An Exposition of Polygonal Approximation of Circle

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Abstract – This article attempts to discuss a journey of creating an infinite number of circles from a single circle, using its tangents with a pattern (P_X; X≥3) and reaching up to a single point from a given circle. The time to get a new circle from its predecessor circle varies with the pattern. The more we repeat the polygonal pattern the better we can observe that the set of all end intersecting points of the tangents becomes similar[1] to the set [2] R. The pattern behind both the radius of the successor circles and predecessor circles is also discussed. Most interestingly, when we apply the P_∞ pattern, then all the infinitely many successor circles almost merge into a single one, almost without taking any time and it takes infinite time to reach up to a single point from the given circle, for that pattern. This whole idea can be applied to astronomical objects. Although we are dealing with a circular path, which is an approximation for the elliptical orbit of a celestial body.

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1 INTRODUCTION.

Polygons, along with lines and circles, constitute the earliest collection of Geometric figures studied by humans. By polygon, we mean a closed figure with a finite number n of sides (n vertices)(n $\geq$ 3) each of which is a line segment. The polygonal approximation of circles is commonly known throughout the Mathematics world. In polygonal approximation, we are interested in lengths and these lengths are approximations to the arc length of the curve. Increasing the value of the number of sub-intervals into which domain is divided, increases the accuracy of the approximation.

The first work of circle approximation dates back to Archimedes of Syracuse (287-212BC), one of the greatest mathematicians of the ancient world. Archimedes approximated the area of a circle by using the Pythagorean theorem to find the areas of two regular polygons: the polygon inscribed within the circle and the polygon within which the

[1] Similar means they have the same potency. Cutting the circumference of a circle and stretching it, gives us an interval on the real axis. Now any non-empty non-singleton interval and R have the same potential. This equipotency in between an arbitrary non-empty non-singleton interval and R can be easily checked using projection .

[2] R means, the set of real numbers.

circle was circumscribed **[2]**. Some mathematicians use parametric polynomial curves and Quadratic B$\acute{e}$zier curves to approximate circular arcs. In 1993, L Yong-Kui has worked on circle approximation and its generation. He introduces a new algorithm for the generation of the circle, using the intersecting polygon instead of the inscribed polygon, which greatly reduces the error **[4]**. In recent years 2011, J$\acute{o}$zef Borkowski had done work on 'Minimization of Maximum Errors In Universal Approximation of The Unit Circle By a Polygon' **[5]**.

Researchers gave many algorithms and interesting ways to approximate circles. A very few worked on the generation of circles. Now some queries!, is it possible to create infinitely many successor and predecessor circles from a single circle, using its tangents? If yes! then how the time varies for getting new circles again and again ? what is the process to reach up into a single point from a given circle? What is the pattern behind the radius of the successor circles and distances between any two nearby successor circles? Also what for the predecessor circles? If there is any pattern then can we use them to understand something else ? In this article, we will try to illustrate and resolve all those queries.

2 PROCEDURE.

Let us consider that, we have a single circle (Figure 1) and any tangents of the circle looks like Figure 2. Here, we will deal with the regular polygonal pattern of tangents.

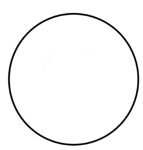

Figure 1: circle

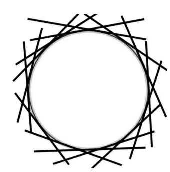

Figure 2: Circle with it's tangents

2.1 SUCCESSOR CIRCLES.

2.1.1 VISUALISATION TO THE PROCESS.

Let's begin with the simplest regular polygonal pattern i.e. the triangular pattern of tangents. First we take three tangents with triangular pattern on the given single circle. Then we have Figure 3.

Define, b_i : for all i belongs to N, as end-intersecting points of tangents on the circle. It is assumed that all end-intersecting points are distributed in a precise way. Meaning, for each n=k (Here, n=k meaning we are applying the mentioned polygon pattern k times), k belongs to N ; $d_u(b_1, b_2) = d_u(b_2, b_3) = ... = d_u(b_i, b_{i+1})$; for all i belongs to N. d_u symbolises the Euclidean distance [3]. Define $d_u(b_i, b_{i+1}) = l_k$ for n=k.

[3] In the whole article, we will use d_u as a notation of Euclidean distance.

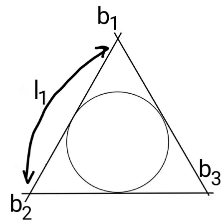

Figure 3: n=1

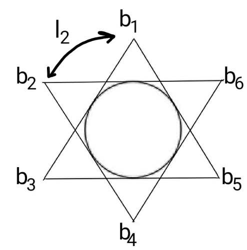

Figure 4: n=2

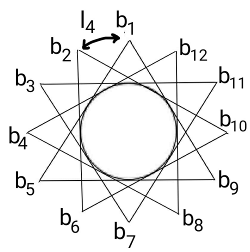

Figure 5: n=4

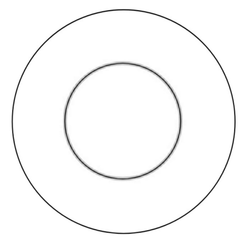

Figure 6: 1st successor circle, $n \longrightarrow \infty$

The above geometric treatments gives us a decreasing sequence here, $(L_n)_{n=1}^{\infty} = \{l_1, l_2, l_3, ..., l_n, ...\}$ where, $l_1 > l_2 > l_3 > ... > l_n > ....$ It can be easily checked that this sequence converges to 0 (using Least Upper Bound property) and then we will have our 1_{st} successor circle and the set of all $b_i{}'s$ becoming similar to the set of real numbers. Now 2_{nd} successor circle can be obtained applying the same idea of tangents pattern (here, triangular) on the 1_{st} successor circle.

Let, c_i : for all i belongs to N, are the end intersecting points of tangents on the 1_{st} successor circle. Assumed that, for each n=k: k belongs to N ; $d_u(c_1, c_2) = d_u(c_2, c_3)$ = ... $= d_u(c_i, c_{i+1})$; for all i belongs to N. Define for n=k, $d_u(c_i, c_{i+1}) = m_k$.

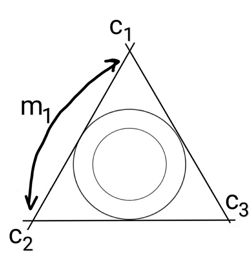

Figure 7: n=1

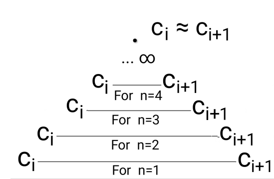

Figure 8: Varying lengths of $m_i's$

Then a decreasing sequence can be obtained and it is defined by, $(M_n)_{n=1}^{\infty} = \{m_1, m_2, m_3, ..., m_n, ...\}$. Now, $d_u(c_i, c_{i+1}) \longrightarrow 0$ when $n \longrightarrow \infty$; for all i belongs to N and it can be easily shown that , this sequence converges to 0. Therefore, the Set of all, $c_i's$ becoming similar to the set R. So, now we have our 2_{nd} successor circle.

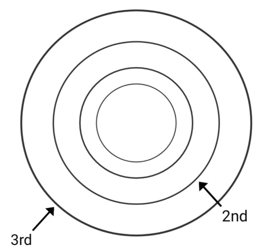

Figure 9: 2nd and 3rd successor circles

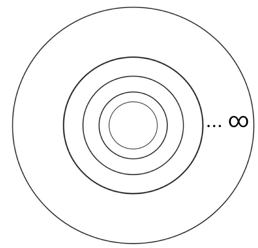

Figure 10: Infinitely many successor circles

Our 3_{rd} successor circle can be obtained, again applying the same procedure. In Fig.8 we attempted to give an idea of how does the Euclidean distance between c_i and c_{i+1} i.e. the lengths of $m_i's$ decrease when the value of n increases. That distance is going to be zero when $n \longrightarrow \infty$. Infinitely many successor circles can be obtained, repeating this sequential process infinitely many times.

Now, we will deal with the 2_{nd} simplest regular polygon i.e. square (basically the square pattern of tangents on circles). Assumed that, the end-intersecting points (say $b_i's$) of tangents are distributed in a precise way. Define, $d_u(b_i, b_{i+1}) = l_k$,

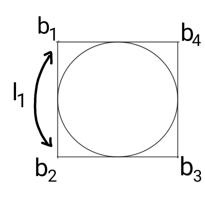

Figure 11: n=1

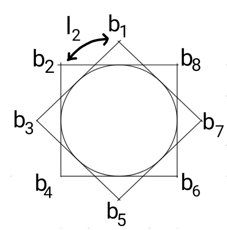

Figure 12: n=2

for n=k : k belongs to N. It can be easily checked that a decreasing sequence of $l_k's$,

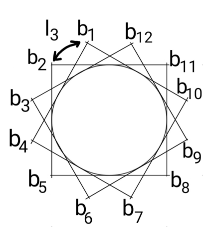

Figure 13: n=3

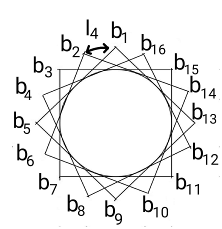

Figure 14: n=4

$(L_n)_{n=1}^{\infty} = \{l_1, l_2, l_3, ..., l_n, ...\}$ can be obtained and converging to 0 and then we will have our 1_{st} successor circle and the set of all $b_i's$ becoming similar to the set R.

Let, c_i : for all i belongs to N, are the end intersecting points of tangents on the 1_{st} successor circle. Assumed that, for each n=k : k belongs to N ; $d_u(c_1, c_2) = d_u(c_2, c_3)$ = ... $= d_u(c_i, c_{i+1})$; for all i belongs to N. Define, $d_u(c_i, c_{i+1}) = m_k$, for n=k; k belongs to N. Here also the below defined sequence $(M_n)_{n=1}^{\infty} = \{m_1, m_2, m_3, ..., m_n, ...\}$

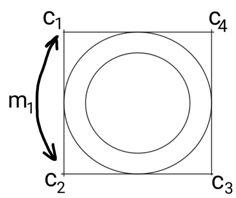

Figure 15: n=1

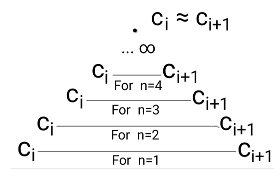

Figure 16: Varying length's of $m_i's$

converges to 0 and the set of all $c_i's$ becoming similar to the set of real numbers i.e. R. Then our 2_{nd} successor circle is obtained. Infinitely many successor circles can be

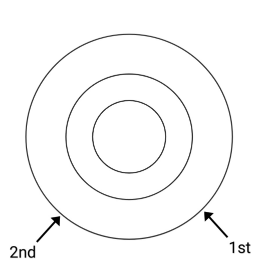

Figure 17: 1_{st} and 2_{nd} successor circle

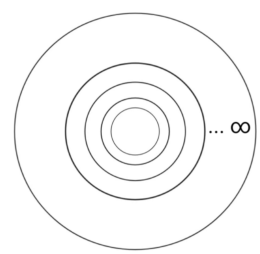

Figure 18: Infinitely many successor circles

obtained, repeating this sequential process infinitely many times.

In previous portions,the triangular pattern and square pattern are discussed. But the interesting fact is, infinite number of circles can be obtained with any polygonal pattern (applying sequential idea). In polygonal approximation, increasing the value of the number of sub-intervals into which the domain is divided, increases the accuracy of approximation.

2.1.2 RATE OF CONVERGENCE.

Let P_X (X $\geq$ 3): regular polygon with X number of vertices.
Let T_1: time to get the 1_{st} successor circle from the given single circle using the P_3 pattern of tangents. Whenever we apply the P_3 pattern for the 1_{st} time, then 3 end-intersecting points arises. Applying P_3 pattern for the 2_{nd} time, we have 6 end-intersecting points. Continuing this process, we will have 9, 12, 15, ... so on. Observe

that, a sequence arises here and it is defined by,

$$(P_{3n})_{n=1}^{\infty} = \{3, 6, 9, 12, ..., 3n, ...\}\ i.e.(3n)_{n=1}^{\infty}$$

Now, T_1 depends on the rate of convergence of sequence $(\frac{1}{P_{3n}})_{n=1}^{\infty}$ [4].
Let, T_2 be the time to get the 1_{st} successor circle using P_4 pattern of tangents. It can be easily checked that a sequence of end-intersecting points of tangents can be obtained and it is defined by,

$$(P_{4n})_{n=1}^{\infty} = \{4, 8, 12, 16, ..., 4n, ...\}\ i.e.(4n)_{n=1}^{\infty}$$

Here, T_2 depends on the rate of convergence of sequence $(\frac{1}{P_{4n}})_{n=1}^{\infty}$.
Now proceeding in a similar way, it can be observed that, if T_K : K belongs to N , be the time to get the 1_{st} successor circle from the single circle using P_{K+2} pattern then T_K depends on the rate of convergence of sequence , $(\frac{1}{P_{(K+2)n}})_{n=1}^{\infty}$. We need to compare all those sequences, on which $T_K{}'s$ depends. It can be easily checked that, for all n belongs to N, $\frac{1}{3n} > \frac{1}{4n} > \frac{1}{5n} > ... > \frac{1}{Kn} > ...$ here, any $\frac{1}{Kn}$ defines the terms of sequence $(\frac{1}{Kn})_{n=1}^{\infty}$. Arranging those above sequences according to its rate of convergence then we have, $(\frac{1}{3n})_{n=1}^{\infty}, (\frac{1}{4n})_{n=1}^{\infty}, (\frac{1}{5n})_{n=1}^{\infty}, ..., (\frac{1}{(Kn)})_{n=1}^{\infty}, ...$ rate of converges decreases when K increases. Now, we construct a sequence of all $T_K{}'S$ defined by,

$$(T_n)_{n=1}^{\infty} = \{T_1, T_2, T_3, ..., T_n, ...\} : T_1 > T_2 > T_3 > ... > T_K > ...$$

So, it is decreasing in order and converging to 0. Thus by increasing n in P_n, the time to get our 1_{st} successor circle from the given single circle, can be reduced and Thus increasing n in P_n, reduces the time to get infinitely many successor circles from the single circle.

Let, T_{P_K}: Time to get an infinite number of circles from a single circle using P_K pattern (K $\geq$ 3). So, $T_{P_3} > T_{P_4} > T_{P_5} > ... > T_{P_K} > ...$

Define, P_∞ : the polygon with infinitely many vertices (here, P_∞ is just a notation). The idea of polygonal approximation of circle tells us that the regular polygon with infinite vertices is an approximation to circle **[2]**.

An interesting fact is that, if we apply P_∞ pattern of the tangents on the circle then almost without taking any time, our 1_{st} successor circle can be obtained. More precisely, for P_∞ on the given single circle, almost without taking any time, we will get infinitely many successor circles and they almost merge into a single one. Basically, the decreasing sequence, $(T_{P_n})_{n=1}^{\infty} = \{T_{P_1}, T_{P_2}, ..., T_{P_n}, ...\}$ converges to 0.

Let's deal with the thought that, how does the distance between any two nearby successive circles depends on the radius of the given single circle and also how does the radius of the successive circles vary from each other.

[4] Rate of convergence is how quickly the terms of the sequence converges to 0

Let, r be the radius of the given single circle.
Let, b_1 , b_2 , b_3 , ... b_n , ... are eip [5] of tangents (for, P_X pattern) and u_1 , u_2 , u_3 , ... u_n, ... are the ctip [6]. The origin of the given single circle. Therefore, we will join those points with lines (we take only those lines which passes through origin). Now joining points are , (i) eip and eip (ii) eip and ctip (iii) ctip and ctip.

Therefore, if we have n joining lines (passing through origin), then the whole circle is divided into 2n equal parts. Interestingly, for the P_X pattern; X=n, we will have 2n equal parts of the circle.
At first, we will deal with the P_3 pattern (Triangular pattern) of tangents. Here, ${b_i}'s$

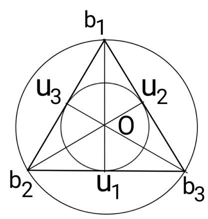

Figure 19: Points position

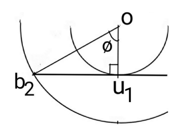

Figure 20

lies on the circumference of the 1_{st} successor circle and ou_1 = ou_2 = ou_3 = r , i.e. the radius of the given single circle and ob_1 = ob_2 = ob_3 = r_1 ; radius of the 1_{st} successor circle. For P_3 pattern we have 3 vertices. So the whole circle would divide into 6 equal parts. One of those equal parts is, $\triangle u_1 o b_2$. It can be observed that, ou_1 is perpendicular to $b_2 u_1$. Now, $\phi = \frac{360}{6} = 60$ degree (Since, 6 equal parts), therefore $\cos\frac{\pi}{3} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{3}}$.

Let, r_2 is the radius of the 2nd successor circle. Here, ob_2 = r_2 , ou_1 = r_1 (From Figure 25). It can be easily checked that,

$$r_2 = \frac{r_1}{\cos\frac{\pi}{3}} = \frac{r}{\left(\cos\frac{\pi}{3}\right)^2}.$$

Basically, we have a sequence of radius of successor circles and it is defined below,

$$\left\{ r, \frac{r}{\cos\frac{\pi}{3}}, \frac{r}{\left(\cos\frac{\pi}{3}\right)^2}, \dots \right\} = \left(\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j-1}} \right)_{j=1}^{\infty}$$

Where, $\frac{r}{\left(\cos\frac{\pi}{3}\right)^j}$, is the radius of the j_{th} successor circle obtained using P_3 pattern. The distance between any two successive circles (including single circle also) is defined

[5]eip: end-intersecting points
[6]ctip: circle-tangent intersecting points

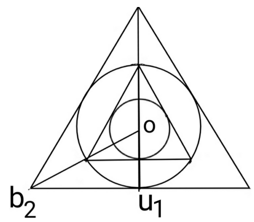

Figure 21

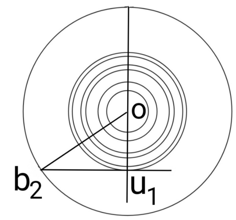

Figure 22: nth successor circle

by,

$$\left[\frac{1}{\left(\cos\frac{\pi}{3}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{3}\right)^{j-1}}\right] r.$$

Now, we will check for P_4 pattern. For P_4 pattern, the whole circle can be divided into

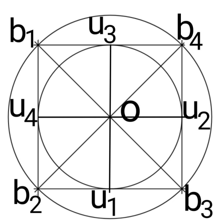

Figure 23: Points position

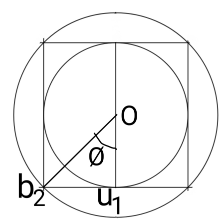

Figure 24

8 equal parts. One of those equal parts is $\triangle u_1 o b_2$. It can be easily seen that, ou_1 is perpendicular to $b_2 u_1$ (here, ou_1 = r, ob_2 = r_1). Now, ϕ = $\frac{360}{8}$ = 45 degree (Since, 8 equal parts) therefore, $\cos\frac{\pi}{4} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{4}}$.

Let, r_2 be the radius of the 2_{nd} successor circle. Here, ob_2 = r_2 , ou_1 = r_1. It can be easily checked that,

$$r_2 = \frac{r_1}{\cos\frac{\pi}{4}} = \frac{r}{\left(\cos\frac{\pi}{4}\right)^{2}}.$$

Here, a sequence of radius of successor circles can be obtained and each radius is depending on the radius of the given single circle. That sequence is defined below,

$$\left\{ r, \frac{r}{\cos\frac{\pi}{4}}, \frac{r}{\left(\cos\frac{\pi}{4}\right)^{2}}, \ldots \right\} = \left(\frac{r}{\left(\cos\frac{\pi}{4}\right)^{j-1}} \right)_{j=1}^{\infty}$$

Where, $\frac{r}{\left(\cos\frac{\pi}{4}\right)^{j}}$ is the radius of the j_{th} successor circle, obtained using P_4 pattern.

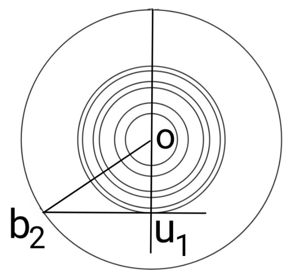

Figure 25: nth successor circle

The distance between any two successive circles (including single circle also) is ,

$$\left[\frac{1}{\left(\cos\frac{\pi}{4}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{4}\right)^{j-1}}\right]r.$$

Now, we want to propose a new theorem here.

Theorem 1. *If* $(P_X; X \geq 3)$ *be any X-regular polygonal pattern of tangents on a single circle, then the* $(j+1)_{th}$ *term of the sequence of radii of generations of circle i.e. the radius of the* j_{th} *successor circle, is defined by,* $\frac{r}{\left(\cos\frac{\pi}{X}\right)^{j}}$ *and the distance between any two nearby successor circles (including single circle also) is defined by,*

$$\left[\frac{1}{\left(\cos\frac{\pi}{X}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{X}\right)^{j-1}}\right]r$$

Now we introduce this above theorem as,
“ PK’s Theorem for Circle (For the polygon in which circle is circumscribed) ”.

Let, the statement of the above theorem be our P(m), for some m belongs to N. To prove this statement we will use ’Induction Hypothesis’.

Proof. By our previous discussions it can be easily checked that, P(1) and P(2) are true for X=3 and X=4 respectively .
Let us assume that, P(K) is true. Means, we are applying here, P_{K+2} pattern of tangents . A sequence of radius of the successor circles depending on the radius of the given circle, can be obtained. Then the radius of the j_{th} successor circle is, $\frac{r}{\left(\cos\frac{\pi}{K+2}\right)^{j}}$ and the distance between any two nearby successor circles (including single circle also) is,

$$\left[\frac{1}{\left(\cos\frac{\pi}{K+2}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{K+2}\right)^{j-1}}\right]r$$

Now, we need to show that, P(K+1) is also true.

Let, $b_1, b_2, b_3, ..., b_{K+1}$ are eip of the tangents of the given single circle. So, all those points lie on the circumference of the 1_{st} successor circle. Let, u_1 be any of the ctip and lies on the circumference of the single circle. Therefore taking lines passing through the origin, we join those points. Joining points are, (i). eip and eip (ii) eip and ctip (iii) ctip and ctip.

Here we are using, P_{K+3} pattern of tangents. Thus, circle can be broken into

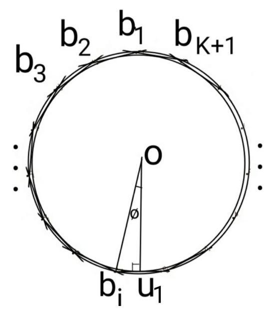

Figure 26

2(K+3) equal parts and each of those equal parts is a 'Right Triangle'. One of those equal part is, $\triangle ou_1b_i$, where. ou_1 = r , $ob_i = r_1$. It can be easily checked that, $\phi = \frac{360}{2(K+3)}$ degree. Therefore, $\cos\frac{\pi}{(K+3)} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{K+3}}$.

r_2 i.e. the radius of the 2_{nd} successor circle, can be obtained repeating the same process on the 1_{st} successor circle. So now,

$$r_2 = \frac{r_1}{\cos\frac{\pi}{K+3}} = \frac{r}{\left(\cos\frac{\pi}{K+3}\right)^2}.$$

Basically, we have a sequence of radius of successor circles and it is defined below,

$$\left\{ r, \frac{r}{\cos\frac{\pi}{K+3}}, \frac{r}{\left(\cos\frac{\pi}{K+3}\right)^2}, ... \right\} = \left(\frac{r}{\left(\cos\frac{\pi}{K+3}\right)^{j-1}} \right)_{j=1}^{\infty}.$$

Here, the radius of the j_{th} successor circle is, $\frac{r}{\left(\cos\frac{\pi}{K+3}\right)^j}$,

and the distance between any two nearby successor circles (including given single circle also) is, $\left[\frac{1}{\left(\cos\frac{\pi}{K+3}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{K+3}\right)^{j-1}} \right] r.$

So, our proposed statement holds for this pattern also. Hence, P(K+1) is true.

Thus, our statement P(m) is true for all m belongs to N.

Hence, **"PK's Theorem for Circle (For incircle of a regular polygon)"** is proved. □

For the P_∞ pattern of tangents on the single circle then we will just take the limit on the radius and distance formulae, (what we have obtained using our $P_X; (X \geq 3)$ pattern. Therefore the radius of the j_{th} successor circle (for P_∞ pattern) is,

$$\lim_{X\to\infty} \frac{r}{\left(\cos\frac{\pi}{X}\right)^j} = \frac{r}{\left(\lim_{X\to\infty}\cos\frac{\pi}{X}\right)^j} = \frac{r}{1} = r$$

and the distance between any two successive circles (including the single circle also) is,

$$\lim_{X\to\infty}\left[\frac{1}{\left(\cos\frac{\pi}{X}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{X}\right)^{j-1}}\right] r = [1-1]r = 0.$$

So, it can be observed that, for P_∞ pattern of tangents, all the infinitely many successor circles has almost the same radius, as what the single circle has, i.e. all the infinitely many successor circles almost [7] merge into a single one, almost without taking any time.

Interestingly, if our given circle has zero radius then whatever P_X pattern we apply on it, it will always remain a single point.

2.2 PREDECESSOR CIRCLES.

In this portion, it is attempted to discuss that, how do we can reach up into a single point from a given circle using polygonal approximation.
Let us assume that, each circle is made up of a polygonal approximation. From the discussions of the previous portion of this article, it follows that we always have a predecessor circle for each circle.

2.2.1 RATE OF CONVERGENCE.

It can be observed that area of any n-regular polygon inscribed in a circle has a smaller (or, almost the same) area than the area of the circle. The more the value of X in P_X increases, the better the regular polygon also turns into a circle. For P_∞ pattern, the area of the regular polygon becomes almost the same as the area of the given circle **[3]**.

Let us consider that, T_1 be the time to reach up into a single point from the given single circle, using P_3 pattern.
T_2 be the required time for using P_4 pattern.
...
T_K be the required time for using P_{K+2} pattern.
...
T_∞ be the required time for using P_∞ pattern. (Here, T_∞ and P_∞ are just a notation)

[7] We mentioned 'almost', because we use polygonal approximation to get such observation and also in Limit, we are not interested in the arrival, rather we are interested in approaching .

Claim: The sequence $(T_n)_{n=1}^{\infty}$ is increasing in order and it is a divergent sequence.

To Establish : r_j for $P_3 < r_j$ for $P_4 < ... < r_j$ for $P_k < ... < r_j$ for $P_n < ...$ and r_j for $P_X \geq r_{j+1}$ for $P_X \geq r_{j+2}$ for $P_X \geq ...$. For P_∞ pattern, $\lim_{X\to\infty} r_j = r$ and $\lim_{X\to\infty}(r_{j-1} - r_j) = 0$, for all j belongs to N .

Proof. Let, A_1 , A_2 , A_3 are the vertices of any inscribed triangle (i.e. P_3) and V_1 , V_2 , V_3 are the 1_{st} predecessor circle and polygon P_3 intersecting points. Since here we

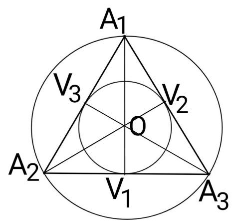

Figure 27

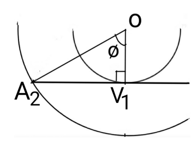

Figure 28

are dealing with the P_3 pattern, we can divide the given circle into 6 equal parts. Each of them is a right triangle. One of those equal parts is, $\triangle$A$_2$OV_1. Now, $\phi = \frac{360}{6} = 60$ degree (Since, 6 equal parts).
Let, OA_2=r : radius of the given single circle and OV_1 = r_1 : radius of the 1_{st} predecessor circle. Therefore, $\cos\frac{\pi}{3} = \frac{r_1}{r} \Rightarrow r_1 = \left(\cos\frac{\pi}{3}\right)r.$
Let, r_2 be the radius of the 2_{nd} predecessor circle. It can be easily checked that,

$$r_2 = \left(\cos\frac{\pi}{3}\right)r_1 \Rightarrow r_2 = \left(\cos\frac{\pi}{3}\right)^2 r.$$

Continuing like this, a sequence of the radius of predecessor circles can be obtained.

$$\left\{r, \big(\cos\frac{\pi}{3}\big)r, \left(\cos\frac{\pi}{3}\right)^2 r, ...\right\} = \left(\big(\cos\frac{\pi}{3}\big)^{j-1} r\right)_{j=1}^{\infty}.$$

The j_{th} term of the above sequence i.e. $\left(\cos\frac{\pi}{3}\right)^j r$, is the radius of the j_{th} predecessor circle and the distance between any two predecessor circles (including given single circle also) is defined by,

$$\left[\left(\cos\frac{\pi}{3}\right)^{j-1} - \left(\cos\frac{\pi}{3}\right)^j\right]r.$$

Let's check for P_4 pattern. Let, A_1 , A_2 , A_3 , A_4 are the vertices of any inscribed square (i.e. P_4) and V_1 , V_2 , V_3 , V_4 are the 1_{st} predecessor circle and polygon P_4 intersecting points. Here given circle can be divided into 8 equal parts. One of those equal parts is, $\triangle$A$_2$OV_1. Now, $\phi = \frac{360}{8} = 45$ degree (Since, 8 equal parts). Let,

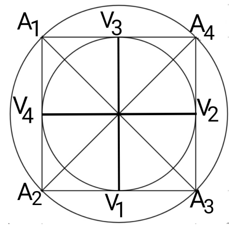

Figure 29

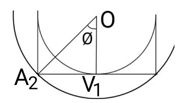

Figure 30

OA_2 = r is the radius of the given single circle and OV_1 =r_1 is the radius of the 1_{st} predecessor circle. Therefore, $\cos\frac{\pi}{4} = \frac{r_1}{r} \Rightarrow r_1 = \left(\cos\frac{\pi}{4}\right)r$.

Let, r_2 be the radius of the 2nd predecessor circle. It can be easily checked that,

$$r_2 = \left(\cos\frac{\pi}{4}\right)r_1 \Rightarrow r_2 = \left(\cos\frac{\pi}{4}\right)^2 r$$

Interestingly, we have a sequence of radius ,

$$\left(\left(\cos\frac{\pi}{4}\right)^{j-1} r\right)_{j=1}^{\infty} = \left\{r, \left(\cos\frac{\pi}{4}\right)r, \left(\cos\frac{\pi}{4}\right)^2 r, ...\right\}$$

It can be easily observed that, $\left(\cos\frac{\pi}{4}\right)^j r$ is the radius of the j_{th} predecessor circle and the distance between any two predecessor circle (including given single circle also) is,

$$\left[\left(\cos\frac{\pi}{4}\right)^{j-1} - \left(\cos\frac{\pi}{4}\right)^j\right]r.$$

Now here, we wants to propose a new theorem.

Theorem 2. *If $(P_X; X \geq 3)$ be any X- regular polygonal pattern of tangents on a single circle, then the $(j+1)_{th}$ term of the sequence of radii of generations of circle i.e. the radius of the j_{th} predecessor circle is defined by, $\left(\cos\frac{\pi}{X}\right)^j r$ and the distance between any two nearby predecessor circle (including given single circle also) is defined by,*

$$\left[\left(\cos\frac{\pi}{X}\right)^{j-1} - \left(\cos\frac{\pi}{X}\right)^j\right]r.$$

Now we introduce this above theorem as,
"PK's Theorem for Circle (For inscribed polygon)".

Let, the statement of the above theorem be our P(m), for some m belongs to N. To prove this statement we will use 'Induction Hypothesis'.

Proof. It can be easily checked that P(1) and P(2) are true for X=3 and X=4 respectively. Let us assume that, P(K) is true. Means, we are applying here, P_{K+2} pattern. Here the radius of the j_{th} predecessor circle is, $\left(\cos\frac{\pi}{K+2}\right)^{j} r$ and the distance between any two nearby predecessor circles (including single circle also) is,

$$\left[\left(\cos\frac{\pi}{K+2}\right)^{j-1} - \left(\cos\frac{\pi}{K+2}\right)^{j}\right] r.$$

Now, we need to show that, P(K+1) is also true.
Let, $A_1, A_2, ..., A_i, ...$ are the vertices of the inscribed polygon (P_{K+3}) of a circle with radius r . Each mid point of the side A_iA_{i+1} : for all i belongs to N , is a polygon-predecessor circle intersecting point. Let, V_1 be one of those polygon- predecessor circle intersecting point. Here in the above figure , OV_1 = r_1 : radius of the 1_{st}

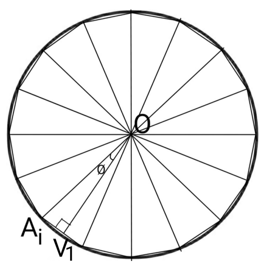

Figure 31

predecessor circle and OA_i = r : radius of the given single circle. The whole given circle can be broken into 2(K+3) equal parts. One of those equal parts is $\triangle A_iOV_1$. Let, ϕ be the angle between OV_1 and OA_i. Now, $\phi = \frac{360}{2(K+3)}$ degree (Since, 2(K+3) equal parts). Therefore $\cos\frac{\pi}{K+3} = \frac{r_1}{r} \Rightarrow r_1 = \left(\cos\frac{\pi}{K+3}\right) r$.
Repeating the same process on the 1_{st} predecessor circle, r_2 i.e. the radius of the 2_{nd} predecessor circle , can be obtained. So now,

$$r_2 = \left(\cos\frac{\pi}{K+3}\right) r_1 \Rightarrow r_2 = \left(\cos\frac{\pi}{K+3}\right)^{2} r.$$

Interestingly, applying this process again and again we will have a sequence of radius

$$\left(\left(\cos\frac{\pi}{K+3}\right)^{j-1} r\right)_{j=1}^{\infty} = \left\{ r, \left(\cos\frac{\pi}{K+3}\right) r, \left(\cos\frac{\pi}{K+3}\right)^{2} r, ... \right\}.$$

Here, the radius of the j_{th} predecessor circle is $\left(\cos\frac{\pi}{K+3}\right)^{j} r$, and the distance between any two nearby predecessor circles (including single circle also) is,

$$\left[\left(\cos\frac{\pi}{K+3}\right)^{j-1} - \left(\cos\frac{\pi}{K+3}\right)^{j}\right] r.$$

So, P(K+1) is true. Thus, our statement P(m) is true for all m belongs to N. Hence, **" PK's Theorem for Circle (For inscribed polygon) "** is proved.

□

For the P_∞ pattern of tangents on the single circle then we will just take the limit on the radius and distance formulae, (what we have obtained using our $P_X; X \geq 3$ pattern) . Therefore the radius of the j_{th} successor circle (for P_∞ pattern) is, $\lim_{X\to\infty} \left(\cos \frac{\pi}{X} \right)^j r = r$, and the distance between any two nearby predecessor circles (including the single circle also) is,

$$\lim_{X\to\infty} \left[\left(\cos \frac{\pi}{X} \right)^{j-1} - \left(\cos \frac{\pi}{X} \right)^j \right] r = [1-1]r = 0.$$

Basically, it can be observed that, $r_j \to r$ and $(r_{j-1} - r_j) \to 0$ when $X \to \infty$. We need to compare, r_1 for $P_3 = r \cos \frac{\pi}{3}$, r_1 for $P_4 = r \cos \frac{\pi}{4}$, ... , r_1 for $P_X = r \cos \frac{\pi}{X}$ ... Since, $r > 0$. So enough to compare the cosine parts. Here, $\lim_{X\to\infty} \cos \frac{\pi}{X} = 1$ and

$$\cos \frac{\pi}{3} < \cos \frac{\pi}{4} < \cos \frac{\pi}{5} < ... < \cos \frac{\pi}{X} < ... \tag{1}$$

$$\Rightarrow (\cos \frac{\pi}{3})^2 < (\cos \frac{\pi}{4})^2 < (\cos \frac{\pi}{5})^2 < ... < (\cos \frac{\pi}{X})^2 < ... \tag{2}$$

$$\cdots$$

$$\Rightarrow (\cos \frac{\pi}{3})^j < (\cos \frac{\pi}{4})^j < (\cos \frac{\pi}{5})^j < ... < (\cos \frac{\pi}{X})^j < ... \tag{3}$$

For all j belongs to N.

From equation (1) we have, r_1 for $P_3 < r_1$ for $P_4 < r_1$ for P_5 <... r_1 for $P_X <$ ... and from equation (2) we have, r_2 for $P_3 < r_2$ for $P_4 < r_2$ for P_5 <... r_2 for $P_X <$ ... and from equation (3) we have, r_j for $P_3 < r_j$ for $P_4 < r_j$ for P_5 <... r_j for $P_X <$ ... Therefore , r_j for $P_X \geq r_{j+1}$ for $P_X \geq r_{j+2}$ for $P_X \geq$ ... since, $\cos x$ is bounded above by 1. Thus, $\cos x \geq (\cos x)^2 \geq (\cos x)^3 \geq ...$.

It can be conclude that, $(T_n)_{n=1}^\infty$ is increasing. For P_∞ pattern, each r_i is almost equals to r. Thus it takes infinite time to reach up, from a given circle into a single point, i.e. the circle with zero radius . Meaning, for P_∞ pattern, T_∞ is infinite.Thus, $(T_n)_{n=1}^\infty$ is not bounded above. More precisely, $(T_n)_{n=1}^\infty$ is divergent.

Hence, our claim is proved. □

3 DISCUSSIONS AND CONCLUIONS.

The article describes how does one can obtain infinitely many circles from a single circle, using its tangents with a pattern, and how a single point can be obtained from a given circle. It also introduces two new theorems regarding the gaps between nearby successor and predecessor circles and their radii. That idea can be applied into computer aided designs and for the astronomical objects. We can consider circular orbit as an approximation of elliptical orbit of celestial bodies **[6]**. Suppose that, we are dealing with our solar system. Let us assume that, we know the distance from Sun to Mercury (say r) and now we want to calculate the distance from Sun to Earth.

We can create infinite no. of successor circular orbits from the approximate circular orbit of Mercury. Since the distance from Mercury and Earth is finite, the circular orbit of Earth will lie in between j_{th} and $(j+1)_{th}$ successor circles of Mercury circular orbit, for some j belongs to N. The radius of the j_{th} and $(j+1)_{th}$ successor circles are $\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j}}$ and $\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j+1}}$ respectively (here we use P_3 pattern, because it is faster than the other patterns). Let,G be the gap between the $j+1_{th}$ and j_{th} successor circular path, G = $\left[\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j+1}} - \frac{r}{\left(\cos\frac{\pi}{3}\right)^{j}}\right]$. Let, P be a circular path with radius, $\frac{\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j+1}}+\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j}}}{2}$. If Earth lies on P then we are done. If not, then orbit of Earth must lie either in between j_{th}, P or in between P, $j+1_{th}$ circular paths. Let, the orbit of Earth lies in between P, $j+1_{th}$ circular paths. Let, P_1 be a circular path with radius $\frac{r_P+r_{j+1}}{2}$. If Earth lies on P_1 then we are done else orbit of Earth must lie either in between P, P_1 or in between P_1, $j+1_{th}$ circular paths. This process can be repeated again and again untill we get a suitable circular path which is an approximation of the Earths orbit. Then the distance between Sun and Earth is, r_j + $\frac{G}{2}$ + ϵ_1 ; $\epsilon_1 \geq 0$, (

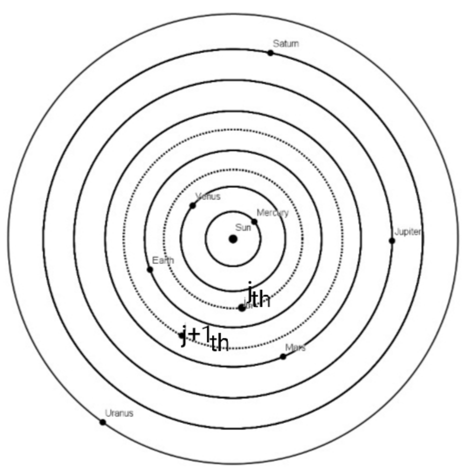

Figure 32: Solar system model with approximate circular orbits

value of ϵ_1 can be obtained by continuing the process and getting the values of P_i's). If the orbit of Earth lies in between j_{th}, P then do the same process. Then the distance between Sun and Earth is,r_j - $\frac{G}{2}$ - ϵ_2 ; $\epsilon_2 \geq 0$, (value of ϵ_2 can be obtained by continuing the process and getting the values of P_i's).

Thus in our solar system we can get the distance between any planet and Sun. More precisely, if we have any system which has a center like thing (as like sun in our solar system). Thus from these ideas we can get the gaps between two planets in solar systems (or gap between two circular paths, centering one particular thing).

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