A Very Brief Note on the Riemann Hypothesis

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Abstract. Robin's criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We also require the properties of *superabundant numbers*, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. In this note, using Robin's inequality on superabundant numbers, we prove that the Riemann Hypothesis is true.

Keywords: Riemann Hypothesis · Robin's inequality · Sum-of-divisors function · Superabundant numbers · Prime numbers.

1 Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n,

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

Proposition 1. *[3, Lemma 1 pp. 2]. Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \cdots < q_m$ with natural numbers $a_1, \ldots, a_m$ as exponents. Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \right) \times \left(\prod_{i=1}^{m} (1 - \frac{1}{q_i^{a_i+1}}) \right).$$

Proposition 2. *[5, Lemma 2.3 pp. 3]. Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Say $\mathsf{Robin}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log \log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm.

Proposition 3. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [4, Theorem 1 pp. 188].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [2, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 4. *If* n *is superabundant, then* n *is a Hardy-Ramanujan integer [1, Theorem 1 pp. 450].*

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \quad \textit{for} \quad (m > 1).$$

Proposition 5. *Every colossally abundant number is superabundant [1, pp. 455].*

Proposition 6. *If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers* $n > 5040$ *such that* $\mathsf{Robin}(n)$ *fails [4, Proposition pp. 204].*

Putting all together yields the proof of the Riemann Hypothesis.

2 Central Lemma

Lemma 1. *If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers* n *such that* $\mathsf{Robin}(n)$ *fails.*

Proof. This is a direct consequence of Propositions 5 and 6.

3 Main Insight

Lemma 2. *Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of* n *as a product of prime numbers* $q_1 < \cdots < q_m$ *with natural numbers* $a_1, \ldots, a_m$ *as exponents. Then,*

$$\left(\prod_{i=1}^{m}(1 - \frac{1}{q_i^{a_i+1}})\right)^2 < \left(\prod_{i=1}^{m}(1 - \frac{1}{q_i^{a_i+1}}) \times (1 + \frac{1}{q_i^{a_i+1}})\right).$$

Proof. This is trivial since

$$(1 + \frac{1}{q_i^{a_i+1}}) > (1 - \frac{1}{q_i^{a_i+1}})$$

and therefore, the proof is done. □

4 Main Theorem

Theorem 1. *The Riemann Hypothesis is true.*

Proof. Let n be a superabundant number. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of this superabundant number n as the product of the first k consecutive primes $q_1 < \cdots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ as exponents, since n must be a Hardy-Ramanujan integer by Proposition 4. We have to show that

$$f(n) < e^{\gamma} \times \log\log n.$$

Suppose that

$$\frac{e^{\gamma} \times \log\log n}{\left(\prod_{i=1}^{k} \frac{q_i}{q_i-1}\right)} < 1.$$

Hence, it is enough to prove that

$$(f(n))^2 < (e^{\gamma} \times \log\log n)^2.$$

We show that

$$(f(n))^2 = \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right)^2 \times \left(\prod_{i=1}^{k} (1 - \frac{1}{q_i^{a_i+1}})\right)^2$$

by Proposition 1. Therefore,

$$(f(n))^2 < \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right)^2 \times \left(\prod_{i=1}^{k} (1 - \frac{1}{q_i^{a_i+1}}) \times (1 + \frac{1}{q_i^{a_i+1}})\right)$$

by Lemma 2. Thus,

$$(f(n))^2 < f(n) \times \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right) \times \left(\prod_{i=1}^{k} (1 + \frac{1}{q_i^{a_i+1}})\right).$$

We only need to show that

$$f(n) \times \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right) \times \left(\prod_{i=1}^{k} (1 + \frac{1}{q_i^{a_i+1}})\right) < (e^{\gamma} \times \log\log n)^2.$$

So,

$$\frac{f(n)}{e^{\gamma} \times \log\log n} < \left(\prod_{i=1}^{k} \frac{q_i^{a_i+1}}{q_i^{a_i+1} + 1}\right) \frac{e^{\gamma} \times \log\log n}{\left(\prod_{i=1}^{k} \frac{q_i}{q_i-1}\right)}.$$

Under our assumption, we have

$$\left(\prod_{i=1}^{k} \frac{q_i^{a_i+1}}{q_i^{a_i+1} + 1}\right) \frac{e^{\gamma} \times \log\log n}{\left(\prod_{i=1}^{k} \frac{q_i}{q_i-1}\right)} < 1.$$

However, that means that $\mathsf{Robin}(n)$ holds because of

$$\frac{f(n)}{e^{\gamma} \times \log\log n} < 1.$$

Now, suppose the inverse inequality

$$\frac{e^{\gamma} \times \log\log n}{\left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right)} \geq 1$$

that is the same as

$$\left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right) \leq e^{\gamma} \times \log\log n.$$

That implies that $\mathsf{Robin}(n)$ holds since

$$f(n) < \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right)$$

by Proposition 2. Finally, the study of this arbitrarily selected superabundant number n has revealed that $\mathsf{Robin}(n)$ holds on anyway. Accordingly, $\mathsf{Robin}(n)$ holds for all superabundant numbers n. This contradicts the fact that there are infinite superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann Hypothesis is false according to Lemma 1. By reductio ad absurdum, we prove that the Riemann Hypothesis is true. □

5 Conclusions

Practical uses of the Riemann Hypothesis include many propositions that are known to be true under the Riemann Hypothesis, and some that can be shown to be equivalent to the Riemann Hypothesis. Indeed, the Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf Hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

Acknowledgments

The author wishes to thank his mother, maternal brother, maternal aunt, and friends Liuva, Yary, Sonia, and Arelis for their support. The author thanks the reviewers for their valuable suggestions and comments that were helpful in significantly improving the quality of the manuscript.

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