

Note on the Odd Perfect Numbers

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and Planat stated that the Riemann Hypothesis is true if and only if the inequality $\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Under the assumption that the Riemann Hypothesis is true and the inequality $\frac{\pi^2}{\beta} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$ is satisfied for infinitely many prime numbers q_n and $\beta \geq 6.0008$, then we prove that there is not any odd perfect number at all.

Keywords: Riemann Hypothesis, Prime numbers, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function

2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

where $d | n$ means the integer d divides n , $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k | n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that $\log$ is the natural logarithm. We know these properties from these functions:

Proposition 1.1. [1]. Let $\prod_{i=1}^r q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_r$ with natural numbers as exponents $a_1, \dots, a_r$. Then,

$$f(n) = \left(\prod_{i=1}^r \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^r \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

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Proposition 1.2. For every prime power q^a , we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p - 1)^2}.$$

Say Robins(n) holds provided

$$G(n) < e^\gamma$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. Robins(n) holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [3].

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [4]. We state the following property about this function:

Proposition 1.4. For every $x \geq 19035709163$ [5]:

$$\theta(x) > \left(1 - \frac{0.15}{\log^3 x}\right) \times x.$$

In mathematics, $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say Dedekinds(q_n) holds provided

$$\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

where q_n is the n th prime number. The importance of this inequality is:

Proposition 1.5. Dedekinds(q_n) holds for all prime numbers $q_n > 3$ if and only if the Riemann Hypothesis is true [6].

Let $q_1 = 2, q_2 = 3, \dots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^k q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [7]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.6. If n is superabundant, then n is an Hardy-Ramanujan integer [8]. Let n be a superabundant number, then $p \parallel n$ where p is the largest prime factor of n except when $n \in \{4, 36\}$ [8]. For large enough superabundant number n , we have that $q^{a_q} < 2^{a_1}$ for $q > 11$ where $q^{a_q} \parallel n$ and $2^{a_1} \parallel n$ [8]. For large enough superabundant number n , we obtain that $\log n < \left(1 + \frac{0.5}{\log p}\right) \times p$ where p is the largest prime factor of n [4]. Let n be a superabundant

number, then $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$ where $\varepsilon(p) = \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})$ and p is the largest prime factor of n [4]. Let n be a superabundant number such that $q^{a_q} \parallel n$ and $2 \leq q \leq p$, then

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq a_q$$

where p is the largest prime factor of n and $\lfloor \dots \rfloor$ is the floor function [8]. If n is superabundant, then

$$p \sim \log n$$

where p is the largest prime factor of n [8]. There are infinitely many superabundant numbers, since the number of superabundant numbers less than x exceeds:

$$\frac{c \times \log x \times \log \log x}{(\log \log \log x)^2}$$

for some constant $c > 0$ [8].

In addition, we will use these properties:

Proposition 1.7. [6], [7]. For $n \geq 2$:

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

Proposition 1.8. It is known that [9]:

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.9. Any odd perfect number N must satisfy the following conditions: $N > 10^{1500}$ and the largest prime factor of N is greater than 10^8 [10], [11].

Now, we state the following conjecture:

Conjecture 1.10. We assume that the Riemann Hypothesis is true and the inequality

$$\frac{\pi^2}{\beta} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers q_n and $\beta \geq 6.0008$.

Under the assumption that the Conjecture 1.10, we prove that there is not any odd perfect number at all.

2. The Main Insight

Theorem 2.1. *For any natural number N , there are always infinitely many large enough superabundant numbers n such that n is a multiple of N .*

Proof. Suppose that p is the largest prime factor of the superabundant number n . We will use the Proposition 1.6 in this proof. We have that there are infinitely many superabundant numbers n . Moreover, we know that

$$p \sim \log n$$

which means that p goes to infinity as long as n goes to infinity. In addition, every prime number q between 2 and p divides n due to n must be an Hardy-Ramanujan integer when n is superabundant. Furthermore, for every prime power $q^{a_q} \parallel n$ and $2 \leq q \leq p$, we obtain that

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq a_q$$

which implies that

$$p \leq q^{a_q+1}.$$

That would mean that a_q goes to infinity as long as p goes to infinity. In this way, for every prime power q^{a_q} , there are always infinitely many large enough superabundant numbers n such that n is a multiple of q^{a_q} . Since every natural number N has a prime factorization within prime powers, then the proof is done. $\square$

3. The Main Theorem

Theorem 3.1. *Under the assumption that the Conjecture 1.10, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Conjecture 1.10 is false. There are always infinitely many large enough superabundant numbers n such that n is a multiple of N because of the Theorem 2.1. We could take the largest prime factor p of n as $p > 10^{10000}$ according to the Proposition 1.9 and the Theorem 2.1. We would have

$$f(n) \leq f(N) \times f\left(\frac{n}{N}\right)$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f\left(\frac{n}{N}\right)$$

since $f(N) = 2$, because N is a perfect number. Hence,

$$\begin{aligned} \frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_1}}) \times f(\frac{n}{2^{a_1}})}{2} \\ &= f\left(\frac{n}{2^{a_1}}\right) \times \frac{(2 - \frac{1}{2^{a_1}})}{2} \\ &= f\left(\frac{n}{2^{a_1}}\right) \times \frac{2^{a_1+1} - 1}{2^{a_1+1}} \end{aligned}$$

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when $2^{a_1} \parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_1}})}{f(\frac{n}{N})} \leq \frac{2^{a_1+1}}{2^{a_1+1} - 1}.$$

However, we know that $p < 2^{a_1}$ because of $p > 10^{10000} > 11$ and the Propositions 1.6 and 1.9. Consequently,

$$\frac{2^{a_1+1}}{2^{a_1+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$\begin{aligned} 2 \times p^2 &\leq (p+1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1 \end{aligned}$$

where this inequality is satisfied for every prime number p . So,

$$\frac{f(\frac{n}{2^{a_1}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.6. Using the Conjecture 1.10, we have that

$$\begin{aligned} e^\gamma &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_1}})}{f(\frac{n}{N}) \times \log \log n} \end{aligned}$$

since $f(\dots)$ is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^\gamma}{f(\frac{n}{2^{a_1}})} \times \log \log n.$$

Under the assumption that the Conjecture 1.10, we deduce that:

$$\frac{\pi^2}{6.0008} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log((\theta(p))^{0.7501}).$$

From the Propositions 1.1 and 1.6, we know that

$$f\left(\frac{n}{2^{a_1}}\right) = \left(\prod_{i=2}^k \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \left(1 + \frac{1}{q_i}\right) \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Conjecture 1.10, we obtain that

$$\begin{aligned} e^\gamma \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.7501}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f\left(\frac{n}{2^{a_1}}\right) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f\left(\frac{n}{2^{a_1}}\right) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.7. Taking into account that $p > 10^{10000} > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} > \frac{e^\gamma}{f\left(\frac{n}{2^{a_1}}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} \times \log \log n.$$

For large enough superabundant number n and $p > 10^{10000}$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.7501})} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^{10000}}}}{\log\left(\left(1 - \frac{0.15}{\log^3 10^{10000}}\right) \times 10^{10000}\right)^{0.7501}} \times \log\left(\left(1 + \frac{0.5}{\log 10^{10000}}\right) \times 10^{10000}\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^{10000}}}}{\log\left(\left(1 - \frac{0.15}{\log^3 10^{10000}}\right) \times 10^{10000}\right)^{0.7501}} \times \log\left(\left(1 + \frac{0.5}{\log 10^{10000}}\right) \times 10^{10000}\right) < 1.999733371.$$

Thus,

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.999733371.$$

For every prime p_j that divides N such that $p_j^{a_j} \parallel N$ and $p_j^{a_j+b_j} \parallel n$ for a_j, b_j two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned} \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_j \left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \frac{1}{f(p)} \\ &= \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \frac{1}{f(p)} \\ &> 1.9999 \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \left(1 - \frac{1}{p+1}\right) \\ &> 1.9999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \times \frac{(1 - \frac{1}{p+1})}{(1 - \frac{1}{2^{a_1+1}})} \\ &> 1.9999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \times \left(1 - \frac{1}{p+1}\right) \\ &> 1.9999 \times \left(1 - \frac{1}{\log 10^{10000}} \times \left(1 + \frac{1.5}{\log 10^{10000}}\right)\right) \times \left(1 - \frac{1}{10^{10000} + 1}\right) \\ &> 1.999733371 \end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

tends to 0 as b_j tends to infinity for every odd prime p_j where

$$\begin{aligned} \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) &\approx \prod_j (f(p_j^{a_j})) \\ &= f(N) \\ &= 2. \end{aligned}$$

Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. In addition, we note that

$$\begin{aligned} \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) &< \prod_{i=1}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \left(1 - \frac{1}{2^{a_1+1}}\right) \end{aligned}$$

after taking into account the Proposition 1.6. However,

$$1.999733371 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.999733371$$

is a contradiction. By contraposition, the number N does not exist under the assumption that the Conjecture 1.10. The smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N . Note that, this proof fails for even perfect numbers or for some other odd numbers N such that $f(N) > 2$ (precisely when we may consider a large enough superabundant number n). $\square$

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