Note on the Riemann Hypothesis

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and and Planat stated that the Riemann Hypothesis is true if and only if the inequality $\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. Using this result, we create a new criterion for the Riemann Hypothesis. We prove the Riemann Hypothesis is true using this new criterion.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. We denote the nth prime number as q_n. We know the following property for the Chebyshev function and the nth prime number:

Proposition 1.1. *For $n \geq 2$ [1, Theorem 1.1]:*

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}).$$

Proposition 1.2. *For $n \geq 8602$ [2, Theorem B (1.11)]:*

$$q_n \leq n \times (\log n + \log \log n - 0.9385).$$

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In mathematics, $\Psi(n) = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. The importance of this inequality is:

Proposition 1.3. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true [3, Theorem 4.2].*

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant. We know the following formula:

Proposition 1.4. *We have that [4, Lemma 2.1 (1)]:*

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k}\right) = \gamma - B = H.$$

In addition, we know this value of the Riemann zeta function:

Proposition 1.5. *It is known that:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

Putting all together yields a proof for the Riemann Hypothesis using the Chebyshev function.

2. What if the Riemann Hypothesis were false?

Theorem 2.1. *If the Riemann Hypothesis is false, then there are infinitely many prime numbers* q_n *for which* $\mathsf{Dedekind}(q_n)$ *does not hold.*

Proof. The Riemann Hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [3, Theorem 4.2]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f is introduced in the Nicolas paper [5, Theorem 3]:

$$f(x) = e^{\gamma} \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann Hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ [5, Theorem 3 (c)]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0 \colon \ \log f(y) \geq k \times y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \times y^{-b} \times \sqrt{y}\right) \times \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \times y^{-b} \times \sqrt{y}\right) = \infty$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0 \colon \ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann Hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

3. A Key Theorem

Theorem 3.1.

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k})\right) = \log(\zeta(2)) - H.$$

Proof. We obtain that

$$\begin{aligned}
\log(\zeta(2)) - H &= \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \times (q_k + 1)})\right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1})\right) - H
\end{aligned}$$

where

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k}) \right) - H \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) \right) - \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) - \log(\frac{q_k}{q_k - 1}) + \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right)
\end{aligned}
$$

and the proof is done. □

4. A New Criterion

Theorem 4.1. Dedekind(q_n) *holds if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is satisfied for the prime number q_n, where the set $S = \{x : x > q_n\}$ contains all the real numbers greater than q_n and χ_S is the characteristic function of the set S (This is the function defined by $\chi_S(x) = 1$ when $x \in S$ and $\chi_S(x) = 0$ otherwise).

Proof. When Dedekind(q_n) holds, we apply the logarithm to the both sides of the inequality:

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log\log\theta(q_n)$$

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

after of using the Theorem 3.1. Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

when Dedekind(q_n) holds. The same happens in the reverse implication. □

5. The Main Insight

Theorem 5.1. *The Riemann Hypothesis is true if the inequality*

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n.

Proof. The inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x>q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is also satisfied, where the set $S = \{x : \ x \geq q_n\}$ contains all the real numbers greater than or equal to q_n. In the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

only change the values of

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes q_n and q_{n+1}. It is enough to show that

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n. Indeed, the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

is the same as

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_{n+1}\}}(q_k)) \times \log(1+\frac{1}{q_k})\right)$$

$$> B + \log\log\theta(q_{n+1}) + \log(1+\frac{1}{q_n}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1})$$

where q_n and q_{n+1} are consecutive primes. From the previous inequality, we note that if

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1}) \geq 0$$

is satisfied, then

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x\geq q_{n+1}\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_{n+1})$$

is also satisfied which means that $\mathsf{Dedekind}(q_{n+1})$ holds according to the Theorem 4.1. Therefore, if the inequality

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1}) \geq 0$$

is always satisfied starting for some natural number n_0, (i.e. it is always satisfied for $n \geq n_0$), then we obtain that $\mathsf{Dedekind}(q_{n+1})$ always holds for $n \geq n_0$. However, this contradicts the fact that if the Riemann Hypothesis is false, then there are infinitely many prime numbers q_{n+1} for which $\mathsf{Dedekind}(q_{n+1})$ does not hold when $n \geq n_0$. We obtain this contradiction as a consequence of the Theorem 2.1. By contraposition (or reductio ad absurdum), we have that the Riemann Hypothesis is true when

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1}) \geq 0$$

is always satisfied starting for some natural number n_0. This last statement would be the same as the result that

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. This is

$$\log\left((1+\frac{1}{q_n}) \times \log\theta(q_n)\right) \geq \log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}} \geq \log\log\theta(q_{n+1}).$$

To sum up, the Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. □

6. The Main Theorem

Theorem 6.1. *The Riemann Hypothesis is true.*

Proof. The Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n because of the Theorem 5.1. That is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

after dividing the both sides of the inequality by $\theta(q_n)$. We would only need to prove that

$$1 + \frac{\log \theta(q_n)}{q_n} \geq 1 + \frac{1}{n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})}$$

because of

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})$$

$$\theta(q_n)^{\frac{1}{q_n}} = e^{\frac{\log \theta(q_n)}{q_n}} \geq 1 + \frac{\log \theta(q_n)}{q_n}.$$

That is equivalent to

$$\left(n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})\right) \times \log \theta(q_n) \geq q_n.$$

Therefore,

$$\left(n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})\right) \times \log \theta(q_n) \geq n \times (\log n + \log \log n - 0.9385)$$

which is

$$\left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right) \times \log \theta(q_n) + 0.9385 \geq \log n + \log \log n$$

$$\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}} \times e^{0.9385} \geq n \times \log n$$

$$e^{0.9385} \geq \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}}.$$

However, we know that

$$\overline{\lim_{n \to \infty}} \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} = \lim_{n \to \infty} \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} = 1$$

since

$$\lim_{n \to \infty} \left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right) = 1$$

$$\theta(q_n) \sim q_n, \quad (n \to \infty)$$

$$q_n \sim n \times \log n, \quad (n \to \infty).$$

Certainly, a sequence of real numbers (x_n) in $[-\infty, \infty]$ converges if and only if

$$\underline{\lim}_{n \to \infty} x_n = \overline{\lim_{n \to \infty}} x_n$$

in which case $\lim_{n \to \infty} x_n$ is equal to their common value, where $-\infty$ or ∞ is not considered as convergence. By definition, the limit superior of a sequence of real numbers x_n is the smallest real number b such that, for any positive real number ε, there exists a natural number m such that $x_n < b + \varepsilon$ for all $n > m$. Hence, for any positive real number ε, there exists a natural number m such that

$$\frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} < 1 + \varepsilon$$

for all $n > m$, because of the definition of limit superior. Moreover, we can see that $e^{0.9385} > 2.5561$. Consequently, it is enough to take any positive real number $\varepsilon \leq 1.5561$. Putting all together yields the proof of the Riemann Hypothesis. □

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