On Some Properties of Determinants of Bicomplex Matrices

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Abstract

In this paper, we have studied Determinants of Bicomplex Matrices and investigated their properties. We have introduced Bicomplex Symmetric Matrix, Bicomplex Skew-Symmetric Matrix, Bicomplex idempotent matrix, Bicomplex Skew-idempotent matrix, Bicomplex Involutory matrix, Bicomplex skew-involutory matrix, three types of Hermetian and Skew-Hermetian Matrix, Bicomplex Orthogonal Matrix and three types of Unitary Matrix and investigated the properties of their determinants.

(**Keywords:** Bicomplex Matrix, Determinant of Bicomplex Matrix, Symmetric Matrix, Idempotent Matrix, Involutory Matrix, Hermetian Matrix, Orthogonal Matrix, Unitary Matrix.)

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1. Introduction

Throughout this paper, the set of Bicomplex numbers is denoted by C_2 and the sets of complex and real numbers are denoted by C_1 and C_0, respectively (cf.**[1]**,**[2]**,**[3]**).

The set of Bicomplex Numbers defined as:

$C_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0, i_1 \neq i_2 \text{ and } i_1^2 = i_2^2 = -1, i_1i_2 = i_2i_1\}$

We shall use the notations $C(i_1)$ and $C(i_2)$ for the following sets:

$C(i_1) = \{x + i_1y : x, y \in C_0\}$

$C(i_2) = \{x + i_2y : x, y \in C_0\}$

1.1 Hyperbolic Numbers:

The set of Hyperbolic Numbers is defined as $H = \{x + i_1i_2y : x, y \in C_0\}$

1.2 Idempotent Elements:

Besides 0 and 1, there are exactly two non – trivial idempotent elements exist in C_2, denoted as e_1 and e_2 and defined as $e_1 = \frac{1+i_1i_2}{2}$ and $e_2 = \frac{1-i_1i_2}{2}$. Note that $e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$.

1.3 Cartesian idempotent set [4]:

Cartesian idempotent set X determined by X_1 and X_2 is denoted as $X_1 \times_e X_2$ and is defined as

$X = X_1 \times_e X_2 = \{\xi \in C_2 : \xi = {}^1\xi\, e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in X_1 \times X_2\}$.

$C_2 = C(i_1) \times_e C(i_1) = C(i_1)e_1 + C(i_1)e_2 = \{\xi \in C_2 : \xi = {}^1\xi\, e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in C(i_1) \times C(i_1)\}$

$C_2 = C(i_2) \times_e C(i_2) = C(i_2)e_1 + C(i_2)e_2 = \{\xi \in C_2 : \xi = \xi_1\, e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in C(i_2) \times C(i_2)\}$.

1.4 Idempotent Representation of Bicomplex Numbers

(I) $C(i_1)$ –idempotent representation of Bicomplex Number

The $C(i_1)$–idempotent representation of Bicomplex Number is given by

$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$

$= z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi\, e_1 + {}^2\xi e_2$,

where ${}^1\xi = z_1 - i_1z_2 = (x_1 + x_4) + i_1(x_2 - x_3)$, ${}^2\xi = z_1 + i_1z_2 = (x_1 - x_4) + i_1(x_2 + x_3) \in C(i_1)$

(II) $C(i_2)$ –idempotent representation of Bicomplex Number

The $C(i_2)$ – idempotent representation of Bicomplex Number is given by

$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_2x_3) + i_1(x_2 + i_2x_4)$

$= w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$,

where $\xi_1 = w_1 - i_2w_2 = (x_1 + x_4) - i_2(x_2 - x_3)$, $\xi_2 = w_1 + i_2w_2 = (x_1 - x_4) + i_2(x_2 + x_3) \in C(i_2)$

1.5 Non-Singular(Invertible) and Singular(Non-invertible) Elements in C_2:

Let $\xi, \eta \in C_2$ s.t. $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ . An element which has inverse is said to be Non-Singular (Invertible), and an element which does not have an inverse is said to be Singular (Non-Invertible).

Non-zero singular elements exist in C_2. Set of all singular elements in C_2 is denoted as O_2. $C_2 \sim O_2$ is the set of all non-singular elements in C_2.

1.5.1 Singular(Non-invertible) Elements in C_2:

(i) The Bicomplex number

$\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2$ is singular if and only if

$z_1^2 + z_2^2 = 0$ or $(z_1 - i_1z_2 = 0$ or $z_1 + i_1z_2 = 0)$ or $({}^1\xi = 0$ or ${}^2\xi = 0)$

(ii) The Bicomplex number

$\xi = w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$ is singular if and only if

$w_1^2 + w_2^2 = 0$ or $(w_1 - i_2w_2 = 0$ or $w_1 + i_2w_2 = 0)$ or $(\xi_1 = 0$ or $\xi_2 = 0)$

1.5.2 Non-Singular(Invertible) Elements in C_2:

(i) The Bicomplex number

$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2$ is non-singular if and only if

$z_1^2 + z_2^2 \neq 0$ or $(z_1 - i_1 z_2 \neq 0$ and $z_1 + i_1 z_2 \neq 0)$ or $({}^1\xi \neq 0$ and ${}^2\xi \neq 0)$

(ii) The Bicomplex number

$\xi = w_1 + i_1 w_2 = (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$ is non-singular if and only if

$w_1^2 + w_2^2 \neq 0$ or $(w_1 - i_2 w_2 \neq 0$ and $w_1 + i_2 w_2 \neq 0)$ or $(\xi_1 \neq 0$ and $\xi_2 \neq 0)$

1.6 Principal Ideals in C_2:

The principal ideals $I_1$and I_2 defined as

$I_1 = \{\xi e_1 : \xi = {}^1\xi\, e_1 + {}^2\xi e_2 \in C_2\} = \{{}^1\xi\, e_1 : {}^1\xi \in C(i_1)\}$

$I_2 = \{\xi e_2 : \xi = {}^1\xi\, e_1 + {}^2\xi e_2 \in C_2\} = \{{}^2\xi e_2 : {}^2\xi \in C(i_1)\}$

Note that $I_1 \cap I_2 = \{0\}$ and $I_1 \cup I_2 = O_2$, set of all singular elements of C_2.

1.7 Zero –Divisors :

If $\xi \neq 0,\ \eta \neq 0 \in C_2$, and $\xi\eta = 0$, then ξ and η are called divisors of zero. The relation $e_1 e_2 = e_2 e_1 = 0$ establishes the existence of zero divisors in C_2. In fact, two Bicomplex numbers ξ and η are divisors of zero if and only if $\xi \in I_1 \sim \{0\}$ and $\eta \in I_2 \sim \{0\}$.

1.8 Norm in C_2

Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2 = {}^1\xi e_1 + {}^2\xi e_2 \in C_2$

The norm in C_2 is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2}$$

or

Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = w_1 + i_1 w_2 = \xi_1 e_1 + \xi_2 e_2 \in C_2$

The norm in C_2 is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \left[\frac{|\xi_1|^2 + |\xi_2|^2}{2}\right]^{1/2}$$

C_2 becomes a modified Banach algebra, in the sense that $\xi, \eta \in C_2$, $\|\xi\,\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$

1.9 Conjugations of Bicomplex Numbers [3]

- $\mathbf{i_1 -}$ **Conjugation**

$\xi^* = (z_1+i_2z_2)^* = \bar{z}_1 + i_2\bar{z}_2, \forall\, z_1, z_2 \in C(i_1)$

$\xi^* = \left({}^1\xi\, e_1 + {}^2\xi e_2\right)^* = \overline{{}^2\xi}\, e_1 + \overline{{}^1\xi}\, e_2 , \forall\, {}^1\xi, {}^2\xi \in C(i_1)$

$\xi^* = (w_1+i_1w_2)^* = w_1 - i_1w_2, \forall\, w_1, w_2 \in C(i_2)$

$\xi^* = \left(\xi_1 e_1 + \xi_2 e_2\right)^* = \xi_2 e_1 + \xi_1 e_2 , \forall\, \xi_1 , \xi_2 \in C(i_2)$

- $\mathbf{i_2 -}$ **Conjugation**

$\xi^\# = (z_1+i_2z_2)^\# = z_1 - i_2z_2, \forall\, z_1, z_2 \in C(i_1)$

$\xi^\# = \left({}^1\xi\, e_1 + {}^2\xi e_2\right)^\# = {}^2\xi\, e_1 + {}^1\xi\, e_2 , \forall\, {}^1\xi, {}^2\xi \in C(i_1)$

$\xi^\# = (w_1+i_1w_2)^\# = \bar{w}_1 + i_1\bar{w}_2, \forall\, w_1, w_2 \in C(i_2)$

$\xi^\# = \left(\xi_1 e_1 + \xi_2 e_2\right)^\# = \overline{\xi_2}\, e_1 + \overline{\xi_1} e_2 , \forall\, \xi_1 , \xi_2 \in C(i_2)$

- $\mathbf{i_1 i_2 -}$ **Conjugation**

$\xi' = (z_1+i_2z_2)' = \bar{z}_1 - i_2\bar{z}_2, \forall\, z_1, z_2 \in C(i_1)$

$\xi' = \left({}^1\xi\, e_1 + {}^2\xi e_2\right)' = \overline{{}^1\xi}\, e_1 + \overline{{}^2\xi}\, e_2 , \forall\, {}^1\xi, {}^2\xi \in C(i_1)$

$\xi' = (w_1+i_1w_2)' = \bar{w}_1 - i_1\bar{w}_2, \forall\, w_1, w_2 \in C(i_2)$

$\xi' = \left(\xi_1 e_1 + \xi_2 e_2\right)' = \overline{\xi_1}\, e_1 + \overline{\xi_2} e_2, \forall\, \xi_1 , \xi_2 \in C(i_2)$

2. Bicomplex Matrix ([5],[6],[7])

We denote $C_2^{m\times n} = \left\{A = \left[\xi_{ij}\right]_{m\times n} : \xi_{ij} \in C_2\right\}$ the set of $m \times n$ matrices with bicomplex entries.

Let $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n} \Rightarrow \xi_{ij} \in C_2$

2.1 Idempotent Representation of Bicomplex Matrix

(I) $\mathbf{C^{m\times n}(i_1)}$ –idempotent representation of Bicomplex Matrix

Let $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n}$

$A = \left[\xi_{ij}\right]_{m\times n} = \left[{}^1\xi_{ij}\right]_{m\times n} e_1 + \left[{}^2\xi_{ij}\right]_{m\times n} e_2 = {}^1A\, e_1 + {}^2A\, e_2$, where ${}^1A = \left[{}^1\xi_{ij}\right]_{m\times n}$, ${}^2A = \left[{}^2\xi_{ij}\right] \in C^{m\times n}(i_1)$

(II) $\mathbf{C^{m\times n}(i_2)}$ –idempotent representation of Bicomplex Matrix

Let $A = \left[\xi_{ij}\right]_{m\times n} \in C_2^{m\times n}$

$A = \left[\xi_{ij}\right]_{m\times n} = \left[\xi_{1,ij}\right]_{m\times n} e_1 + \left[\xi_{2,ij}\right]_{m\times n} e_2 = A_1\, e_1 + A_2\, e_2$, where $A_1 = \left[\xi_{1,ij}\right]_{m\times n}$, $A_2 = \left[\xi_{2,ij}\right]_{m\times n} \in C^{m\times n}(i_2)$

2.2 Determinant of Bicomplex Matrices

As, only square matrices can have determinant

So let, $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$

Then determinant of A is denoted as det(A).

Theorem2.1: Let $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$, then its determinant is given by

$\det(A) = \det({}^1A)\, e_1 + \det({}^2A)\, e_2$ or $\det(A) = \det(A_1)\, e_1 + \det(A_2)\, e_2$.

Corollary 2.1:

For $A, B \in C_2^{n\times n}$, $\det(AB) = \det(A)\det(B)$.

2.3 Algebraic Structure of Set of Bicomplex Matrices:

$C_2^{m\times n}(C_0)$, $C_2^{m\times n}(C(i_1))$ and $C_2^{m\times n}(C(i_2))$ are Linear Space; $C_2^{m\times n}(C_2)$ is a C_2 −Module;
$C_2^{n\times n}(C_0)$, $C_2^{n\times n}(C(i_1))$ and $C_2^{n\times n}(C(i_2))$ are Algebra.

Theorem 2.2:

$A = {}^1A\, e_1 + {}^2A\, e_2 = A_1 e_1 + A_2 e_2 \in C_2^{n\times n}$ is invertible iff ${}^1A$ and ${}^2A$ are invertible in $C^{n\times n}(i_1)$ and A_1 and A_2 are invertible in $C^{n\times n}(i_2)$.

Corollary 2.2:

$A \in C_2^{n\times n}$ is invertible iff det (A) $\notin O_2$.

2.4 Non-Singular (Invertible) and Singular (Non-invertible) Bicomplex Matrix

A matrix $A = \left[\xi_{ij}\right]_{n\times n} \in C_2^{n\times n}$ is said to be Non-Singular (Invertible) if $\det(A) \notin O_2$. It is said to be Singular (Non-invertible) matrix if $\det(A) \in O_2$.

2.5 Conjugation of Bicomplex Matrix

- **$i_1 -$ Conjugation**

$A^* = ({}^1A\, e_1 + {}^2A\, e_2)^* = \overline{{}^2A}\, e_1 + \overline{{}^1A}\; e_2\,, \forall\; {}^1A = \left[{}^1\xi_{ij}\right]_{m\times n}, {}^2A = \left[{}^2\xi_{ij}\right] \in C^{m\times n}(i_1)$

$A^* = (\,A_1\, e_1 + \;A_2\, e_2)^* = \;A_2\, e_1 + A_1\; e_2\,, \forall\; A_1 = \left[\,\xi_{1,ij}\right]_{m\times n}, A_2 = \left[\,\xi_{2,ij}\right]_{m\times n} \;\in C^{m\times n}(i_2)$

- **$i_2 -$ Conjugation**

$A^{\#} = ({}^1A\, e_1 + {}^2A\, e_2)^{\#} = {}^2A\, e_1 + {}^1A\, e_2, \forall\; {}^1A = \left[{}^1\xi_{ij}\right]_{m\times n}, {}^2A = \left[{}^2\xi_{ij}\right] \in C^{m\times n}(i_1)$

$A^{\#} = (\,A_1\, e_1 + \;A_2\, e_2)^{\#} = \;\overline{A}_2\, e_1 + \overline{A}_1\, e_2\,, \forall\, A_1 = \left[\,\xi_{1,ij}\right]_{m\times n}, A_2 = \left[\,\xi_{2,ij}\right]_{m\times n} \;\in C^{m\times n}(i_2)$

- **$i_1 i_2 -$ Conjugation**

$A' = ({}^1A\, e_1 + {}^2A\, e_2)' = \overline{{}^1A}\; e_1 + \overline{{}^2A}\, e_2\,, \forall\; {}^1A = \left[{}^1\xi_{ij}\right]_{m\times n}, {}^2A = \left[{}^2\xi_{ij}\right] \in C^{m\times n}(i_1)$

$A' = (\,A_1\, e_1 + \;A_2\, e_2)' = \;\overline{A}_1\, e_1 + \overline{A}_2\, e_2\,, \forall\, A_1 = \left[\,\xi_{1,ij}\right]_{m\times n}, A_2 = \left[\,\xi_{2,ij}\right]_{m\times n} \;\in C^{m\times n}(i_2)$

2.6 Adjoint of a Bicomplex Square Matrix

Let $A = {}^1A\, e_1 + {}^2A\, e_2 \in C_2^{n\times n}$, where ${}^1A\,, {}^2A \in C^{n\times n}(i_1)$. Then adjoint of the bicomplex matrix A is given by

$\mathrm{adj}(A) = \mathrm{adj}({}^1A)e_1 + \mathrm{adj}({}^2A)e_2.$

Let $A = \;A_1 e_1 + \;A_2 e_2 \in C_2^{n\times n}$, where $\;A_1, A_2 \in C^{n\times n}(i_2)$ .Then adjoint of the bicomplex matrix A is given by

$\mathrm{adj}(A) = \mathrm{adj}(A_1)e_1 + \mathrm{adj}(A_2)e_2.$

Theorem 2.3: $A \in C_2^{n\times n}$, then $A\mathrm{adj}(A) = \mathrm{adj}(A)A = \det(A)I$

Theorem 2.4: Inverse of a Bicomplex Square Matrix

Let $A \in C_2^{n\times n}$ and $\det(A) \notin O_2$. Then $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Theorem 2.5: Let $A \in C_2^{n\times n}$ and $\det(A) \notin O_2$. Then $A^{-1} = \frac{1}{\det({}^1A)}\mathrm{adj}({}^1A)e_1 + \frac{1}{\det({}^2A)}\mathrm{adj}({}^2A)e_2$ or $A^{-1} = \frac{1}{\det(A_1)}\mathrm{adj}(A_1)e_1 + \frac{1}{\det(A_2)}\mathrm{adj}(A_2)e_2$.

Theorem 2.6: Let $A \in C_2^{n\times n}$ and $\det(A) \notin O_2$. Then $A^{-1} = ({}^1A)^{-1}e_1 + ({}^2A)^{-1}e_2$
or $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$.

Theorem 2.7: A matrix $A \in C_2^{n\times n}$ is invertible then following holds

(iii) $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$

(iv) $A^{-1} = \frac{1}{\det({}^1A)}\operatorname{adj}({}^1A)e_1 + \frac{1}{\det({}^2A)}\operatorname{adj}({}^2A)e_2$

(v) $A^{-1} = \frac{1}{\det(A_1)}\operatorname{adj}(A_1)e_1 + \frac{1}{\det(A_2)}\operatorname{adj}(A_2)e_2$

(vi) $A^{-1} = ({}^1A)^{-1}e_1 + ({}^2A)^{-1}e_2$

(vii) $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$

(viii) $\det(A) \notin O_2$

(ix) $\det({}^1A) \neq 0$ and $\det({}^2A) \neq 0$

(x) $\det(A_1) \neq 0$ and $\det(A_2) \neq 0$

(xi) $\det(A^{-1}) = \frac{1}{\det(A)}$

(xii) $\det(A^{-1}) = \frac{1}{\det({}^1A)}\ e_1 + \frac{1}{\det({}^2A)}\ e_2$

(xiii) $\det(A^{-1}) = \frac{1}{\det(A_1)}\ e_1 + \frac{1}{\det(A_2)}\ e_2$

Corollary 2.3: A matrix $A \in C_2^{n\times n}$, then following statements are equivalent

(i) A is singular

(ii) $\det(A) \in O_2$

(iii) $\det({}^1A) = 0$ or $\det({}^2A) = 0$

(iv) $\det(A_1) = 0$ or $\det(A_2) = 0$

3. Determinant of Some Special Bicomplex Matrices:

Symmetric Matrix 3.1:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be symmetric matrix if $A^t = A$.

Skew-Symmetric Matrix 3.2:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew-Symmetric matrix if $A^t = -A$.

Bicomplex Idempotent Matrix 3.3:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be idempotent matrix if $A^2 = A$.

Bicomplex Skew-Idempotent Matrix 3.4:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew-idempotent matrix if $A^2 = -A$.

Bicomplex Involutory Matrix 3.5:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be involutory matrix if $A^2 = I$.

Bicomplex Skew-Involutory Matrix 3.6:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be Skew- involutory matrix if $A^2 = -I$.

Bicomplex Hermitian Matrix 3.7:

- $\mathbf{i_1 -}$ **Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 -$ Hermitian Matrix if

$$(A^t)^* = (A^*)^t = A$$

Note 3.1: If A is $i_1 -$ Hermitian Matrix then $A^* = A^t$

- $\mathbf{i_2 -}$ **Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 -$ Hermitian Matrix if

$$(A^t)^\# = \left(A^\#\right)^t = A$$

Note 3.2: If A is $i_2 -$ Hermitian Matrix then $A^\# = A^t$

- $\mathbf{i_1\, i_2 -}$ **Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\, i_2 -$ Hermitian Matrix if

$$(A^t)' = (A')^t = A$$

Note 3.3: If A is $i_1\, i_2 -$ Hermitian Matrix then $A' = A^t$

Bicomplex Skew- Hermitian Matrix 3.8:

- $\mathbf{i_1 -}$ **Skew $-$ Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 -$ Skew $-$ Hermitian Matrix if

$$(A^t)^* = (A^*)^t = -A$$

Note 3.4: If A is $i_1 -$ Skew $-$ Hermitian Matrix then $A^* = -A^t$

- $\mathbf{i_2 -}$ **Skew $-$ Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 -$ Skew $-$ Hermitian Matrix if

$$(A^t)^\# = \left(A^\#\right)^t = -A$$

Note 3.5: If A is $i_2 -$ Skew $-$ Hermitian Matrix then $A^\# = -A^t$

- $\mathbf{i_1\, i_2 -}$ **Skew $-$ Hermitian Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\, i_2 -$ Skew $-$ Hermitian Matrix if

$$(A^t)' = (A')^t = -A$$

Note 3.6: If A is $i_1\, i_2 -$ Skew $-$ Hermitian Matrix then $A' = -A^t$

Bicomplex Orthogonal Matrix 3.9:
A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be orthogonal matrix if

$$A^tA = AA^t = I$$

Bicomplex Unitary Matrix 3.10:

- **$i_1 -$ Unitary Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1 -$ Unitary Matrix if

$$A^{*t}A = A^{*t}A = I$$

- **$i_2 -$ Unitary Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_2 -$ Unitary Matrix if

$$A^{\#t}A = A^{\#t}A = I$$

- **$i_1\ i_2 -$ Unitary Matrix**

A Bicomplex matrix $A \in C_2^{n\times n}$ is said to be $i_1\ i_2 -$ Unitary Matrix if

$$A'^{t}A = A'^{t}A = I$$

Proposition 3.1: Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-symmetric matrix of odd order. Then $\det(A) = 0$.

Proof: Let $A \in C_2^{n\times n}$ be a Skew-symmetric matrix of odd order

$\Rightarrow A^t = -A$

$\Rightarrow \det(A^t) = \det(-A)$

$\Rightarrow \det(A) = (-1)^n \det(A)$

$\Rightarrow \det(A) = -\det(A)$ $\quad (\because (-1)^n = -1, n \text{ is odd})$

$\Rightarrow 2\det(A) = 0$

$\Rightarrow \det(A) = 0$

Proposition 3.2: Let $A \in C_2^{n\times n}$ be a Bicomplex idempotent matrix. Then $\det(A) \in \{0,1,e_1,e_2\}$.

Proof: Let $A \in C_2^{n\times n}$ be an idempotent matrix

$\Rightarrow A^2 = A$

$\Rightarrow \det(A^2) = \det(A)$

$\Rightarrow \det(AA) = \det(A)$

$\Rightarrow \det(A)\det(A) = \det(A)$

$\Rightarrow \{\det(A)\}^2 = \det(A)$

$\Rightarrow \{\det(^1A)\, e_1 + \det(^2A)\, e_2\}^2 = \det(^1A)\, e_1 + \det(^2A)\, e_2$

$\Rightarrow \{\det(^1A)\}^2 e_1 + \{\det(^2A)\}^2 e_2 = \det(^1A)\, e_1 + \det(^2A)\, e_2$

$\Rightarrow \{\det(^1A)\}^2 = \det(^1A)$ and $\{\det(^2A)\}^2 = \det(^2A)$

$\Rightarrow \det(^1A) = 0,1$ and $\det(^2A) = 0,1$

Hence $\det(A) \in \{0,1,e_1,e_2\}$.

Note 3.7: If A is a non-singular Bicomplex idempotent matrix then $A = I$.

Proposition 3.3: Let $A \in C_2^{n\times n}$ be a Bicomplex skew-idempotent matrix. Then $\det(A) \in \{0,1,e_1,e_2\}$ when n is even and $\det(A) \in \{0,-1,-e_1,-e_2\}$ when n is odd.

Proof: Let $A \in C_2^{n\times n}$ be a Bicomplex skew-idempotent matrix

$\Rightarrow A^2 = -A$

$\Rightarrow \det(A^2) = \det(-A)$

$\Rightarrow \det(AA) = (-1)^n \det(A)$

$\Rightarrow \det(A)\det(A) = (-1)^n \det(A)$

$\Rightarrow \{\det(A)\}^2 = (-1)^n \det(A)$

Case(i) When n is even

$\Rightarrow \{\det(A)\}^2 = \det(A)$ $\qquad (\because (-1)^n = 1, n \text{ is even})$

The result directly follows **Proposition 3.2**

Case(ii) When n is odd

$\Rightarrow \{\det(A)\}^2 = -\det(A)$ $\qquad (\because (-1)^n = -1, n \text{ is odd})$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -\det({}^1A)\, e_1 - \det({}^2A)\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = -\det({}^1A)$ and $\{\det({}^2A)\}^2 = -\det({}^2A)$

$\Rightarrow \det({}^1A) = 0, -1$ and $\det({}^2A) = 0, -1$

Hence $\det(A) \in \{0,-1,-e_1,-e_2\}$.

Note 3.8: If A is a non-singular Bicomplex Skew-idempotent matrix then $A = -I$.

Proposition 3.4: Let $A \in C_2^{n\times n}$ such that $A^2 = \eta A, \eta \in C_2$. Then $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$.

Proof: Let $A \in C_2^{n\times n}$ such that $A^2 = \eta A, \eta \in C_2$

$\Rightarrow A^2 = \eta A$

$\Rightarrow \det(A^2) = \det(\eta A)$

$\Rightarrow \det(AA) = (\eta)^n \det(A)$

$\Rightarrow \det(A)\det(A) = (\eta)^n \det(A)$

$\Rightarrow \{\det(A)\}^2 = (\eta)^n \det(A)$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = \{({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2\}\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = ({}^1\eta)^n \det({}^1A)\, e_1 + ({}^2\eta)^n \det({}^2A)\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n \det({}^1A)$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n \det({}^2A)$

$\Rightarrow \det({}^1A) = 0, ({}^1\eta)^n$ and $\det({}^2A) = 0, ({}^2\eta)^n$

Hence $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$.

Note 3.9: **(i)** For $\eta = 1$, we get **Proposition 3.2**

(ii) For $\eta = -1$, we get **Proposition 3.3**

(iii) If A is non-singular, then $A = \eta I$

Proposition 3.5: Let $A \in C_2^{n\times n}$ be a Bicomplex involutory matrix. Then $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Proof: Let A be any Bicomplex involutory matrix

$\Rightarrow A^2 = I$

$\Rightarrow \det(A^2) = \det(I)$

$\Rightarrow \det(AA) = 1$

$\Rightarrow \det(A)\det(A) = 1$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$

$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$

Hence $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Note 3.10: Bicomplex involutory matrices are always non-singular and $A^{-1} = A$.

Proposition 3.6: Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-involutory matrix. Then $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$ when n is even and $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$ when n is odd.

Proof: Let $A \in C_2^{n\times n}$ be a Bicomplex Skew-involutory matrix

$\Rightarrow A^2 = -I$

$\Rightarrow \det(A^2) = \det(-I)$

$\Rightarrow \det(AA) = (-1)^n$

$\Rightarrow \det(A)\det(A) = (-1)^n$

$\Rightarrow \{\det(A)\}^2 = (-1)^n$

Case(i) When n is even

$\Rightarrow \{\det(A)\}^2 = 1 \qquad (\because (-1)^n = 1, \text{n is even})$

The result directly follows **Proposition 3.5**

Case(ii) When n is odd

$\Rightarrow \{\det(A)\}^2 = -1 \qquad (\because (-1)^n = -1, \text{n is odd})$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -1e_1 - 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -1e_1 - 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = -1$ and $\{\det({}^2A)\}^2 = -1$

$\Rightarrow \det({}^1A) = i_1, -i_1$ and $\det({}^2A) = i_1, -i_1$

Hence $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$.

Note 3.11: Bicomplex skew-involutory matrices are always non-singular and $A^{-1} = -A$.

Proposition 3.7: Let $A \in C_2^{n\times n}$ such that $A^2 = \eta I, \eta \in C_2$. Then

$$\det(A) \in \left\{ \begin{matrix} \sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, \\ \sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2 \end{matrix} \right\}.$$

Proof: Let $A \in C_2^{n\times n}$ be any Bicomplex matrix such that $A^2 = \eta I$

$\Rightarrow A^2 = \eta I$

$\Rightarrow \det(A^2) = \det(\eta I)$

$\Rightarrow \det(AA) = \eta^n \det(I)$

$\Rightarrow \det(A)\det(A) = \eta^n$

$\Rightarrow \{\det(A)\}^2 = \eta^n$

$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = ({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = ({}^1\eta)^n\, e_1 + ({}^2\eta)^n\, e_2$

$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n$

$\Rightarrow \det({}^1A) = \sqrt{({}^1\eta)^n}, -\sqrt{({}^1\eta)^n}$ and $\det({}^2A) = \sqrt{({}^2\eta)^n}, -\sqrt{({}^2\eta)^n}$

Hence

$$\det(A) \in \left\{ \begin{matrix} \sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, \\ \sqrt{({}^1\eta)^n}\, e_1 - \sqrt{({}^2\eta)^n}\, e_2, -\sqrt{({}^1\eta)^n}\, e_1 + \sqrt{({}^2\eta)^n}\, e_2 \end{matrix} \right\}.$$

Note 3.12: **(i)** For $\eta = 1$, we get **Proposition 3.5**

(ii) For $\eta = -1$, we get **Proposition 3.6**

Proposition 3.8: Let $A \in C_2^{n\times n}$ be a i_1 $-$Hermitian matrix. Then

(i) $\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

(ii) $\det(A) = \det(A_1) = \det(A_2)$

(iii) $\det(A) \in C(i_2)$

Proof: **(i)** Let $A \in C_2^{n\times n}$ be a i_1 $-$Hermitian matrix

$\Rightarrow A^* = A^t$

$\Rightarrow \det(A^*) = \det(A^t)$

$\Rightarrow \det(A^*) = \det(A)$ $\qquad\qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = \det(A)$ $\qquad (\because \det(A^*) = [\det(A)]^*)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = \det({}^1A)e_1 + \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = \det({}^1A)e_1 + \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ and $\overline{\det({}^1A)} = \det({}^2A)$

$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$

(ii) Let $A \in C_2^{n\times n}$ be a i_1 −Hermitian matrix

$\Rightarrow A^* = A^t$

$\Rightarrow \det(A^*) = \det(A^t)$

$\Rightarrow \det(A^*) = \det(A)$ $\qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = \det(A)$ $\qquad (\because \det(A^*) = [\det(A)]^*)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_1)e_2 + \det(A_1)e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2) = \det(A_1)$ and $\det(A_1) = \det(A_2)$

$\Rightarrow \det(A_2) = \det(A_1)$

$\Rightarrow \det(A) = \det(A_1) = \det(A_2)$

(iii) From **(ii)**

$\det(A) = \det(A_1) = \det(A_2)$

Also, $\det(A_1) = \det(A_2) \in C(i_2)$

$\Rightarrow \det(A) \in C(i_2)$

Proposition 3.9: Let $A \in C_2^{n\times n}$ be a i_2 −Hermitian matrix. Then

(i) $\det(A) = \det({}^1A) = \det({}^2A)$

(ii) $\overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iii) $\det(A) \in C(i_1)$

Proof:

(i) Let $A \in C_2^{n\times n}$ be a i_2 −Hermitian matrix

$\Rightarrow A^\# = A^t$

$\Rightarrow \det(A^\#) = \det(A^t)$

$\Rightarrow \det(A^\#) = \det(A)$ $\qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^\# = \det(A)$ $\qquad (\because \det(A^\#) = [\det(A)]^\#)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^\# = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^1A)e_2 + \det({}^2A)e_1 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^2A)e_1 + \det({}^1A)e_2 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \det({}^2A) = \det({}^1A)$ and $\det({}^1A) = \det({}^2A)$

$\Rightarrow \det({}^2A) = \det({}^1A)$

$\Rightarrow \det(A) = \det({}^1A) = \det({}^2A)$

(ii) Let $A \in C_2^{n\times n}$ be a i_2 −Hermitian matrix

$\Rightarrow A^{\#} = A^t$

$\Rightarrow \det(A^{\#}) = \det(A^t)$

$\Rightarrow \det(A^{\#}) = \det(A)$ $\quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^{\#} = \det(A)$ $\quad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#} = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ and $\overline{\det(A_1)} = \det(A_2)$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iii) From **(i)**

$\det(A) = \det({}^1A) = \det({}^2A)$

Also $\det({}^1A) = \det({}^2A) \in C(i_1)$

$\Rightarrow \det(A) \in C(i_1)$

Proposition 3.10: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ −Hermitian matrix. Then $\det(A) \in H$ or equivalently $\det(A) = \overline{\det(A_1)}\, e_1 + \overline{\det(A_2)}\, e_2$

Proof: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ −Hermitian matrix

$\Rightarrow A' = A^t$

$\Rightarrow \det(A') = \det(A^t)$

$\Rightarrow \det(A') = \det(A)$ $\quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = \det(A)$ $\quad (\because \det(A') = [\det(A)]')$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]' = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = \det({}^1A)$ and $\overline{\det({}^2A)} = \det({}^2A)$

$\Rightarrow \det({}^1A)$ is a real and $\det({}^2A)$ is a real

$\Rightarrow \det(A) \in H$

Now,

$\overline{\det({}^1A)} = \det({}^1A)$ and $\overline{\det({}^2A)} = \det({}^2A)$

$\Rightarrow \det({}^1A) = \overline{\det({}^1A)}$ and $\det({}^2A) = \overline{\det({}^2A)}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

$\Rightarrow \det(A) = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

Proposition 3.11: Let $A \in C_2^{n\times n}$ be a $i_1 -$ Skew $-$Hermitian matrix. Then

(a) When n is even

(i) $\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$;

(ii) $\det(A) = \det(A_1) = \det(A_2)$

(iii) $\det(A) \in C(i_2)$

(b) When n is odd

(i) $\overline{\det({}^2A)} = -\det({}^1A)$ or equivalently $\overline{\det({}^1A)} = -\det({}^2A)$;

(ii) $\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

(iii) $\det(A) \in i_1 C(i_2)$

Proof: Let $A \in C_2^{n\times n}$ be a $i_1 -$ Skew $-$Hermitian matrix

$\Rightarrow A^* = -A^t$

$\Rightarrow \det(A^*) = \det(-A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A)$ $\quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = (-1)^n \det(A)$ $\quad (\because \det(A^*) = [\det(A)]^*)$

(a) When n is even

$[\det(A)]^* = \det(A)$ $\quad (\because (-1)^n = 1, n \text{ is even})$

All the results directly follow **Proposition 3.8**

(b) When n is odd

$[\det(A)]^* = -\det(A)$ $\quad (\because (-1)^n = 1, n \text{ is odd})$

(i) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = -\det({}^1A)e_1 - \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = -\det({}^1A)e_1 - \det({}^2A)\, e_2$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ and $\overline{\det({}^1A)} = -\det({}^2A)$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ or equivalently $\overline{\det({}^1A)} = -\det({}^2A)$

(ii) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_1)e_2 + \det(A_1)e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2) = -\det(A_1)$ and $\det(A_1) = -\det(A_2)$

$\Rightarrow \det(A_2) = -\det(A_1)$

$\Rightarrow \det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

(iii) From **(ii)**

$\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

$\det(A_1) \in C(i_2) \Rightarrow i_2 \det(A_1) \in C(i_2) \Rightarrow i_1 i_2 \det(A_1) \in i_1 C(i_2)$

$\Rightarrow \det(A) \in i_1 C(i_2)$

Proposition 3.12: Let $A \in C_2^{n\times n}$ be a $i_2 -$ Skew $-$Hermitian matrix. Then

(a) When n is even

(i) $\det(A) = \det({}^1A) = \det({}^2A)$

(ii) $\overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iii) $\det(A) \in C(i_1)$

(b) When n is odd

(i) $\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$;

(ii) $\overline{\det(A_2)} = -\det(A_1)$ or equivalently $\overline{\det(A_1)} = -\det(A_2)$

(iii) $\det(A) \in i_2 C(i_1)$

Proof:

Let $A \in C_2^{n\times n}$ be a $i_2 -$ Skew $-$Hermitian matrix

$\Rightarrow A^{\#} = -A^t$

$\Rightarrow \det(A^{\#}) = \det(-A^t)$

$\Rightarrow \det(A^{\#}) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^{\#}) = (-1)^n \det(A)$ $\quad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^{\#} = (-1)^n \det(A)$ $\quad (\because \det(A^{\#}) = [\det(A)]^{\#})$

(a) When n is even

$[\det(A)]^{\#} = \det(A)$ $\quad (\because (-1)^n = 1, \text{n is even})$

All the results directly follow **Proposition 3.9**

(b) When n is odd

$[\det(A)]^{\#} = -\det(A)$ $\quad (\because (-1)^n = -1, \text{n is odd})$

(i) $[\det(A)]^{\#} = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^{\#} = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^1A)e_2 + \det({}^2A)e_1 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A)e_1 + \det({}^1A)e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A) = -\det({}^1A)$ and $\det({}^1A) = -\det({}^2A)$

$\Rightarrow \det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

(ii) $[\det(A)]^{\#} = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#} = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}\, e_1 + \overline{\det(A_1)}\, e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ and $\overline{\det(A_1)} = -\det(A_2)$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ or equivalently $\overline{\det(A_1)} = -\det(A_2)$

(iii) From **(i)**

$\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

$\det({}^1A) \in C(i_1) \Rightarrow i_1 \det({}^1A) \in C(i_1) \Rightarrow i_1 i_2 \det({}^1A) \in i_2 C(i_1)$

$\Rightarrow \det(A) \in i_2 C(i_1)$

Proposition 3.13: Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ – Skew –Hermitian matrix. Then

(i) When n is even $\det(A) \in H$ or equivalently $\det(A) = \overline{\det(A_1)}\, e_1 + \overline{\det(A_2)}\, e_2$

(ii) When n is odd $\det(A) \in i_1 H$ or $i_2 H$ or equivalently $\det(A) = -\overline{\det(A_1)}\, e_1 - \overline{\det(A_2)}\, e_2$

Proof:

Let $A \in C_2^{n\times n}$ be a $i_1 i_2$ – Skew –Hermitian matrix

$\Rightarrow A' = -A^t$

$\Rightarrow \det(A') = \det(-A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A)$ $\qquad (\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = (-1)^n \det(A)$ $\qquad (\because \det(A') = [\det(A)]')$

(i) When n is even

$[\det(A)]' = \det(A)$ $\qquad (\because (-1)^n = 1, \text{n is even})$

The results directly follow **Proposition 3.10**

(ii) When n is odd

$[\det(A)]' = -\det(A)$ $\qquad (\because (-1)^n = -1, \text{n is odd})$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]' = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det({}^1A)$ is Pure imaginary and $\det({}^2A)$ is Pure imaginary

$\Rightarrow \det(A) \in i_1H$ or i_2H

Now,

$\overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det({}^1A) = -\overline{\det({}^1A)}$ and $\det({}^2A) = -\overline{\det({}^2A)}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

$\Rightarrow \det(A) = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

Proposition 3.14: Let $A \in C_2^{n\times n}$ be an orthogonal matrix. Then $\det(A) \in \{1, -1, i_1i_2, -i_1i_2\}$.

Proof: Let $A \in C_2^{n\times n}$ be an orthogonal matrix

$\Rightarrow A^tA = AA^t = I$

$\Rightarrow \det(A^tA) = \det(I)$

$\Rightarrow \det(A^t)\det(A) = 1$

$\Rightarrow \det(A)\det(A) = 1$ $\qquad (\because \det(A^t) = \det(A))$

$\Rightarrow \{\det(A)\}^2 = 1$

$\Rightarrow \{\det({}^1A)\, e_1 + \det({}^2A)\, e_2\}^2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = 1e_1 + 1e_2$

$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$

$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$

Hence $\det(A) \in \{1, -1, i_1i_2, -i_1i_2\}$.

Note 3.13: Bicomplex orthogonal Matrix is invertible and $A^{-1} = A^t$.

Proposition 3.15: Let $A \in C_2^{n\times n}$ be a i_1 − Unitary Matrix. Then

(i) $\overline{\det({}^2A)}\det({}^1A) = 1$ or equivalently $\overline{\det({}^1A)}\det({}^2A) = 1$

(ii) $\det(A) = \frac{1}{\overline{\det({}^2A)}}e_1 + \frac{1}{\overline{\det({}^1A)}}e_2$

Proof:

(i) Let $A \in C_2^{n\times n}$ be a i_1 − Unitary Matrix

$\Rightarrow A^{*t}A = AA^{*t} = I$

$\Rightarrow \det(A^{*t}A) = \det(I)$

$\Rightarrow \det(A^{*t})\det(A) = 1$

$\Rightarrow \det(A^*)\det(A) = 1$ $\qquad (\because \det(A^{*t}) = \det(A^*))$

$\Rightarrow [\det(A)]^* \det(A) = 1$ $\qquad (\because \det(A^*) = [\det(A)]^*)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\, e_2]^*[\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow \left[\overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}\, e_1\right][\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow \left[\overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}e_2\right][\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow \overline{\det({}^2A)}\det({}^1A)e_1 + \overline{\det({}^1A)}\det({}^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \overline{\det({}^2A)}\det({}^1A) = 1$ and $\overline{\det({}^1A)}\det({}^2A) = 1$

$\Rightarrow \overline{\det({}^2A)}\det({}^1A) = 1$ or equivalently $\overline{\det({}^1A)}\det({}^2A) = 1$

From **(i)**

$\overline{\det({}^2A)}\det({}^1A) = 1$ and $\overline{\det({}^1A)}\det({}^2A) = 1$

$\Rightarrow \det({}^1A) = \dfrac{1}{\overline{\det({}^2A)}}$ and $\det({}^2A) = \dfrac{1}{\overline{\det({}^1A)}}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \frac{1}{\overline{\det({}^2A)}}e_1 + \frac{1}{\overline{\det({}^1A)}}e_2$

$\Rightarrow \det(A) = \frac{1}{\overline{\det({}^2A)}}e_1 + \frac{1}{\overline{\det({}^1A)}}e_2$

Note 3.14: $i_1 -$ Unitary Matrix is invertible and $A^{-1} = A^{*t}$.

Proposition 3.16: Let $A = {}^1A\, e_1 + {}^2A\, e_2 \in C_2^{n\times n}$ be a $i_2 -$ Unitary Matrix. Then

(i) $\det({}^1A)\det({}^2A) = 1$

(ii) $\det(A) = \frac{1}{\det({}^2A)}e_1 + \frac{1}{\det({}^1A)}e_2$

Proof: Let $A = {}^1A\, e_1 + {}^2A\, e_2 \in C_2^{n\times n}$ be a $i_2 -$ Unitary Matrix

$\Rightarrow A^{\#t}A = AA^{\#t} = I$

$\Rightarrow \det(A^{\#t}A) = \det(I)$

$\Rightarrow \det(A^{\#t})\det(A) = 1$

$\Rightarrow \det(A^{\#})\det(A) = 1$ $\qquad (\because \det(A^{\#t}) = \det(A^{\#}))$

$\Rightarrow [\det(A)]^{\#}\det(A) = 1$ $\qquad (\because \det(A^{\#}) = [\det(A)]^{\#})$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)\, e_2]^{\#}[\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow [\det({}^2A)\, e_1 + \det({}^1A)e_2][\det({}^1A)e_1 + \det({}^2A)\, e_2] = 1$

$\Rightarrow \det({}^2A)\det({}^1A)e_1 + \det({}^1A)\det({}^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \det({}^2A)\det({}^1A) = 1$ and $\det({}^1A)\det({}^2A) = 1$

$\Rightarrow \det({}^1A)\det({}^2A) = 1$

(ii) From **(i)** $\det({}^1A)\det({}^2A) = 1$

$\Rightarrow \det({}^1A) = \frac{1}{\det({}^2A)}$ and $\det({}^2A) = \frac{1}{\det({}^1A)}$

$\Rightarrow \det(^1A)e_1 + \det(^2A)e_2 = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

$\Rightarrow \det(A) = \frac{1}{\det(^2A)}e_1 + \frac{1}{\det(^1A)}e_2$

Note 3.15: $i_2 -$ Unitary Matrix is invertible and $A^{-1} = A^{\#t}$.

Proposition 3.17: Let $A \in C_2^{n\times n}$ be a $i_1 i_2 -$ Unitary Matrix. Then $\|\det(A)\| = 1$.

Proof: Let $A \in C_2^{n\times n}$ be a $i_1 i_2 -$ Unitary Matrix

$\Rightarrow A'^t A = AA'^t = I$

$\Rightarrow \det(A'^t A) = \det(I)$

$\Rightarrow \det(A'^t)\det(A) = 1$

$\Rightarrow \det(A')\det(A) = 1 \qquad (\because \det(A'^t) = \det(A'))$

$\Rightarrow [\det(A)]' \det(A) = 1 \qquad (\because \det(A') = [\det(A)]')$

$\Rightarrow [\det(^1A)e_1 + \det(^2A)\, e_2]'[\det(^1A)e_1 + \det(^2A)\, e_2] = 1$

$\Rightarrow [\,\overline{\det(^1A)}\, e_1 + \overline{\det(^2A)}\, e_2][\det(^1A)e_1 + \det(^2A)\, e_2] = 1$

$\Rightarrow \overline{\det(^1A)}\det(^1A)e_1 + \overline{\det(^2A)}\det(^2A)e_2 = 1e_1 + 1e_2$

$\Rightarrow \overline{\det(^1A)}\det(^1A) = 1$ and $\overline{\det(^2A)}\det(^2A) = 1$

$\Rightarrow |\det(^1A)|^2 = 1$ and $|\det(^2A)|^2 = 1$

$\Rightarrow \|\det(A)\| = \left[\frac{|\det(^2A)|^2 + |\det(^2A)|^2}{2}\right]^{\frac{1}{2}} = 1$

Note 3.16: $i_1 i_2 -$ Unitary Matrix is invertible and $A^{-1} = A'^t$.

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Reference

[1] Price, G. B. (1991) "An introduction to multicomplex space and Functions" *Marcel Dekker*

[2] Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D. C., Vajiac, A.(2015) "Bicomplex Holomorphic Functions:The Algebra, Geometry and Analysis of Bicomplex Numbers" *Springer International Publishing*.

[3] Kumar, J.(2018) **"**On Some Properties of Bicomplex Numbers •Conjugates •Inverse •Modulii**"** Journal of *Emerging Technologies and Innovative Research (JETIR),* .5(9), 475-499

[4] Srivastava, Rajiv K. (2008) "Certain Topological Aspects of Bicomplex Space". *Bull. Pure & Appl. Math,* 222-234.

[5] Alpay, D., Luna-Elizarrarás, M. E., Shapiro, M. , Struppa, D. C. (2014) "Basics of Functional Analysis with Bicomplex Scalars and Bicomplex Schur Analysis" *Springer International Publishing*, 19-30.
[6] Kumar,J. (2016) **"**Conjugation of Bicomplex Matrix**"** *J. of Science and Tech. Res. (JSTR),* 1(1), 24-28.
[7] Beezer, Robert A. (2012) "A First Course in Linear Algebra" *Congruent Press, Gig Harbor, Washington, USA.*