Calculus and Computation for Geometric Series with Binomial Coefficients

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Abstract: In today's world, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical, combinatorial, and numerical equations for solving today's scientific problems and challenges. Combinatorics involves integers, factorials, binomial expansions, geometric series with binomial coefficients, and computation for finding solutions to the problems in computing and engineering science. This paper presents computational techniques and differential and integral calculus for the summation of geometric series with binomial coefficients in an innovative way. Also, it presents theorems, binomial expansions, and relationship between the binomial expansions and geometric series. These computing techniques refer to the methodological advances which are useful for researchers who are working in computational science. Computational science is a rapidly growing multi- and inter-disciplinary area where science, engineering, computation, mathematics, and collaboration use advance computing capabilities to understand and solve the most complex real life problems.

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1. Introduction

In the earlier days, geometric series served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. Combinatorics involves integers, factorials, binomial coefficients and binomial expansions, and the multiple summations of geometric series produce an innovative binomial coefficient and its binomial identities. Nowadays, the binomial expansions and geometric series with binomial coefficients [1-18] have significant applications in science, engineering, economics, queuing theory, computation, combinatorics, and management.

1.1 Computation of Geometric Series with Powers of Two

Let us develop the sum of geometric series [5-7] with exponents of 2 independently as follows:

$$2^n = 2^{n-1} + 2^{n-1} = 2^{n-1} + 2^{n-2} + 2^{n-2} = \cdots = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots 2^k + 2^k$$

$$\Rightarrow 2^k + 2^{k+2} + 2^{k+3} + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 2^k \Rightarrow \sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k.$$

In the geometric series $if\ \ k = 0$, then $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1, \qquad (k, n\ \in N),$

where $N = \{0, 1, 2, 3, \cdots\}$ is set of natural numbers including zero element.

1.2 Traditional Binomial Coefficient

The factorial function or factorial [29, 30] of a nonnegative integer *n*, denoted by *n*!, is the product of all positive integers less than or equal to *n*. For examples, 3! = 1×2× 3 = 6 and 0! = 1.

A binomial coefficient is always an integer that denotes $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$, where $n, r \in N$.

Here, $\binom{n+r}{r} = \dfrac{(n+r)}{r!\,n!} \implies (n+r) = l \times r!\,n!$, where l is an integer.

2. Binomial Expansions and Geometric Series with its Derivatives

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind for establishing a novel binomial series along with an innovative binomial coefficient [15-20].

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \ \& \ \ V_r^n = \frac{(r+1)(r+2)(r+3)\cdots\cdots(r+n-1)(r+n)}{n!},$$

where $n \geq 1, r \geq 0$ *and* $n, r \in N$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ and V_r^n refer to the binomial sereis and binomial coefficient respectively.

Let us compare the binomial coefficient V_x^y with the traditional binomial coefficient as follows:

Let $z = x + y$. Then, $\binom{z}{x} = zC_x = \dfrac{z!}{x!\,y!}$. Here, $V_x^y = V_y^x \implies zC_x = zC_y, (x, y, z \in N)$.

For example, $V_3^5 = V_5^3 = (5+3)C_3 = (5+3)C_5 = 56$.

Also, $V_n^0 = V_0^n = nC_0 = nC_n = \dfrac{n!}{n!\,0!} = 1$ *and* $V_0^0 = 0C_0 = \dfrac{0!}{0!} = 1 (\because 0! = 1)$.

2.1 The First Derivative of Geometric Series

Differentiation is the derivative [20] of a function with respect to an independent variable. In this section, a geometric series is considered as the function of independent variable *x*.

The function of geometric sereis is $f(x) = \sum_{i=0}^{r} x^i = 1 + x + x^2 + x^3 + \cdots + x^r = \dfrac{x^{r+1}-1}{x-1}$.

The first derivative of geometric series is built as follows:

$$f^1(x) = 1 + 2x + 3x^2 + 4x^3 \cdots + rx^{r-1} = f^1\left(\frac{x^{r+1}-1}{x-1}\right) = \frac{(rx-r-1)x^r+1}{(x-1)^2}$$

$$\implies V_0^1 + V_1^1 x + V_2^1 x^2 + V_3^1 x^3 \cdots + V_{r-1}^1 x^{r-1} = \frac{(rx-r-1)x^r+1}{(x-1)^2}, (x \neq 1).$$

By substituting $x = 2$ in $f^1(x)$, we get the mathematical equation as follows:

$$1 + 2(2) + 3(2)^2 + 4(2)^3 + \cdots + r2^{r-1} = \frac{(r-1)2^r+1}{(2-1)^2} = (r-1)2^r + 1.$$

Similarly, we get the following equations by substituting the values of x:

For $x = 3$, $$1 + 2(3) + 3(3)^2 + 4(3)^3 + \cdots + r3^{r-1} = \frac{(2r-1)3^r+1}{(3-1)^2} = \frac{(2r-1)3^r+1}{2^2}.$$

For $x = 4$, $$1 + 2(4) + 3(4)^2 + 4(4)^3 \cdots + r4^{r-1} = \frac{(3r-1)4^r+1}{(4-1)^2} = \frac{(3r-1)4^r+1}{3^2}.$$

For any number k that is equal to x, we get the equation $\sum_{i=0}^{r-1} V_i^1 k^i = \frac{(kr-r-1)k^r+1}{(k-1)^2}$.

2.2 The n^th Derivative of Geometric Series

$y = f(x) = \sum_{i=0}^{r} x^i = \frac{x^{r+1}-1}{x-1}$. The derivatives of y are given below.

$$\frac{1}{1!}\frac{dy}{dx} = \sum_{i=0}^{r-1} V_i^1 x^i \Rightarrow \frac{1}{2!}\frac{d^2y}{dx^2} = \sum_{i=0}^{r-2} V_i^2 x^i \Rightarrow \frac{1}{3!}\frac{d^3y}{dx^3} = \sum_{i=0}^{r-3} V_i^3 x^i \Rightarrow \cdots \frac{1}{n!}\frac{d^ny}{dx^n} = \sum_{i=0}^{r-n} V_i^n x^i.$$

The nth derivative [20] of geometric series is

$$\frac{1}{n!}\frac{d^ny}{dx^n} = \sum_{i=0}^{r-n} V_i^n x^i = \frac{1}{n!} f^n(x) = \frac{1}{n!} f^n\left(\frac{x^{r+1}-1}{x-1}\right).$$

$$\sum_{i=0}^{r-1} V_i^1 x^i = \frac{1}{1!} f^1\left(\frac{x^{r+1}-1}{x-1}\right); \sum_{i=0}^{r-2} V_i^2 x^i = \frac{1}{2!} f^2\left(\frac{x^{r+1}-1}{x-1}\right); \ \& \sum_{i=0}^{r-3} V_i^3 x^i = \frac{1}{3!} f^3\left(\frac{x^{r+1}-1}{x-1}\right)$$

are first, second, and third derivatives respectively.

2.3 Binomial Expansions equal to Multiple of 2

Let us develop some series of binomial coefficients or binomial expansions [15-16] which are equal to the multiple of 2 or exponents of 2 or both.

$$(1) \sum_{i=0}^{n} V_i^{n-i} = 2^n. \qquad (2) \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}. \qquad (3) \sum_{i=0}^{n} (i+1)V_i^{n-i} = (n+2)2^{n-1}.$$

$$(4) \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = (\mathrm{n}-2)2^{\mathrm{n}-1}, \quad V_r^n = \prod_{i=1}^{n} \frac{(r+i)}{n!}, (n \geq 1, r \geq 0 \ \& \ n, r \in N).$$

2.4 Relations between Binomial Expansion and Geometric Series

Relation 1: $\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}$.

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 1) as follows:

$(i+1)V_i^{n-i} + (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = 2i\mathrm{V}_\mathrm{i}^{\mathrm{n-i}}$. This idea can be applied to Relation 1.

$$\sum_{i=0}^{n} (i+1)V_i^{n-i} + \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = 2\sum_{i=0}^{n} iV_i^{n-i} = (n+2)2^{n-1} + (\mathrm{n}-2)2^{\mathrm{n}-1} = 2n2^{\mathrm{n}-1}.$$

$$\text{Then, } 2\sum_{i=0}^{n} iV_i^{n-i} = 2n2^{\mathrm{n}-1} \Rightarrow \sum_{i=0}^{n} iV_i^{n-i} = n2^{\mathrm{n}-1}.$$

Hence, Relation 1 is proved.

Relation 2: $\sum_{i=0}^{n} (i+1)V_i^{n-i} - \sum_{\mathrm{i}=0}^{\mathrm{n}} (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = \sum_{i=0}^{n} V_i^{n-i} = 2^n$.

Proof: Let us simplify the general terms in the two parts of binomial expansions (Relation 2) as follows:

$(i+1)V_i^{n-i} - (\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = 2\mathrm{V}_\mathrm{i}^{\mathrm{n-i}}$. This idea can be applied to Relation 2.

$$\sum_{i=0}^{n}(i+1)V_i^{n-i} - \sum_{\mathrm{i}=0}^{\mathrm{n}}(\mathrm{i}-1)\mathrm{V}_\mathrm{i}^{\mathrm{n-i}} = 2\sum_{i=0}^{n} V_i^{n-i} = (n+2)2^{n-1} - (\mathrm{n}-2)2^{\mathrm{n-1}} = 4\times 2^{\mathrm{n-1}}.$$

$$\text{Then, } 2\sum_{i=0}^{n} V_i^{n-i} = 22^{\mathrm{n}} \Rightarrow \sum_{i=0}^{n} V_i^{n-i} = 2^{\mathrm{n}}.$$

Hence, Relation 2 is proved.

2.5 Annamalai's Binomial Expansion

Let $n, r \in N = \{0,1,2,3,\cdots\}$. The Annamalai′s binomial identity [15] is given below:

$$V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1} \Leftrightarrow V_n^0 + V_n^1 + V_n^2 + \cdots + V_n^r = V_{n+1}^r, (\because V_n^r = V_r^n).$$

From the binomial identity $V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1}$, we can derive the following binomial expansions:

$$(1). \sum_{i=0}^{n} V_i^0 = \sum_{i=0}^{n} 1 = 1+1+1+1+\cdots+1+1 = \frac{(n+1)}{1!}.$$

$$(2). \sum_{i=0}^{n} V_i^1 = \sum_{i=0}^{n} \frac{(i+1)}{1!} = 1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2!}.$$

$$(3). \sum_{i=0}^{n} V_i^2 = \sum_{i=0}^{n} \frac{(i+1)(i+2)}{2!} = 1+3+\cdots+\frac{(n+1)(n+2)}{2!} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

$$(4). \sum_{i=0}^{n} V_i^3 = \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)}{3!} = \frac{(n+1)(n+2)(n+3)(n+4)}{4!}.$$

$$(5). \sum_{i=0}^{n} V_i^4 = \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)(i+4)}{4!} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!}.$$

Similarly, we can continue this process up to *r* times. The *r*th binomial expansion is as follows:

$$(r). \sum_{i=0}^{n} V_i^r = \sum_{i=0}^{n} \frac{(i+1)(i+2)(i+3)\cdots(i+r)}{r!} = \frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!},$$

From the binomial identity $V_n^0 + V_n^1 + V_n^2 + \cdots + V_n^r = V_{n+1}^r$, we can derive the following binomial expansions.

$$1 + \frac{(n+1)}{1!} + \frac{(n+1)(n+2)}{2!} + \frac{(n+1)(n+2)(n+3)}{3!} + \cdots + \frac{(n+1)(n+2)\cdots(n+r)}{r!} = \frac{(n+2)(n+3)\cdots(n+r)(n+r+1)}{r!}.$$

These expressions are called Annamalai's binomial expansions.

The following theorem is derived from the Annamalai's binomial series [15].

Theorem 2.1: $\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n} V_{i-n}^r x^i.$

Proof: Let's show that the computation of summations of the binomial series (right-hand side of the theorem) is equal to the binomial series (left- hand side of the theorem).

$$\sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

$$= (V_0^r + V_1^r x + V_2^r x^2 + V_3^r x^3 + \cdots + V_n^r x^n) + (V_0^r x + V_1^r x^2 + V_2^r x^3 + V_3^r x^4 + \cdots + V_{n-1}^r x^n)$$

$$+ (V_0^r x^2 + V_1^r x^3 + V_2^r x^4 + V_3^r x^5 + \cdots + V_{n-2}^r x^n) + \cdots + (V_0^r x^{n-1} + V_1^r x^n) + V_0^r x^n$$

$$= V_0^r + (V_0^r + V_1^r)x + (V_0^r + V_1^r + V_2^r)x^2 + \cdots + (V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_n^r)x^n$$

(Note that $V_0^p + V_1^p + V_2^p + \cdots + V_r^p = V_r^{p+1}$ for $r = 0, 1, 2, 3, \cdots$, and $V_0^p = V_0^{p+1} = 1$)

$$= V_0^{r+1} + V_1^{r+1} x + V_2^{r+1} x^2 + V_3^{r+1} x^3 + V_4^{r+1} x^4 + \cdots + V_{n-1}^{r+1} x^{n_1} + V_n^{r+1} x^n = \sum_{i=0}^{n} V_i^{r+1} x^i.$$

3. Binomial Expansion equal to the Sum of Geometric Series

Binomial expansion denotes a series of binomial coefficients. In this section, we focus on the summation of multiple binomial expansions or summation of multiple series of binomial coefficients.

Theorem 3.1: $\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 1.$

This binomial theorem states that the sum of multiple summations of series of binomial coefficients [15-19] is equal to the sum of a geometric series with exponents of 2.

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step\ 0$: $\binom{0}{0} = \frac{0!}{0!} = 1 \Rightarrow \sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 2^0.$

$Step\ 1$: $\sum_{i=0}^{1} \binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1.$

$Step\ 2$: $\sum_{i=0}^{2} \binom{2}{i} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 = 2^2.$

$Step\ 3$: $\sum_{i=0}^{3} \binom{3}{i} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=0}^{n} \binom{n}{i} = 2^n.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\sum_{i=0}^{n}2^{i},$$

where $\sum_{i=0}^{n}2^{i}=\dfrac{2^{n+1}-1}{2-1}=2^{n+1}-1$ is the geometric sereis with exponents of two.

$$\therefore\ \sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=2^{n+1}-1.$$

Hence, theorem is proved.

Some results of Theorem 3.1 are given below:

$$(a)\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}=2^{p}-1,\text{where } 1\le p\in N.$$

$$(b)\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{p-1}\binom{q-1}{i}=2^{q}-1,\text{where } 1\le q\in N.$$

By subtracting (a) from (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{q-1}\binom{q-1}{i}\right)-\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}\right)=2^{q}-2^{p},$$

$$i.e.,\sum_{i=0}^{p}\binom{p}{i}+\sum_{i=0}^{p+1}\binom{p+1}{i}+\sum_{i=0}^{p+2}\binom{p+2}{i}+\cdots+\sum_{i=0}^{q-2}\binom{q-2}{i}+\sum_{i=0}^{q-1}\binom{q-1}{i}=2^{q}-2^{p},$$

where p $< q$ & $p,q\ \in N$.

By adding (a) and (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}\right)+\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{q-1}\binom{q-1}{i}\right)=2^{p}+2^{q}-2,$$

$$If\ p=q,\qquad then\ \ 2\left(\sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}\right)=22^{p}-2=2(2^{p}-1),$$

$$i.e.,\qquad \sum_{i=0}^{0}\binom{0}{i}+\sum_{i=0}^{1}\binom{1}{i}+\sum_{i=0}^{2}\binom{2}{i}+\sum_{i=0}^{3}\binom{3}{i}+\cdots+\sum_{i=0}^{p-1}\binom{p-1}{i}=2^{p}-1,\text{where } 1\le q\in N.$$

Theorem 3.2: $\sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k+1}\binom{k+1}{i}+\sum_{i=0}^{k+2}\binom{k+2}{i}+\cdots+\sum_{i=0}^{n}\binom{n}{i}=\ 2^{n+1}-2^{k},$

where $k\le n$ & $k,n\ \in N$.

Proof. The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k.$$

Then, $\sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k+1}{i} + \sum_{i=0}^{k+2} \binom{k+2}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=k}^{n} 2^i.$

$\therefore \sum_{i=0}^{k} \binom{k}{i} + \sum_{i=0}^{k+1} \binom{k+1}{i} + \sum_{i=0}^{k+2} \binom{k+2}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^k.$

Hence, theorem is proved.

Some results of Theorem 3.2 are given below:

(i) $\sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^n = 2^n.$ $(ii) \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n-1}(2^2 - 1) = 3(2^{n-1}).$

$(iii) \sum_{i=0}^{n-2} \binom{n-2}{i} + \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^{n-2} = 2^{n-2}(2^3 - 1) = 7(2^{n-2}).$

$(iv) \sum_{i=0}^{n-3} \binom{n-3}{i} + \sum_{i=0}^{n-2} \binom{n-2}{i} + \sum_{i=0}^{n-1} \binom{n-1}{i} + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 2^{n-3} = 15(2^{n-3}).$

These results can be generalized as follows:

$$\sum_{i=0}^{p} \binom{p}{i} + \sum_{i=0}^{p+1} \binom{p+1}{i} + \sum_{i=0}^{p+2} \binom{p+2}{i} + \cdots + \sum_{i=0}^{q-1} \binom{q-1}{i} + \sum_{i=0}^{q} \binom{q}{i} = 2^p(2^{q-p+1} - 1),$$

where $0 \leq p \leq q \ and \ p, q \in N$.

Theorem 3.3: $\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = (n-1)2^n + 1.$

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step\ 1$: $1\binom{1}{1} = \binom{1}{1} = \frac{1!}{1!\,0!} = 1 \Rightarrow \sum_{i=1}^{1} i\binom{1}{i} = 1 = 1 \times 2^0,$ $(0! = 1).$

$Step\ 2$: $\sum_{i=1}^{2} i\binom{2}{i} = 1\binom{2}{1} + 2\binom{2}{2} = 2 + 2 = 4 = 2 \times 2^1.$

$Step\ 3$: $\sum_{i=1}^{3} i\binom{2}{i} = 1\binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3} = 3 + 6 + 3 = 12 = 2 \times 2^1.$

$Step\ 4$: $\sum_{i=1}^{4} i\binom{2}{i} = 1\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4} = 4 + 12 + 12 + 4 = 4 \times 2^3.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=1}^{n} i \times 2^{i-1}.$$

where $\sum_{i=1}^{n} i \times 2^{i} = (n-1)2^{n} + 1$.

$$\therefore \sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = (n-1)2^{n} + 1.$$

Hence, theorem is proved.

Some results of Theorem 3.3 are given below

$$\left\{\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k+1}{i} + \cdots + \sum_{i=1}^{n+1} i\binom{n+1}{i}\right\}$$
$$- \left\{\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{k} i\binom{k}{i}\right\} = n2^{n+1} - k2^{k+1}$$

$\Rightarrow \sum_{i=1}^{k+1} i\binom{k+1}{i} + \sum_{i=1}^{k+2} i\binom{k+2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} + \sum_{i=1}^{n+1} i\binom{n+1}{i} = 2(n2^{n} - k2^{k})$ and

$\sum_{i=1}^{k} i\binom{k}{i} + \sum_{i=1}^{k+1} i\binom{k+1}{i} + \cdots + \sum_{i=1}^{n-1} i\binom{n-1}{i} + \sum_{i=1}^{n} i\binom{n}{i} = 2\{(n-1)2^{n-1} - (k-1)2^{k-1}\},$

where $k < n$ & $k, n \in N$.

Theorem 3.4: $(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \; dx + C = (p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \; dx + 1 = \sum_{i=0}^{n} V_i^{p} x^i$,

where C is the constant of Integration and C =1 because 1 is the first term of geometric series.

Proof. Let us prove the theorem on integral calculus using the following binomial expansions.

$$\sum_{i=0}^{n} V_i^{p} x^i = 1 + \frac{(p+1)}{1!}x + \frac{(p+1)(p+2)}{2!}x^2 + \cdots + \frac{(n+1)(n+2)\ldots(n+p)}{p!}x^n .$$

$$\sum_{i=0}^{n-1} V_i^{p+1} x^i = 1 + \frac{(p+2)}{1!}x + \frac{(p+2)(p+3)}{2!}x^2 + \cdots + \frac{n(n+1)(n+2)\ldots(n+p)}{(p+1)!}x^{n-1}.$$

Let's prove that the integration (left-hand side of the theorem) is equal to the binomial series (right- hand side of the theorem).

$$\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx = x + \frac{(p+2)}{1!}\frac{x^2}{2} + \frac{(p+2)(p+3)}{2!}\frac{x^3}{3} + \cdots + \frac{n(n+1)\dots(n+p)}{(p+1)!}\frac{x^n}{n} + C.$$

$$(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx = 1 + \frac{(p+1)}{1!}x + \frac{(p+1)(p+2)}{2!}x^2 + \frac{(p+1)(p+2)(p+3)}{3!}x^3 + \cdots + \frac{(n+1)(n+2)\dots(n+p)}{p!}x^n, \qquad \text{where } C = 1.$$

$$(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + C = (p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + 1 = \sum_{i=0}^{n} V_i^p x^i \ .$$

Hence, theorem is proved.

Some results of Theorem 3.4 are given below:

Let $p = 0$. Then $(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + 1 = \int \sum_{i=0}^{n-1} V_i^1 x^i \ dx + 1 = \sum_{i=0}^{n} x^i \ = \frac{x^{n+1}-1}{x-1}.$

Let $p = 1$. Then $2\int \sum_{i=0}^{n-1} V_i^2 x^i \ dx + 1 = \sum_{i=0}^{n} V_i^1 x^i \ \Rightarrow \sum_{i=0}^{n-1} V_i^1 x^i = \frac{(rx-r-1)x^r + 1}{(x-1)^2},$

which is the first derivative of geometric series. More details about the first derivative of geometric series are given in Section 2.1.

In general, the integration of summation of geometric series is constituted as follows:

$$(p+1)\int \sum_{i=k}^{n-1} V_{i-k}^{p+1} x^i \ dx + C = \sum_{i=k+1}^{n} V_{i-(k+1)}^p x^i + V_{i-k}^p x^i = \sum_{i=k}^{n} V_{i-k}^p x^i \ ,$$

where the integral constant *is* $C = V_{i-k}^p x^i$ because it is the first term of the series.

4. Conclusion

In this article, the nth derivative [20-28] of geometric series has been introduced and its applications used in combinatorics including binomial expansions. Also, computation of the summation of series of binomial expansions and geometric series were derived in an innovative way. Theorems and relations between the binomial expansions and geometric series have been developed for researchers who are working in science, economics, engineering, management, and medicine [31].

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