric series served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. Nowadays, the geometric series and its sums [1-18] have significant applications in science, engineering, economics, queuing theory, computation, combinatorics, and management.

1.1 Computation of Geometric Series with Powers of Two

Let us develop the sum of geometric series [5-7] with exponents of 2 independently.

$$2^n = 2^{n-1} + 2^{n-1} = 2^{n-1} + 2^{n-2} + 2^{n-2} = \cdots = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots 2^k + 2^k$$

$$\Rightarrow 2^k + 2^{k+2} + 2^{k+3} + \cdots + 2^{n-1} = 2^n - 2^k \Rightarrow \sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k,$$

where $k \leq n \ \ and \ \ k, n \in N$. In the geometric series $if \ \ k = 0$, then $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$

1.2 Traditional Binomial Coefficient

The factorial or factorial function [23, 24] of a nonnegative integer *n*, denoted by *n*!, is the product of all positive integers less than or equal to *n*.

A binomial coefficient is always an integer that denotes $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, where $n, r \in N$.

Here, $\binom{n+r}{r} = \frac{(n+r)}{r!\,n!} \Longrightarrow (n+r) = l \times r!\,n!,$ where l is an integer.

2. Binomial Expansions and Geometric Series with its Derivatives

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind for establishing a novel binomial series along with an innovative binomial coefficient [15-20].

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \;\&\; V_r^n = \frac{(r+1)(r+2)(r+3)\cdots\cdots(r+n-1)(r+n)}{n!},$$

where $n \geq 1, r \geq 0$ *and* $n, r \in N$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ and V_r^n refer to the binomial sereis and binomial coefficient respectively.

Let us compare the binomial coefficient V_x^y with the traditional binomial coefficient as follows:

Let $z = x + y$. Then, $zC_x = \frac{z!}{x!\,y!}$. Here, $V_x^y = V_y^x \Longrightarrow zC_x = zC_y, \qquad (x, y, z \in N)$.

For example,

$$V_3^5 = V_5^3 = (5+3)C_3 = (5+3)C_5 = 56.$$

Also, $V_n^0 = V_0^n = nC_0 = nC_n = \frac{n!}{n!\,0!} = 1$ *and* $V_0^0 = 0C_0 = \frac{0!}{0!} = 1(\because 0! = 1)$.

2.1 The n^th Derivative of Geometric Series

Differentiation is the derivative [20-22] of a function with respect to an independent variable. In this section, a geometric series is considered as the function of independent variable *x* as follows:

The function of geometric sereis is $f(x) = \sum_{i=0}^{r} x^i = 1 + x + x^2 + x^3 + \cdots + x^r = \frac{x^{r+1}-1}{x-1}$.

Let $N = \{0, 1, 2, 3, 4, 5, \cdots\}$ be the set of natural numbers including zero element.

For our convenience, the n^{th} derivative of $f(x)$ is defined as $\sum_{i=0}^{r-1} V_i^n x^i = f^n\left(\frac{x^{r+1}-1}{x-1}\right)$,

$$i.e., \qquad V_0^n + V_1^n x + V_2^n x^2 + V_3^n x^3 + \cdots + V_{r-1}^n x^{r-1} = f^n\left(\frac{x^{r+1}-1}{x-1}\right), (x \neq 1 \;\&\; n, r \in N),$$

$$Where\;\; V_r^n = \frac{(r+1)(r+2)(r+3)\cdots(r+n)}{n!}, \qquad (n \geq 1, r \geq 0 \;\&\; n, r \in N).$$

If n =1, then it is first derivative of the function $f(x)$, that is,

$$V_0^1 + V_1^1 x + V_2^1 x^2 + V_3^1 x^3 \cdots + V_{r-1}^1 x^{r-1} = f^1\left(\frac{x^{r+1}-1}{x-1}\right) = \frac{(rx-r-1)x^r+1}{(x-1)^2},$$

where f^1 denotes the first derivative of the function $f(x)$ of geometric series.

Now, $1 + 2x + 3x^2 + 4x^3 \cdots + rx^{r-1} = \frac{(rx-r-1)x^r+1}{(x-1)^2}, (x \neq 1)$.

Let $x = 2$. we get $1 + 2(2) + 3(2)^2 + 4(2)^3 \cdots + r2^{r-1} = \frac{(r-1)2^r+1}{(2-1)^2} = (r-1)2^r + 1$.

Let $x = 3.$ we get $1 + 2(3) + 3(3)^2 + 4(3)^3 \cdots + r3^{r-1} = \frac{(2r-1)3^r + 1}{(3-1)^2} = \frac{(2r-1)3^r + 1}{2^2}.$

Let $x = 4.$ we get $1 + 2(4) + 3(4)^2 + 4(4)^3 \cdots + r4^{r-1} = \frac{(3r-1)4^r + 1}{(4-1)^2} = \frac{(3r-1)4^r + 1}{3^2}.$

Similarly, by substitute the values of x in $\sum_{i=0}^{r-1} V_i^1 = \frac{(rx-r-1)x^r + 1}{(x-1)^2}$,

we can obtain more new results like above results for $x = 2, 3, \& 4.$

The second derivative of the function $f(x)$ is given below:

$V_0^2 + V_1^2 x + V_2^2 x^2 + V_3^2 x^3 \cdots + V_{r-1}^2 x^{r-1} = f^2\left(\frac{x^{r+1}-1}{x-1}\right)$. We can find likewise upto f^n ,

that is, $\sum_{i=0}^{r-1} V_i^n x^i = f^n\left(\frac{x^{r+1}-1}{x-1}\right)$ refers to the result of n^{th} derivative and

x is any number.

2.2 Binomial Expansions equal to Multiple of 2

Let us develop some series of binomial coefficients or binomial expansions [15-16] which are equal to the multiple of 2 or exponents of 2 or both.

(1) $\sum_{i=0}^{n} V_i^{n-i} = 2^n.$ (2) $\sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}.$ (3) $\sum_{i=0}^{n} (i+1)V_i^{n-i} = (n+2)2^{n-1}.$

(4) $\sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}} = (\mathrm{n}-2)2^{\mathrm{n}-1}$, $V_r^n = \prod_{i=1}^{n} \frac{(r+i)}{n!}$, $(n \geq 1, r \geq 0 \,\&\, n, r \in N).$

2.3 Relations between Binomial Expansion and Geometric Series

Relation 1: $\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}} = \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}.$

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 1) as follows:

$(i+1)V_i^{n-i} + (\mathrm{i}-1)\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}} = 2i\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}}.$ This idea can be applied to Relation 1.

$$\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}} = 2\sum_{i=0}^{n} iV_i^{n-i} = (n+2)2^{n-1} + (\mathrm{n}-2)2^{\mathrm{n}-1} = 2n2^{\mathrm{n}-1}.$$

$$\text{Then, } 2\sum_{i=0}^{n} iV_i^{n-i} = 2n2^{\mathrm{n}-1} \Rightarrow \sum_{i=0}^{n} iV_i^{n-i} = n2^{\mathrm{n}-1}.$$

Relation 2: $\sum_{i=0}^{n} (i+1)V_i^{n-i} - \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_{\mathrm{i}}^{\mathrm{n-i}} = \sum_{i=0}^{n} V_i^{n-i} = 2^n.$

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 2) as follows:

$(i+1)V_i^{n-i} - (\mathrm{i}-1)\mathrm{V_i^{n-i}} = 2\mathrm{V_i^{n-i}}$. This idea can be applied to Relation 2.

$$\sum_{i=0}^{n}(i+1)V_i^{n-i} - \sum_{\mathrm{i}=0}^{\mathrm{n}}(\mathrm{i}-1)\mathrm{V_i^{n-i}} = 2\sum_{i=0}^{n}V_i^{n-i} = (n+2)2^{n-1} - (\mathrm{n}-2)2^{\mathrm{n}-1} = 4 \times 2^{\mathrm{n}-1}.$$

$$\text{Then, } 2\sum_{i=0}^{n}V_i^{n-i} = 22^{\mathrm{n}} \Rightarrow \sum_{i=0}^{n}V_i^{n-i} = 2^{\mathrm{n}}.$$

Hence, two relations are proved.

2.4 Annamalai's Binomial Expansion

The following binomial expansions [15-20], named as Annamalai's binomial expansions, are derived from the Annamalai's (iii) binomial identity $\sum_{i=0}^{r} V_i^p = V_r^{p+1}$.

(1). $$\sum_{i=0}^{n}\frac{(i+1)}{1!} = 1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2!}.$$

(2). $$\sum_{i=0}^{n}\frac{(i+1)(i+2)}{2!} = 1+3+6+\cdots+\frac{(n+1)(n+2)}{2!} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

(3). $$\sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)}{3!} = \frac{(n+1)(n+2)(n+3)(n+4)}{4!}.$$

(4). $$\sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)(i+4)}{4!} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!}.$$

Similarly, this process continues up to *r* times. The r^{th} binomial expansion is as follows:

(r). $$\sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)\cdots(i+r)}{r!} = \frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!}$$

$$i.e., \sum_{i=0}^{n}\prod_{j=1}^{r}\frac{i+j}{r!} = \prod_{i=1}^{r+1}\frac{n+i}{(r+1)!}.$$

This Annamalai's binomial expansion [15-17] is used to create the Annamalai's binomial series as follows.

$$\sum_{i=0}^{n}V_i^r x^i = \sum_{i=0}^{n}\prod_{j=1}^{r}\frac{i+j}{r!}x^i.$$

The following theorem is derived from the Annamalai's binomial series [15].

Theorem 2.1: $\sum_{i=0}^{n}V_i^{r+1}x^i = \sum_{i=0}^{n}V_i^r x^i + \sum_{i=1}^{n}V_{i-1}^r x^i + \sum_{i=2}^{n}V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n}V_{i-n}^r x^i.$

Proof: Let's show that the computation of summations of the binomial series (right-hand side of the theorem) is equal to the binomial series (left- hand side of the theorem).

$$\sum_{i=0}^{n}V_i^r x^i + \sum_{i=1}^{n}V_{i-1}^r x^i + \sum_{i=2}^{n}V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n}V_{i-(n-1)}^r x^i + \sum_{i=n}^{n}V_{i-n}^r x^i$$

$$= (V_0^r + V_1^r x + V_2^r x^2 + V_3^r x^3 + \cdots + V_n^r x^n) + (V_0^r x + V_1^r x^2 + V_2^r x^3 + V_3^r x^4 + \cdots + V_{n-1}^r x^n)$$

$$+(V_0^r x^2 + V_1^r x^3 + V_2^r x^4 + V_3^r x^5 + \cdots + V_{n-2}^r x^n) + \cdots + (V_0^r x^{n-1} + V_1^r x^n) + V_0^r x^n$$

$$= V_0^r + (V_0^r + V_1^r)x + (V_0^r + V_1^r + V_2^r)x^2 + \cdots + (V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_n^r)x^n$$

(Note that $V_0^p + V_1^p + V_2^p + \cdots + V_r^p = V_r^{p+1}$ for $r = 1, 2, 3, \cdots$, and $V_0^p = V_0^{p+1} = 1$)

$$= V_0^{r+1} + V_1^{r+1}x + V_2^{r+1}x^2 + V_3^{r+1}x^3 + V_4^{r+1}x^4 + \cdots + V_{n-1}^{r+1}x^{n-1} + V_n^{r+1}x^n$$

$$= \sum_{i=0}^{n} V_i^{r+1} x^i$$

Hence, theorem is proved.

3. Binomial Expansion equal to the Sum of Geometric Series

Binomial expansion denotes a series of binomial coefficients. In this section, we focus on the summation of multiple binomial expansions or summation of multiple series of binomial coefficients.

Theorem 3.1: $\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 1.$

This binomial theorem states that the sum of multiple summations of series of binomial coefficients [15-19] is equal to the sum of a geometric series with exponents of 2.

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step\ 0$: $\binom{0}{0} = \frac{0!}{0!} = 1 \Rightarrow \sum_{i=0}^{0}\binom{0}{i} = \binom{0}{0} = 2^0.$

$Step\ 1$: $\sum_{i=0}^{1}\binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1.$

$Step\ 2$: $\sum_{i=0}^{2}\binom{2}{i} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 = 2^2.$

$Step\ 3$: $\sum_{i=0}^{3}\binom{3}{i} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=0}^{n}\binom{n}{i} = 2^n$.

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = \sum_{i=0}^{n} 2^i,$$

where $\sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1$ is the geometric sereis with exponents of two.

$$\therefore \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1}-1.$$

Hence, theorem is proved.

Some results of Theorem 3.1 are given below:

$$(a) \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i} = 2^p - 1, \text{where } 1 \le p \in N.$$

$$(b) \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{q-1}{i} = 2^q - 1, \text{where } 1 \le q \in N.$$

By subtracting (a) from (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{q-1}\binom{q-1}{i}\right) - \left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) = 2^q - 2^p,$$

$$i.e., \sum_{i=0}^{p}\binom{p}{i} + \sum_{i=0}^{p+1}\binom{p+1}{i} + \sum_{i=0}^{p+2}\binom{p+2}{i} + \cdots + \sum_{i=0}^{q-2}\binom{q-2}{i} + \sum_{i=0}^{q-1}\binom{q-1}{i} = 2^q - 2^p,$$

where p $< q$ & $p, q \in N$.

By adding (a) and (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) + \left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{q-1}\binom{q-1}{i}\right) = 2^p + 2^q - 2,$$

$$If\ p = q, \quad then\ 2\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) = 22^p - 2 = 2(2^p - 1),$$

$$i.e., \quad \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i} = 2^p - 1, \text{where } 1 \le q \in N.$$

Theorem 3.2: $\sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k+1}\binom{k+1}{i} + \sum_{i=0}^{k+2}\binom{k+2}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^k,$

where $k \le n$ & $k, n \in N$.

Proof. The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k.$$

$$\text{Then, } \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=k}^{n}2^{i}.$$

$$\therefore \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{k}.$$

Hence, theorem is proved.

Some results of Theorem 3.2 are given below:

(i) $\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n}=2^{n}.$ $(ii)\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n-1}(2^{2}-1)=3(2^{n-1}).$

$(iii)\sum_{i=0}^{n-2}\binom{n-2}{i}+\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n-2}= 2^{n-2}(2^{3}-1)=7(2^{n-2}).$

$(iv)\sum_{i=0}^{n-3}\binom{n-3}{i}+\sum_{i=0}^{n-2}\binom{n-2}{i}+\sum_{i=0}^{n-1}\binom{n-1}{i}+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-2^{n-3}=15(2^{n-3}).$

These results can be generalized as follows:

$$\sum_{i=0}^{p}\binom{p}{i}+\sum_{i=0}^{p+1}\binom{p+1}{i}+\sum_{i=0}^{p+2}\binom{p+2}{i}+\cdots+\sum_{i=0}^{q-1}\binom{q-1}{i}+\sum_{i=0}^{q}\binom{q}{i}=2^{p}(2^{q-p+1}-1),$$

where $0\le p\le q \text{ and } p,q\in N.$

Theorem 3.3: $\sum_{i=1}^{1} i\binom{1}{i}+\sum_{i=1}^{2} i\binom{2}{i}+\sum_{i=1}^{3} i\binom{3}{i}+\cdots+\sum_{i=1}^{n} i\binom{n}{i}=(n-1)2^{n}+1.$

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step\ 1$: $1\binom{1}{1}=\binom{1}{1}=\frac{1!}{1!\,0!}=1\Rightarrow\sum_{i=1}^{1} i\binom{1}{i}=1=1\times 2^{0},\qquad (0!=1).$

$Step\ 2$: $\sum_{i=1}^{2} i\binom{2}{i}=1\binom{2}{1}+2\binom{2}{2}=2+2=4=2\times 2^{1}.$

$Step\ 3$: $\sum_{i=1}^{3} i\binom{2}{i}=1\binom{3}{1}+2\binom{3}{2}+3\binom{3}{3}=3+6+3=12=2\times 2^{1}.$

$Step\ 4$: $\sum_{i=1}^{4} i\binom{2}{i}=1\binom{4}{1}+2\binom{4}{2}+3\binom{4}{3}+4\binom{4}{4}=4+12+12+4=4\times 2^{3}.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=1}^{n} i\binom{n}{i}=n2^{n-1}.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=1}^{n} i \times 2^{i-1}.$$

where $\sum_{i=1}^{n} i \times 2^{i} = (n-1)2^{n} + 1$. Its proof is expressed in Theorem 1.1.

$$\therefore \sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = (n-1)2^{n} + 1.$$

Hence, theorem is proved.

Some results of Theorem 3.3 are given below

$$\left\{\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k+1}{i} + \cdots + \sum_{i=1}^{n+1} i\binom{n+1}{i}\right\} - \left\{\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{k} i\binom{k}{i}\right\} = n2^{n+1} - k2^{k+1}$$

$$\Rightarrow \sum_{i=1}^{k+1} i\binom{k+1}{i} + \sum_{i=1}^{k+2} i\binom{k+2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=1}^{n+1} i\binom{n+1}{i} = 2(n2^{n} - k2^{k}) \text{ and}$$

$$\sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k+1}{i} + \cdots + \sum_{i=1}^{n-1} i\binom{n-1}{i} + \sum_{i=1}^{n} i\binom{n}{i} = 2\{(n-1)2^{n-1} - (k-1)2^{k-1}\},$$

where $k < n$ & $k, n \in N$.

4. Conclusion

In this article, the n^{th} derivative of geometric series has been introduced and its applications used in combinatorics including binomial expansions. Also, computation of the summation of series of binomial expansions and geometric series were derived in an innovative way. Theorems and relations between the binomial expansions and geometric series have been developed for researchers who are working in science, economics, engineering, management, and medicine [25].

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