Note on the Riemann Hypothesis

International Conference on Recent Developments in Mathematics (ICRDM 2022)

Frank Vega, CopSonic France

vega.frank@gmail.com

August 24-26, 2022

Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 1984, Guy Robin stated a new criterion for the Riemann Hypothesis. We prove the Riemann Hypothesis is true using the Robin's criterion.

Note

This result is an extension of my article "Robin's criterion on divisibility" published in **The Ramanujan Journal** (03 May 2022).

References

This presentation and references can be found at my *ResearchGate Project*: `https://www.researchgate.net/project/The-Riemann-Hypothesis`.

Introduction

The Riemann Hypothesis is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859).

The Riemann Hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems.

This is one of the Clay Mathematics Institute's Millennium Prize Problems.

Objectives

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n.

Define $f(n)$ as $\frac{\sigma(n)}{n}$.

We provide a proof for the Riemann Hypothesis using the properties of the f function.

Methods

We say that Robin(n) holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm.

The Ramanujan's Theorem stated that if the Riemann Hypothesis is true, then the previous inequality holds for large enough n. Next, we have the Robin's Theorem:

Proposition 1

Robin(n) *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true (Robin, 1984, Theorem 1 pp. 188).*

In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers.

A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer (Choie et al., 2007, pp. 367). If n is superabundant, then n is a Hardy-Ramanujan integer (Alaoglu and Erdős, 1944, Theorem 1 pp. 450).

Several analogues of the Riemann Hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann Hypothesis might be false.

Lemma 2

If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers n such that $\text{Robin}(n)$ *fails (i.e.* $\text{Robin}(n)$ *does not hold).*

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \quad \textit{for} \quad (m > 1).$$

If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that Robin(n) fails (Robin, 1984, Proposition pp. 204). Every colossally abundant number is superabundant (Alaoglu and Erdős, 1944, pp. 455). In this way, the proof is complete. ■

A Strictly Decreasing Sequence

For every prime number q_k, we define the sequence

$$Y_k = \frac{e^{\frac{0.2}{\log^2(q_k)}}}{(1 - \frac{1}{\log(q_k)})}.$$

As the prime number q_k increases, the sequence Y_k is strictly decreasing (Vega, 2022, Lemma 6.1 pp. 6).

A Known Result

We use the following Proposition:

Proposition 3

(Nazardonyavi and Yakubovich, 2013, Lemma 3.3 pp. 8). Let $x \geq 11$*. For* $y > x$*, we have*

$$\frac{\log \log y}{\log \log x} < \sqrt{\frac{y}{x}}.$$

The following is a key Lemma.

Lemma 4

Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents. Suppose that $\mathrm{Robin}(n)$ *fails. Then,*

$$n < \alpha^2 \cdot (N_k)^{Y_k},$$

where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k and $\alpha = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$.

Remark

When n is superabundant and Robin(n) fails, then we have

$$\frac{\log \log n}{\log \log (N_k)^{Y_k}} \leq \alpha$$

as a consequence of the reference (Vega, 2022, Theorem 6.6 pp. 8).

We assume that $(N_k)^{Y_k} > n > 5040 > 11$ since $0 < \alpha < 1$. Consequently,

$$\sqrt{\frac{n}{(N_k)^{Y_k}}} < \frac{\log \log n}{\log \log (N_k)^{Y_k}}.$$

As result, we obtain that

$$n < \alpha^2 \cdot (N_k)^{Y_k}$$

and therefore, the proof is done. ■

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We also use the following Proposition:

Proposition 5

(Nazardonyavi and Yakubovich, 2013, Proposition 4.12. pp. 14). For large enough superabundant number n

$$\log n < 2^{\nu_2(n)}.$$

A proof of the Riemann Hypothesis is considered as the Holy Grail of Mathematics by several authors.

Theorem 6

The Riemann Hypothesis is true.

We know there are infinitely many superabundant numbers by the reference (Alaoglu and Erdős, 1944, Theorem 9 pp. 454).

For every prime q, $\nu_q(n)$ goes to infinity as long as n goes to infinity when n is superabundant according to the references (Nazardonyavi and Yakubovich, 2013, Theorem 4.4 pp. 12) and (Alaoglu and Erdős, 1944, Theorem 7 pp. 454).

Since Y_k is strictly decreasing and $0 < \alpha^2 < 1$, then we deduce that the following inequality $n \geq \alpha^2 \cdot (N_k)^{Y_k}$ is always satisfied for a sufficiently large superabundant number n.

Let n_k be a superabundant number such that q_k is the largest prime factor of n, then

$$\lim_{k \to \infty} \frac{n_k}{N_k} = \infty,$$

where N_k is the primorial number of order k. Certainly, for large enough superabundant number n_k, we can see that $\frac{n_k}{N_k} > 2^{\nu_2(n_k)} > \log n_k$. Hence, it is enough to show that

$$\lim_{k \to \infty} \log n_k = \infty.$$

Moreover, we would have

$$\lim_{k \to \infty} \frac{(N_k)^{Y_k}}{N_k} = 1,$$

since we only need to verify that $\lim_{k \to \infty} Y_k = 1$.

Accordingly, Robin(n) holds for all large enough superabundant numbers n.

This contradicts the fact that there are infinite superabundant numbers n, such that Robin(n) fails when the Riemann Hypothesis is false.

By reductio ad absurdum, we prove that the Riemann Hypothesis is true. ■

The Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Large Prime Gap Conjecture, etc.

Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

References

Leonidas Alaoglu and Paul Erdős. On highly composite and similar numbers. *Transactions of the American Mathematical Society*, 56(3):448–469, 1944. URL https://doi.org/10.2307/1990319.

YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19 (2):357–372, 2007. URL https://doi.org/10.5802/jtnb.591.

Sadegh Nazardonyavi and Semyon Yakubovich. Superabundant numbers, their subsequences and the Riemann hypothesis. *arXiv preprint arXiv:1211.2147v3*, 2013. version 3 (Submitted on 26 Feb 2013). arXiv URL https://arxiv.org/pdf/1211.2147v3.pdf.

Guy Robin. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl*, 63(2):187–213, 1984. URL http://zakuski.utsa.edu/~jagy/Robin_1984.pdf.

Frank Vega. Robin's criterion on divisibility. *The Ramanujan Journal*, pages 1–11, 2022. URL https://doi.org/10.1007/s11139-022-00574-4.