ON SOME PROPERTIES OF DETERMINANTS OF BICOMPLEX MATRICES

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ABSTRACT. In this paper, we have studied determinants of bicomplex matrices and investigated their properties. We have introduced bicomplex symmetric matrix, bicomplex skew-symmetric matrix, bicomplex idempotent matrix, bicomplex skew-idempotent matrix, bicomplex involutory matrix, bicomplex skew-involutory matrix, three types of Hermetian and skew-Hermetian matrix, bicomplex orthogonal matrix and three types of unitary matrix and investigated the properties of their determinants.

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1. INTRODUCTION

Throughout this paper, the set of Bicomplex numbers is denoted by $\mathbb{C}_2$ and the sets of complex and real numbers are denoted by $\mathbb{C}_1$ and $\mathbb{C}_0$,respectively. For detail of the theory (cf. [1, 2, 3]).

Definition 1.1 (Bicomplex numbers). The set of bicomplex numbers defined as:

$$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

where $i_1 \neq i_2, i_1^2 = i_2^2 = -1$ and, $i_1i_2 = i_2i_1$.

We shall use the notations $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ for the following sets:
$\mathbb{C}(i_1) = \{x + i_1y : x, y \in \mathbb{C}_0\}$
$\mathbb{C}(i_2) = \{x + i_2y : x, y \in \mathbb{C}_0\}$

Definition 1.2 (Hyperbolic numbers). The set of Hyperbolic Numbers is defined as

$$\mathbb{H} = \{x + i_1i_2y : x, y \in \mathbb{C}_0\}$$

Definition 1.3 (Idempotent elements). Besides 0 and 1, there are exactly two non-trivial idempotent elements exist in $\mathbb{C}_2$, denoted as e_1 and e_2 and defined as $e_1 = \frac{1+i_1i_2}{2}$ and $e_2 = \frac{1-i_1i_2}{2}$. Note that $e_1 + e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$.

Definition 1.4 (Cartesian idempotent set). [4] Cartesian idempotent set X determined by X_1 and X_2 is denoted as $X_1 \times_e X_2$ and is defined as

$$X = X_1 \times_e X_2 = \{\xi \in X : \xi = ae_1 + be_2, (a, b) \in X_1 \times X_2\}$$

(i) The Cartesian idempotent set $\mathbb{C}_2$ determined by $\mathbb{C}(i_1)$ is given as :

$$\begin{aligned}\mathbb{C}_2 &= \mathbb{C}(i_1) \times_e \mathbb{C}(i_1) = \mathbb{C}(i_1)e_1 + \mathbb{C}(i_1)e_2 \\ &= \{\xi \in \mathbb{C}_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2, ({}^1\xi, {}^2\xi) \in \mathbb{C}(i_1) \times \mathbb{C}(i_1)\}\end{aligned}$$

(ii) The Cartesian idempotent set $\mathbb{C}_2$ determined by $\mathbb{C}(i_2)$ is given as :

$$\begin{aligned}\mathbb{C}_2 &= \mathbb{C}(i_2) \times_e \mathbb{C}(i_2) = \mathbb{C}(i_2)e_1 + \mathbb{C}(i_2)e_2 \\ &= \{\xi \in \mathbb{C}_2 : \xi = \xi_1 e_1 + \xi_2 e_2, (\xi_1, \xi_2) \in \mathbb{C}(i_2) \times \mathbb{C}(i_2)\}\end{aligned}$$

1.1. **Idempotent Representation of Bicomplex Numbers.** The bicomplex numbers can be represented in two different idempotent forms w.r.t. the elements from $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$, explained as follows :

(a) The $\mathbb{C}(i_1)$– idempotent representation of Bicomplex Numbers is given by

$$\begin{aligned}\xi &= x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4) \\ &= z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2,\end{aligned}$$

where, ${}^1\xi = z_1 - i_1z_2 = (x_1 + x_4) + i_1(x_2 - x_3)$ and
${}^2\xi = z_1 + i_1z_2 = (x_1 - x_4) + i_1(x_2 + x_3) \in \mathbb{C}(i_1)$

(b) The $\mathbb{C}(i_2)$–idempotent representation of Bicomplex Numbers is given by

$$\begin{aligned}\xi &= x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = (x_1 + i_2x_3) + i_1(x_2 + i_2x_4) \\ &= w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1e_1 + \xi_2e_2,\end{aligned}$$

where,
$\xi_1 = w_1 - i_2w_2 = (x_1 + x_4) - i_2(x_2 - x_3)$,
$\xi_2 = w_1 + i_2w_2 = (x_1 - x_4) + i_2(x_2 + x_3) \in \mathbb{C}(i_2)$

1.2. **Singular Elements in $\mathbb{C}_2$:** Let $\xi,\eta \in \mathbb{C}_2$ such that $\xi\eta = \eta\xi = 1$, then η is said to be a multiplicative inverse of ξ. The invertible elements are also called as non-singular element. The set of all singular elements in $\mathbb{C}_2$ is denoted as $\mathbb{O}_2$ and $\mathbb{C}_2 \setminus \mathbb{O}_2$ is the set of all non-singular elements in $\mathbb{C}_2$.

1.2.1. *Singular Elements in* $\mathbb{C}_2$. We are providing some conditions for the singularity of the bicomplex number as follows:

(i) $\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2$ is singular if and only if $z_1^2 + z_2^2 = 0$ or ($z_1 - i_1z_2 = 0$ or $z_1 + i_1z_2 = 0$) or (${}^1\xi = 0$ or ${}^2\xi = 0$)

(ii) $\xi = w_1 + i_1w_2 = (w_1 - i_2w_2)e_1 + (w_1 + i_2w_2)e_2 = \xi_1e_1 + \xi_2e_2$ is singular if and only if $(w_1^2 + w_2^2 = 0)$ or $(w_1 - i_2w_2 = 0$ or $w_1 + i_2w_2 = 0)$ or $(\xi_1 = 0$ or $\xi_2 = 0)$

Definition 1.5 (**Principal Ideals in $\mathbb{C}_2$**). There are two principal ideals in $\mathbb{C}_2$ viz., $\mathbb{I}_1$ and $\mathbb{I}_2$ defined as

$$\begin{aligned}\mathbb{I}_1 &= \{\xi e_1 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2\} = \{{}^1\xi e_1 : {}^1\xi \in \mathbb{C}(i_1)\}\\ \mathbb{I}_2 &= \{\xi e_2 : \xi = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2\} = \{{}^2\xi e_2 : {}^2\xi \in \mathbb{C}(i_1)\}\end{aligned}$$

Note that $\mathbb{I}_1 \cap \mathbb{I}_2 = \{0\}$ and $\mathbb{I}_1 \cup \mathbb{I}_2 = \mathbb{O}_2$.

Definition 1.6 (**Zero-divisors**). Two non-zero bicomplex numbers ξ and η are said to be zero-divisors if $\xi\,\eta = 0$.

The relation $e_1 e_2 = e_2 e_1 = 0$ establishes the existence of zero divisors in $\mathbb{C}_2$.

Proposition 1.1. *The bicomplex numbers ξ and η are divisors of zero if and only if $\xi \in \mathbb{I}_1 \setminus \{0\}$ and $\eta \in \mathbb{I}_2 \setminus \{0\}$.*

Definition 1.7 (**Norm on the bicomplex space**).

(i) Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2 = {}^1\xi e_1 + {}^2\xi e_2 \in \mathbb{C}_2$. Then the norm in $\mathbb{C}_2$ is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}}$$

(ii) Let $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = w_1 + i_1 w_2 = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2$. Then the norm in $\mathbb{C}_2$ is defined as

$$\|\xi\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|w_1|^2 + |w_2|^2} = \sqrt{\frac{|\xi_1|^2 + |\xi_2|^2}{2}}$$

Note that $\mathbb{C}_2$ becomes a modified Banach algebra, in the sense that for any $\xi, \eta \in \mathbb{C}_2$,

$$\|\xi\,\eta\| \leq \sqrt{2}\,\|\xi\|\,\|\eta\|.$$

1.3. **Conjugations of Bicomplex Numbers.** There are three different types of conjugations given on the bicomplex space and are explained as follows [3]:

(a) i_1-Conjugation

$$\begin{aligned}\xi^* &= (z_1 + i_2 z_2)^* = \overline{z_1} + i_2\overline{z_2}, \forall\ z_1, z_2 \in \mathbb{C}(i_1)\\ &= ({}^1\xi e_1 + {}^2\xi e_2)^* = \overline{{}^2\xi} e_1 + \overline{{}^1\xi} e_2, \forall\ {}^1\xi, {}^2\xi \in \mathbb{C}(i_1)\\ &= (w_1 + i_1 w_2)^* = w_1 - i_1 w_2, \forall\ w_1, w_2 \in \mathbb{C}(i_2)\\ &= (\xi_1 e_1 + \xi_2 e_2)^* = \xi_2 e_1 + \xi_1 e_2, \forall\ \xi_1, \xi_2 \in \mathbb{C}(i_2)\end{aligned}$$

(b) i_2-Conjugation

$$\begin{aligned}
\xi^{\#} &= (z_1 + i_2 z_2)^{\#} = z_1 - i_2 z_2, \forall\ z_1, z_2 \in \mathbb{C}(i_1) \\
&= ({}^1\xi e_1 + {}^2\xi e_2)^{\#} = {}^2\xi e_1 + {}^1\xi e_2, \forall\ {}^1\xi, {}^2\xi \in \mathbb{C}(i_1) \\
&= (w_1 + i_1 w_2)^{\#} = \overline{w_1}\ + i_1 \overline{w_2}\ , \forall\ w_1, w_2 \in \mathbb{C}(i_2) \\
&= (\xi_1 e_1 + \xi_2 e_2)^{\#} = \overline{\xi_2}\ e_1 + \overline{\xi_1}\ e_2, \forall\ \xi_1, \xi_2 \in \mathbb{C}(i_2).
\end{aligned}$$

(c) $i_1 i_2$-Conjugation

$$\begin{aligned}
\xi^{'} &= (z_1 + i_2 z_2)^{'} = \overline{z_1}\ - i_2 \overline{z_2}, \forall\ z_1, z_2 \in \mathbb{C}(i_1) \\
&= ({}^1\xi e_1 + {}^2\xi e_2)^{'} = \overline{{}^1\xi}\ e_1 + \overline{{}^2\xi} e_2, \forall\ {}^1\xi, {}^2\xi \in \mathbb{C}(i_1) \\
&= (w_1 + i_1 w_2)^{'} = \overline{w_1}\ - i_1 \overline{w_2}, \forall\ w_1, w_2 \in \mathbb{C}(i_2) \\
&= (\xi_1 e_1 + \xi_2 e_2)^{'} = \overline{\xi_1}\ e_1 + \overline{\xi_2} e_2, \forall\ \xi_1, \xi_2 \in \mathbb{C}(i_2)
\end{aligned}$$

2. Bicomplex Matrix

In this section, we discussed about the bicomplex matrices along with their properties and some results. We established the results with respect to the conjugations of the bicomplex matrices [5, 6, 7].

Here, we denote $\mathbb{C}_2^{m\times n} = \{A = [\xi_{ij}]_{m\times n} : \xi_{ij} \in \mathbb{C}_2\}$ as the set of $m \times n$ matrices with bicomplex entries. Let $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n} \Rightarrow \xi_{ij} \in \mathbb{C}_2$

2.1. **Idempotent Representation of Bicomplex Matrix.** We can represent the Bicomplex Matrix in two different idempotent forms in terms of $\mathbb{C}^{m\times n}(i_1)$ and $\mathbb{C}^{m\times n}(i_2)$, explained as follows :

(i) The $\mathbb{C}^{m\times n}(i_1)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ is given by

$$A = [\xi_{ij}]_{m\times n} = [{}^1\xi_{ij}]_{m\times n} e_1 + [{}^2\xi_{ij}]_{m\times n} e_2 = {}^1A e_1 + {}^2A e_2,$$

where ${}^1A = [{}^1\xi_{ij}]_{m\times n}, {}^2A = [{}^2\xi_{ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_1)$

(ii) The $\mathbb{C}^{m\times n}(i_2)$-idempotent representation of Bicomplex Matrix $A = [\xi_{ij}]_{m\times n} \in \mathbb{C}_2^{m\times n}$ is given by

$$A = [\xi_{ij}]_{m\times n} = [\xi_{1,ij}]_{m\times n} e_1 + [\xi_{2,ij}]_{m\times n} e_2 = A_1 e_1 + A_2 e_2,$$

where $A_1 = [\xi_{1,ij}]_{m\times n}, A_2 = [\xi_{2,ij}]_{m\times n} \in \mathbb{C}^{m\times n}(i_2)$

2.2. **determinant of Bicomplex Matrices.** As, only square matrices can have determinant, so let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$. Then determinant of A is denoted as $\det(A)$.

Theorem 2.1. *Let $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$,then its determinant is given by*
$\det(A) = \det({}^1A)e_1 + \det({}^2A)e_2$ *or* $\det(A) = \det(A_1)e_1 + \det(A_2)e_2$.

Corollary 2.2. *For* $A, B \in \mathbb{C}_2^{n\times n}$, $\det(AB) = \det(A)\det(B)$.

2.3. **Algebraic Structure of Bicomplex Matrices.** Bicomplex Matrices have the following algebraic structures.

$\mathbb{C}_2^{m\times n}(\mathbb{C}_0), \mathbb{C}_2^{m\times n}(\mathbb{C}(i_1))$ and $\mathbb{C}_2^{m\times n}(\mathbb{C}(i_2))$ are linear space; $\mathbb{C}_2^{m\times n}(\mathbb{C}_2)$ is a $\mathbb{C}_2$-module; $\mathbb{C}_2^{n\times n}(\mathbb{C}_0), \mathbb{C}_2^{n\times n}(\mathbb{C}(i_1))$ and $\mathbb{C}_2^{n\times n}(\mathbb{C}(i_2))$ are algebra.

Theorem 2.3. $A = {}^1Ae_1 + {}^2Ae_2 = A_1e_1 + A_2e_2 \in \mathbb{C}_2^{n\times n}$ *is invertible iff* ${}^1A$ *and* ${}^2A$ *are invertible in* $\mathbb{C}^{n\times n}(i_1)$ *and* A_1 *and* A_2 *are invertible in* $\mathbb{C}^{n\times n}(i_2)$.

Corollary 2.4. $A \in \mathbb{C}_2^{n\times n}$ *is invertible iff* $\det(A) \notin \mathbb{O}_2$.

2.4. **Non-Singular and Singular Bicomplex Matrix.**

A matrix $A = [\xi_{ij}]_{n\times n} \in \mathbb{C}_2^{n\times n}$ is said to be Non-Singular (Invertible) if $\det(A) \notin \mathbb{O}_2$. It is said to be Singular (Non-invertible) matrix if $\det(A) \in \mathbb{O}_2$.

2.5. **Conjugation of Bicomplex Matrix.** Here, we have given three different types of conjugations of the bicomplex matrices. Let $A = [\xi]_{m\times n} \in \mathbb{C}_2^{n\times n}$ be a bicomplex matrix. Then, its conjugations are defined as follows:

(a) i_1**-Conjugation**

$A^* = ({}^1Ae_1 + {}^2Ae_2)^* = \overline{{}^2A}e_1 + \overline{{}^1A}e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^* = (A_1e_1 + A_2e_2)^* = A_2e_1 + A_1e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(b) i_2**-Conjugation**

$A^\# = ({}^1Ae_1 + {}^2Ae_2)^\# = {}^2Ae_1 + {}^1Ae_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A^\# = (A_1e_1 + A_2e_2)^\# = \overline{A}_2e_1 + \overline{A}_1e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

(c) i_1i_2**-Conjugation**

$A' = ({}^1Ae_1 + {}^2Ae_2)' = \overline{{}^1A}e_1 + \overline{{}^2A}e_2$ when ${}^1A, {}^2A \in \mathbb{C}^{m\times n}(i_1)$,

$A' = (A_1e_1 + A_2e_2)' = \overline{A}_1e_1 + \overline{A}_2e_2$ when $A_1, A_2 \in \mathbb{C}^{m\times n}(i_2)$.

2.6. **Adjoint of a Bicomplex Square Matrix.**

Let $A = {}^1Ae_1 + {}^2Ae_2 = A_1e_1 + A_2e_2 \in \mathbb{C}_2^{n\times n}$. Then adjoint of the bicomplex matrix A is given by $adj(A) = adj({}^1A)e_1 + adj({}^2A)e_2$ or $adj(A) = adj(A_1)e_1 + adj(A_2)e_2$.

Theorem 2.5. *If* A *is a bicomplex square matrix,* $A.adj(A) = adj(A).A = \det(A).I$.

Theorem 2.6 (Inverse of a Bicomplex Square Matrix)**.** *Let* A *be a non-singular bicomplex square matrix of order* n*, then* $A^{-1} = \frac{1}{\det(A)}adj(A)$.

Theorem 2.7. *Let* A *be a non-singular bicomplex square matrix of order* n*, then*

$A^{-1} = \frac{1}{\det({}^1A)}adj({}^1A)e_1 + \frac{1}{\det({}^2A)}adj({}^2A)e_2$ *or*

$A^{-1} = \frac{1}{\det(A_1)}adj(A_1)e_1 + \frac{1}{\det(A_2)}adj(A_2)e_2$.

Theorem 2.8. *Let* A *be a non-singular bicomplex square matrix of order* n*, then*

$A^{-1} = ({}^1A)^{-1}e_1 + ({}^2A)^{-1}e_2$ *or* $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$.

Theorem 2.9. *Let A be a non-singular bicomplex square matrix of order n, then following holds:*

(i) A *is invertible.*
(ii) $A^{-1} = \frac{1}{\det(A)} adj(A)$
(iii) $A^{-1} = \frac{1}{\det(^1A)} adj(^1A)e_1 + \frac{1}{\det(^2A)} adj(^2A)e_2$
(iv) $A^{-1} = \frac{1}{\det(A_1)} adj(A_1)e_1 + \frac{1}{\det(A_2)} adj(A_2)e_2$.
(v) $A^{-1} = (^1A)^{-1}e_1 + (^2A)^{-1}e_2$
(vi) $A^{-1} = (A_1)^{-1}e_1 + (A_2)^{-1}e_2$
(vii) $\det(A) \notin \mathbb{O}_2$
(viii) $\det(^1A) \neq 0$ *and* $\det(^2A) \neq 0$
(ix) $\det(A_1) \neq 0$ *and* $\det(A_2) \neq 0$
(x) $\det(A^{-1}) = \frac{1}{\det(A)}$
(xi) $\det(A^{-1}) = \frac{1}{\det(^1A)}e_1 + \frac{1}{\det(^2A)}e_2$
(xii) $\det(A^{-1}) = \frac{1}{\det(A_1)}e_1 + \frac{1}{\det(A_2)}e_2$

Corollary 2.10. *A matrix $A \in \mathbb{C}_2^{n\times n}$, then following conditions are equivalent*

(i) A *is singular.*
(ii) $\det(A) \in \mathbb{O}_2$.
(iii) $\det(^1A) = 0$ *or* $\det(^2A) = 0$.
(iv) $\det(A_1) = 0$ *or* $\det(A_2) = 0$.

3. determinant of Some Special Bicomplex Matrices

In this section, we defined and studied some special type of bicomplex square matrices and their determinant.

Definition 3.1 (**Symmetric Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be symmetric matrix if $A^t = A$.

Definition 3.2 (**Skew-Symmetric Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew-Symmetric matrix if $A^t = -A$.

Definition 3.3 (**Bicomplex Idempotent Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be idempotent matrix if $A^2 = A$.

Definition 3.4 (**Bicomplex Skew-Idempotent Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew-idempotent matrix if $A^2 = -A$.

Definition 3.5 (**Bicomplex Involutory Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be involutory matrix if $A^2 = I$.

Definition 3.6 (**Bicomplex Skew-Involutory Matrix**)**.** A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be Skew- involutory matrix if $A^2 = -I$.

Definition 3.7 (**Bicomplex Hermitian Matrix**)**.** There are three types of Bicomplex Hermitian Matrix, defined as follows:

(i) i_1-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -Hermitian matrix if $(A^t)^* = (A^*)^t = A$.

(ii) i_2-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -Hermitian matrix if $(A^t)^\# = (A^\#)^t = A$

(iii) i_1i_2-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1i_2 -Hermitian matrix if $(A^t)' = (A')^t = A$.

Note 3.1. There are some observations as given follows:

(a) If A is i_1-Hermitian Matrix then $A^* = A^t$.
(b) If A is i_2-Hermitian matrix then $A^\# = A^t$.
(c) If A is i_1i_2-Hermitian matrix then $A' = A^t$.

Definition 3.8 (**Bicomplex Skew- Hermitian Matrix**)**.** There are three types of bicomplex skew-Hermitian matrix, defined as follows:

(i) i_1-Skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -skew Hermitian matrix if $(A^t)^* = (A^*)^t = -A$

(ii) i_2-skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -skew Hermitian matrix if $(A^t)^\# = (A^\#)^t = -A$

(iii) i_1i_2-Skew-Hermitian Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1i_2-skew Hermitian matrix if $(A^t)' = (A')^t = -A$.

Note 3.2. Some observations about the definitions are given as follows:

(a) If A is i_1-skew-Hermitian matrix then $A^* = -A^t$.
(b) If A is i_2-skew-Hermitian matrix then $A^\# = -A^t$.
(c) If A is i_1i_2-skew-Hermitian matrix then $A' = -A^t$.

Definition 3.9 (**Bicomplex Orthogonal Matrix**)**.** A bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be orthogonal matrix if $A^tA = AA^t = I$.

Definition 3.10 (**Bicomplex Unitary Matrix**)**.** There are three types of Bicomplex Unitary Matrix, defined as follows:

(i) i_1-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1 -Unitary Matrix if

$A^{*t}A = A^{*t}A = I.$

(ii) i_2-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_2 -Unitary Matrix if
$A^{\#t}A = A^{\#t}A = I.$

(iii) i_1i_2-Unitary Matrix

A Bicomplex matrix $A \in \mathbb{C}_2^{n\times n}$ is said to be i_1i_2 -Unitary Matrix if
$A'^{t}A = A'^{t}A = I.$

Proposition 3.1. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-symmetric matrix of odd order. Then* $\det(A) = 0$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a Skew-symmetric matrix of odd order
$\Rightarrow A^t = -A$
$\Rightarrow \det(A^t) = \det(-A)$
$\Rightarrow \det(A) = (-1)^n \det(A)$ ($\because \det(A^t) = \det(A)$)
$\Rightarrow \det(A) = -\det(A)$ ($\because (-1)^n = -1$,n is odd)
$\Rightarrow 2\det(A) = 0.$
$\Rightarrow \det(A) = 0.$ □

Proposition 3.2. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex idempotent matrix. Then* $\det(A) \in \{0, 1, e_1, e_2\}$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be an idempotent matrix
$\Rightarrow A^2 = A$
$\Rightarrow \det(A^2) = \det(A)$
$\Rightarrow \det(AA) = \det(A)$
$\Rightarrow \det(A)\det(A) = \det(A)$
$\Rightarrow \{\det(A)\}^2 = \det(A)$
$\Rightarrow \{\det(^1A)e_1 + \det(^2A)e_2\}^2 = \det(^1A)e_1 + \det(^2A)e_2$
$\Rightarrow \{\det(^1A)\}^2e_1 + \{\det(^2A)\}^2e_2 = \det(^1A)e_1 + \det(^2A)e_2$
$\Rightarrow \{\det(^1A)\}^2 = \det(^1A)$ and $\{\det(^2A)\}^2 = \det(^2A)$
$\Rightarrow \det(^1A) = 0, 1$ and $\det(^2A) = 0, 1$
Hence $\det(A) \in \{0, 1, e_1, e_2\}$. □

Note 3.3. If A is a non-singular bicomplex idempotent matrix then $A = I$.

Proposition 3.3. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex skew-idempotent matrix. Then* $\det(A) \in \{0, 1, e_1, e_2\}$ *when n is even and* $\det(A) \in \{0, -1, -e_1, -e_2\}$ *when n is odd.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex skew-idempotent matrix
$\Rightarrow A^2 = -A$
$\Rightarrow \det(A^2) = \det(-A)$
$\Rightarrow \det(AA) = (-1)^n \det(A)$

$\Rightarrow \det(A)\det(A) = (-1)^n \det(A)$
$\Rightarrow \{\det(A)\}^2 = (-1)^n \det(A)$
Case(i) When n is even.
$\Rightarrow \{\det(A)\}^2 = \det(A)$ ($\because (-1)^n = 1, n$ is even)
The result directly follows Proposition 3.2.
Case(ii) When n is odd.
$\Rightarrow \{\det(A)\}^2 = -\det(A)$ ($\because (-1)^n = -1, n$ is odd)
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$
$\Rightarrow \{\det({}^1A)\}^2 = -\det({}^1A)$ and $\{\det({}^2A)\}^2 = -\det({}^2A)$
$\Rightarrow \det({}^1A) = 0, -1$ and $\det({}^2A) = 0, -1$
Hence $\det(A) \in \{0, -1, -e_1, -e_2\}$. □

Note 3.4. If A is a non-singular Bicomplex Skew-idempotent matrix then $A = -I$.

Proposition 3.4. *Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$. Then* $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta A, \eta \in \mathbb{C}_2$
$\Rightarrow A^2 = \eta A$
$\Rightarrow \det(A^2) = \det(\eta A)$
$\Rightarrow \det(AA) = (\eta)^n \det(A)$
$\Rightarrow \det(A)\det(A) = (\eta)^n \det(A))$
$\Rightarrow \{\det(A)\}^2 = (\eta)^n \det(A)$
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = \{{}^1\eta e_1 + {}^2\eta e_2\}^n\{\det({}^1A)e_1 + \det({}^2A)e_2\}$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = \{({}^1\eta)^n e_1 + ({}^2\eta)^n e_2\}\{\det({}^1A)e_1 + \det({}^2A)e_2\}$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = ({}^1\eta)^n \det({}^1A)e_1 + ({}^2\eta)^n \det({}^2A)e_2$
$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n \det({}^1A)$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n \det({}^2A)$
$\Rightarrow \det({}^1A) = 0, ({}^1\eta)^n$ and $\det({}^2A) = 0, ({}^2\eta)^n$
Hence $\det(A) \in \{0, (\eta)^n, ({}^1\eta)^n e_1, ({}^2\eta)^n e_2\}$. □

Note 3.5. **(i)** For $\eta = 1$, we get Proposition 3.2.
(ii) For $\eta = -1$, we get Proposition 3.3.
(iii) If A is non-singular, then $A = \eta I$

Proposition 3.5. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex involutory matrix. Then* $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

Proof. Let A be any bicomplex involutory matrix
$\Rightarrow A^2 = I$
$\Rightarrow \det(A^2) = \det(I)$
$\Rightarrow \det(AA) = 1$

$\Rightarrow \det(A)\det(A) = 1$
$\Rightarrow \{\det(A)\}^2 = 1$
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = 1e_1 + 1e_2$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = 1e_1 + 1e_2$
$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$
$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$
Hence $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$.

□

Note 3.6. Bicomplex involutory matrices are always non-singular and $A^{-1} = A$.

Proposition 3.6. *Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix. Then* $\det(A) \in \{1, -1, i_1 i_2, -i_1 i_2\}$ *when n is even and* $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$ *when n is odd.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a Bicomplex Skew-involutory matrix
$\Rightarrow A^2 = -I$
$\Rightarrow \det(A^2) = \det(-I)$
$\Rightarrow \det(AA) = (-1)^n$
$\Rightarrow \det(A)\det(A) = (-1)^n$
$\Rightarrow \{\det(A)\}^2 = (-1)^n$
Case(i) When n is even
$\Rightarrow \{\det(A)\}^2 = 1$ ($\because (-1)^n = 1, n$ is even)
The result directly follows Proposition 3.5.
Case(ii) When n is odd
$\Rightarrow \{\det(A)\}^2 = -1$ ($\because (-1)^n = -1, n$ is odd)
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = -1e_1 - 1e_2$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = -1e_1 - 1e_2$
$\Rightarrow \{\det({}^1A)\}^2 = -1$ and $\{\det({}^2A)\}^2 = -1$
$\Rightarrow \det({}^1A) = i_1, -i_1$ and $\det({}^2A) = i_1, -i_1$
Hence $\det(A) \in \{i_1, -i_1, i_2, -i_2\}$. □

Note 3.7. Bicomplex skew-involutory matrices are always non-singular and $A^{-1} = -A$.

Proposition 3.7. *Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta I, \eta \in \mathbb{C}_2$. Then* $\det(A) \in \{\sqrt{({}^1\eta)^n}e_1 + \sqrt{({}^2\eta)^n}e_2, -\sqrt{({}^1\eta)^n}e_1 - \sqrt{({}^2\eta)^n}e_2, \sqrt{({}^1\eta)^n}e_1 - \sqrt{({}^2\eta)^n}e_2, -\sqrt{({}^1\eta)^n}e_1 + \sqrt{({}^2\eta)^n}e_2\}$

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ such that $A^2 = \eta I, \eta \in \mathbb{C}_2$
$\Rightarrow A^2 = \eta I$
$\Rightarrow \det(A^2) = \det(\eta I)$
$\Rightarrow \det(AA) = (\eta)^n \det(I)$

$\Rightarrow \det(A)\det(A) = (\eta)^n$
$\Rightarrow \{\det(A)\}^2 = (\eta)^n$
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = ({}^1\eta e_1 + {}^2\eta e_2)^n$
$\Rightarrow \{\det({}^1A)\}^2 e_1 + \{\det({}^2A)\}^2 e_2 = ({}^1\eta)^n e_1 + ({}^2\eta)^n e_2$
$\Rightarrow \{\det({}^1A)\}^2 = ({}^1\eta)^n$ and $\{\det({}^2A)\}^2 = ({}^2\eta)^n$
$\Rightarrow \det({}^1A) = \sqrt{({}^1\eta)^n}, -\sqrt{({}^1\eta)^n}$ and $\det({}^2A) = \sqrt{({}^1\eta)^n}, -\sqrt{({}^1\eta)^n}$
Hence
$\det(A) \in \{\sqrt{({}^1\eta)^n}e_1+\sqrt{({}^2\eta)^n}e_2, -\sqrt{({}^1\eta)^n}e_1-\sqrt{({}^2\eta)^n}e_2, \sqrt{({}^1\eta)^n}e_1-\sqrt{({}^2\eta)^n}e_2, -\sqrt{({}^1\eta)^n}e_1+\sqrt{({}^2\eta)^n}e_2\}$

□

Note 3.8. (i) For $\eta = 1$, the Proposition 3.5 holds.
(ii) For $\eta = -1$, the Proposition 3.6 holds.

Proposition 3.8. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1- Hermitian matrix. Then*
(i) $\overline{\det({}^2A)} = \det({}^1A)$ *or equivalently* $\overline{\det({}^1A)} = \det({}^2A)$.
(ii) $\det(A) = \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2$.
(iii) $\det(A_1) = \det(A_2)$.
(iv) $\det(A) = \det(A_1) = \det(A_2)$.
(v) $\det(A) \in \mathbb{C}(i_2)$.

Proof.
(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1-Hermitian matrix
$\Rightarrow A^* = A^t$
$\Rightarrow \det(A^*) = \det(A^t)$
$\Rightarrow \det(A^*) = \det(A)$ $(\because \det(A^t) = \det(A))$
$\Rightarrow [\det(A)]^* = \det(A)$ $(\because \det(A^*) = [\det(A)]^*)$
$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = \det({}^1A)e_1 + \det({}^2A)e_2$
$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = \det({}^1A)e_1 + \det({}^2A)e_2$
$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = \det({}^1A)e_1 + \det({}^2A)e_2$
$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ and $\overline{\det({}^1A)} = \det({}^2A)$
$\Rightarrow \overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$
(ii) From(i)
$\overline{\det({}^2A)} = \det({}^1A)$ or equivalently $\overline{\det({}^1A)} = \det({}^2A)$
$\Rightarrow \det({}^1A) = \overline{\det({}^2A)}$ and $\det({}^2A) = \overline{\det({}^1A)}$
$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2$
$\Rightarrow \det(A) = \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2$
(iii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1-Hermitian matrix
$\Rightarrow A^* = A^t$
$\Rightarrow \det(A^*) = \det(A^t)$

$\Rightarrow \det(A^*) = \det(A)$ ($\because \det(A^t) = \det(A)$)

$\Rightarrow [\det(A)]^* = \det(A)$ ($\because \det(A^*) = [\det(A)]^*$)

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_1)e_2 + \det(A_2)e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \det(A_2) = \det(A_1)$ and $\det(A_1) = \det(A_2)$

$\Rightarrow \det(A_1) = \det(A_2)$

(iv) From (iii)

$\det(A_1) = \det(A_2)$

$\Rightarrow \det(A) = \det(A_1) = \det(A_2)$

(v) From (iv) $\det(A) = \det(A_1) = \det(A_2)$.

Also, $\det(A_1) = \det(A_2) \in \mathbb{C}(i_2)$

$\Rightarrow \det(A) \in \mathbb{C}(i_2)$. □

Proposition 3.9. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2-Hermitian matrix. Then*

(i) $\det(^1A) = \det(^2A)$

(ii) $\det(A) = \det(^1A) = \det(^2A)$

(iii) $\overline{\det(A_2)} = \det(A_1)$ *or equivalently* $\overline{\det(A_1)} = \det(A_2)$

(iv) $\det(A) = \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2$

(v) $\det(A) \in \mathbb{C}(i_1)$.

Proof.

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2-Hermitian matrix

$\Rightarrow A^{\#} = A^t$

$\Rightarrow \det(A^{\#}) = \det(A^t)$

$\Rightarrow \det(A^{\#}) = \det(A)$ ($\because \det(A^t) = \det(A)$)

$\Rightarrow [\det(A)]^{\#} = \det(A)$ ($\because \det(A^{\#}) = [\det(A)]^{\#}$)

$\Rightarrow [\det(^1A)e_1 + \det(^2A)e_2]^{\#} = \det(^1A)e_1 + \det(^2A)e_2$

$\Rightarrow \det(^1A)e_2 + \det(^2A)e_1 = \det(^1A)e_1 + \det(^2A)e_2$

$\Rightarrow \det(^2A)e_1 + \det(^1A)e_2 = \det(^1A)e_1 + \det(^2A)e_2$

$\Rightarrow \det(^2A) = \det(^1A)$ and $\det(^1A) = \det(^2A)$

$\Rightarrow \det(^1A) = \det(^2A)$

(ii) From(i) $\det(^1A) = \det(^2A)$

$\Rightarrow \det(A) = \det(^1A) = \det(^2A)$

(iii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2-Hermitian matrix

$\Rightarrow A^{\#} = A^t$

$\Rightarrow \det(A^{\#}) = \det(A^t)$

$\Rightarrow \det(A^{\#}) = \det(A)$ ($\because \det(A^t) = \det(A)$)

$\Rightarrow [\det(A)]^{\#} = \det(A)$ ($\because \det(A^{\#}) = [\det(A)]^{\#}$)

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#} = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ and $\overline{\det(A_1)} = \det(A_2)$

$\Rightarrow \overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

(iv) From(iii)

$\overline{\det(A_2)} = \det(A_1)$ or equivalently $\overline{\det(A_1)} = \det(A_2)$

$\Rightarrow \det(A) = \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2$

(v) From (ii)

$\det(A) = \det({}^1A) = \det({}^2A)$

Also, $\det({}^1A) = \det({}^2A) \in \mathbb{C}(i_1)$

$\Rightarrow \det(A) \in \mathbb{C}(i_1)$. □

Proposition 3.10. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 -Hermitian matrix. Then*

(i) $\det({}^1A)$ *and* $\det({}^2A)$ *both are real*

(ii) $\det(A) = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

(iii) $\det(A_1)$ *and* $\det(A_2)$ *both are real*

(iv) $\det(A) = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

(v) $\det(A) \in \mathbb{H}$.

Proof.

(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Hermitian matrix

$\Rightarrow A' = A^t$

$\Rightarrow \det(A') = \det(A^t)$

$\Rightarrow \det(A') = \det(A)$ $(\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = \det(A)$ $(\because \det(A') = [\det(A)]')$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]' = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2 = \det({}^1A)e_1 + \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = \det({}^1A)$ and $\overline{\det({}^2A)} = \det({}^2A)$

$\Rightarrow \det({}^1A)$ is real and $\det({}^2A)$is real

(ii) From(i)

$\det({}^1A) = \overline{\det({}^1A)}$ and $\det({}^2A) = \overline{\det({}^2A)}$

$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

$\Rightarrow \det(A) = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

(iii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2-Hermitian matrix

$\Rightarrow A' = A^t$

$\Rightarrow \det(A') = \det(A^t)$

$\Rightarrow \det(A') = \det(A)$ $(\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = \det(A)$ $(\because \det(A') = [\det(A)]')$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]' = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2 = \det(A_1)e_1 + \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)} = \det(A_1)$ and $\overline{\det(A_2)} = \det(A_2)$

$\Rightarrow \det(A_1)$ is real and $\det(A_2)$ is real

(iv) From(iii)

$\det(A_1) = \overline{\det(A_1)}$ and $\det(A_2) = \overline{\det(A_2)}$

$\Rightarrow \det(A_1)e_1 + \det(A_2)e_2 = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

$\Rightarrow \det(A) = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

(v) From(i)

$\det({}^1A)$ and $\det({}^2A)$ both are real

$\Rightarrow \det(A) \in \mathbb{H}$. □

Proposition 3.11. *If $A \in \mathbb{C}_2^{n\times n}$ is a i_1-skew Hermitian matrix. Then the following conditions hold:*

(a) When n is even.

(i) $\overline{\det({}^2A)} = \det({}^1A)$ *or equivalently* $\overline{\det({}^1A)} = \det({}^2A)$.

(ii) $\det(A) = \overline{\det({}^2A)}\, e_1 + \overline{\det({}^1A)}\, e_2$.

(iii) $\det(A_1) = \det(A_2)$.

(iv) $\det(A) = \det(A_1) = \det(A_2)$.

(v) $\det(A) \in \mathbb{C}(i_2)$.

(b) When n is odd.

(i) $\overline{\det({}^2A)} = -\det({}^1A)$ *or equivalently* $\overline{\det({}^1A)} = -\det({}^2A)$.

(ii) $\det(A) = -\overline{\det({}^2A)}e_1 - \overline{\det({}^1A)}e_2$.

(iii) $\det(A_1) = -\det(A_2)$.

(iv) $\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$.

(v) $\det(A) \in i_1\mathbb{C}(i_2)$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 Skew - Hermitian matrix

$\Rightarrow A^* = -A^t$

$\Rightarrow \det(A^*) = \det(-A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^*) = (-1)^n \det(A)$ $(\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^* = (-1)^n \det(A)$ $(\because \det(A^*) = [\det(A)]^*)$

(a) When n is even

$[\det(A)]^* = \det(A)$ $(\because (-1)^n = 1,\ n$ is even$)$

All the results directly follow Proposition 3.8.

(b) When n is odd

$[\det(A)]^* = -\det(A)$ $(\because (-1)^n = -1$, n is odd $)$

(i) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^* = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ and $\overline{\det({}^1A)} = -\det({}^2A)$

$\Rightarrow \overline{\det({}^2A)} = -\det({}^1A)$ or equivalently $\overline{\det({}^1A)} = -\det({}^2A)$

(ii) From(i) $\overline{\det({}^2A)} = -\det({}^1A)$ or equivalently $\overline{\det({}^1A)} = -\det({}^2A)$

$\Rightarrow \det(A) = -\overline{\det({}^2A)}e_1 - \overline{\det({}^1A)}e_2$

(iii) $[\det(A)]^* = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^* = -\{\det(A_1)e_1 + \det(A_2)e_2\}$

$\Rightarrow \det(A_1)e_2 + \det(A_2)e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2)e_1 + \det(A_1)e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \det(A_2) = -\det(A_1)$ and $\det(A_1) = -\det(A_2)e_2$

$\Rightarrow \det(A_1) = -\det(A_2)$

(iv) From(iii)

$\det(A_1) = -\det(A_2)$

$\Rightarrow \det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 det(A_2)$

(v) From (ii)

$\det(A) = i_1 i_2 \det(A_1) = -i_1 i_2 \det(A_2)$

Also, $\det(A_1) \in \mathbb{C}(i_2)$

$\Rightarrow i_2 \det(A_1) \in \mathbb{C}(i_2)$

$\Rightarrow i_1 i_2 \det(A_1) \in i_1\mathbb{C}(i_2)$

$\Rightarrow \det(A) \in i_1\mathbb{C}(i_2)$. □

Proposition 3.12. *If $A \in \mathbb{C}_2^{n\times n}$ is a i_2-skew Hermitian matrix. Then the following results hold:*

(a) When n is even

(i) $\det({}^1A) = \det({}^2A)$

(ii) $\det(A) = \det({}^1A) = \det({}^2A)$

(iii) $\overline{\det(A_2)} = \det(A_1)$ *or equivalently* $\overline{\det(A_1)} = \det(A_2)$

(iv) $\det(A) = \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2$

(v) $\det(A) \in \mathbb{C}(i_1)$.

(b) When n is odd

(i) $\det({}^1A) = -\det({}^2A)$

(ii) $\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

(iii) $\overline{\det(A_2)} = -\det(A_1)$ *or equivalently* $\overline{\det(A_1)} = -\det(A_2)$

(iv) $\det(A) = -\overline{\det(A_2)}e_1 - \overline{\det(A_1)}e_2$

(v) $\det(A) \in i_2\mathbb{C}(i_1)$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Skew -Hermitian matrix

$\Rightarrow A^{\#} = -A^t$

$\Rightarrow \det(A^{\#}) = \det(-A^t)$

$\Rightarrow \det(A^{\#}) = (-1)^n \det(A^t)$

$\Rightarrow \det(A^{\#}) = (-1)^n \det(A)$ $(\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]^{\#} = (-1)^n \det(A)$ $(\because \det(A^{\#}) = [\det(A)]^{\#})$

(a) When n is even

$[\det(A)]^{\#} = \det(A)(\because (-1)^n = 1, n \text{ is even})$

All the results directly follow Proposition 3.9.

(b) When n is odd

$[\det(A)]^{\#} = -\det(A)$ $(\because (-1)^n = -1, n\text{is odd})$

(i) $[\det(A)]^{\#} = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^{\#} = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \det({}^1A)e_2 + \det({}^2A)e_1 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A)e_1 + \det({}^1A)e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \det({}^2A) = -\det({}^1A)$ and $\det({}^1A) = -\det({}^2A)$

$\Rightarrow \det({}^2A) = -\det({}^1A)$

(ii) From(i)

$\det({}^2A) = -\det({}^1A)$

$\Rightarrow \det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

(iii) $[\det(A)]^{\#} = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#} = -\{\det(A_1)e_1 + \det(A_2)e_2\}$

$\Rightarrow \overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ and $\overline{\det(A_1)} = -\det(A_2)$

$\Rightarrow \overline{\det(A_2)} = -\det(A_1)$ or equivalently $\overline{\det(A_1)} = -\det(A_2)$

(iv) From(iii)

$\overline{\det(A_2)} = -\det(A_1)$ or equivalently $\overline{\det(A_1)} = -\det(A_2)$

$\Rightarrow \det(A) = -\overline{\det(A_2)}e_1 - \overline{\det(A_1)}e_2$

(v) From (ii)

$\det(A) = i_1 i_2 \det({}^1A) = -i_1 i_2 \det({}^2A)$

Also, $\det({}^1A) \in \mathbb{C}(i_1)$

$\Rightarrow i_1 \det(A) \in \mathbb{C}(i_1)$

$\Rightarrow i_1 i_2 \det(A) \in i_2\mathbb{C}(i_1)$

$\Rightarrow \det(A) \in i_2\mathbb{C}(i_1)$. □

Proposition 3.13. *If $A \in \mathbb{C}_2^{n\times n}$ is a $i_1 i_2$-skew-Hermitian matrix. Then the following results hold:*

(a) When n is even

(i) $\det({}^1A)$ *and* $\det({}^2A)$ *both are real*

(ii) $\det(A) = \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2$

(iii) $\det(A_1)$ *and* $\det(A_2)$ *both are real*

(iv) $\det(A) = \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2$

(v) $\det(A) \in \mathbb{H}$

(b) When n is odd

(i) $\det({}^1A)$ *and* $\det({}^2A)$ *both are pure imaginary*

(ii) $\det(A) = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

(iii) $\det(A_1)$ *and* $\det(A_2)$ *both are pure imaginary*

(iv) $\det(A) = -\overline{\det(A_1)}e_1 - \overline{\det(A_2)}e_2$

(v) $\det(A) \in i_1\mathbb{H}$ *or* $i_2\mathbb{H}$

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2 - Skew-Hermitian matrix

$\Rightarrow A' = -A^t$

$\Rightarrow \det(A') = \det(-A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A^t)$

$\Rightarrow \det(A') = (-1)^n \det(A)$ $(\because \det(A^t) = \det(A))$

$\Rightarrow [\det(A)]' = (-1)^n \det(A)$ $(\because \det(A') = [\det(A)]')$

(a) When n is even

$[\det(A)]' = \det(A)$ $\quad (\because (-1)^n = 1,\ n \text{ is even})$

The results directly follow Proposition 3.10.

(b) When n is odd

$[\det(A)]' = -\det(A)$ $(\because (-1)^n = -1,\ n \text{ is odd})$

(i) $[\det(A)]' = -\det(A)$

$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]' = -\{\det({}^1A)e_1 + \det({}^2A)e_2\}$

$\Rightarrow \overline{\det({}^1A)}e_1 + \overline{\det({}^2A)}e_2 = -\det({}^1A)e_1 - \det({}^2A)e_2$

$\Rightarrow \overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det({}^1A)$ is pure imaginary and $\det({}^2A)$is pure imaginary

(ii) From(i) $\det({}^1A)$ and $\det({}^2A)$ both are pure imaginary

$\Rightarrow \overline{\det({}^1A)} = -\det({}^1A)$ and $\overline{\det({}^2A)} = -\det({}^2A)$

$\Rightarrow \det(A) = -\overline{\det({}^1A)}e_1 - \overline{\det({}^2A)}e_2$

(iii) $[\det(A)]' = -\det(A)$

$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]' = -\{\det(A_1)e_1 + \det(A_2)e_2\}$

$\Rightarrow \overline{\det(A_1)}e_1 + \overline{\det(A_2)}e_2 = -\det(A_1)e_1 - \det(A_2)e_2$

$\Rightarrow \overline{\det(A_1)} = -\det(A_1)$ and $\overline{\det(A_2)} = -\det(A_2)$

$\Rightarrow \det(A_1)$ is pure imaginary and $\det(A_2)$ is pure imaginary

(iv) From(iii) $\det(A_1)$ and $\det(A_2)$ both are pure imaginary

$\Rightarrow \overline{\det(A_1)} = -\det(A_1)$ and $\overline{\det(A_2)} = -\det(A_2)$

$\Rightarrow \det(A) = -\overline{\det(A_1)}e_1 - \overline{\det(A_2)}e_2$

(v) From(i) $\det({}^1A)$ and $\det({}^2A)$ both are pure imaginary

$\Rightarrow \det(A) \in i_1\mathbb{H}$ or $i_2\mathbb{H}$. □

Proposition 3.14. *Let* $A \in \mathbb{C}_2^{n\times n}$ *be an orthogonal matrix. Then* $\det(A) \in \{1, -1, i_1i_2, -i_1i_2\}$.

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be an orthogonal matrix
$\Rightarrow A^tA = AA^t = I$
$\Rightarrow \det(A^tA) = \det(I)$
$\Rightarrow \det(A^tA) = 1$
$\Rightarrow \det(A^t)\det(A) = 1$
$\Rightarrow \det(A)\det(A) = 1$ ($\because \det(A^t) = \det(A)$)
$\Rightarrow \{\det(A)\}^2 = 1$
$\Rightarrow \{\det({}^1A)e_1 + \det({}^2A)e_2\}^2 = 1$
$\Rightarrow \{\det({}^1A)\}^2e_1 + \{\det({}^2A)\}^2e_2 = 1e_1 + 1e_2$
$\Rightarrow \{\det({}^1A)\}^2 = 1$ and $\{\det({}^2A)\}^2 = 1$
$\Rightarrow \det({}^1A) = 1, -1$ and $\det({}^2A) = 1, -1$.
Hence $\det(A) \in \{1, -1, i_1i_2, -i_1i_2\}$. □

Note 3.9. Every Bicomplex orthogonal matrix A is invertible and $A^{-1} = A^t$.

Proposition 3.15. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1-unitary Matrix. Then*
(i) $\overline{\det({}^2A)}\det({}^1A) = 1$ *or equivalently* $\overline{\det({}^1A)}\det({}^2A) = 1$
(ii) $\det(A) = \frac{1}{\det({}^2A)}e_1 + \frac{1}{\det({}^1A)}e_2$
(iii) $\det(A_1)\det(A_2) = 1$
(iv) $\det(A) = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2$

Proof.
(i) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Unitary Matrix
$\Rightarrow A^{*t}A = AA^{*t} = I$
$\Rightarrow \det(A^{*t}A) = \det(I)$
$\Rightarrow \det(A^{*t})det(A) = 1$
$\Rightarrow [\det(A)]^*\det(A) = 1$ ($\because \det(A^{*t}) = [\det(A)]^*$)
$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^*[\det({}^1A)e_1 + \det({}^2A)e_2] = 1$
$\Rightarrow [\overline{\det({}^1A)}e_2 + \overline{\det({}^2A)}e_1][\det({}^1A)e_1 + \det({}^2A)e_2] = 1$
$\Rightarrow [\overline{\det({}^2A)}e_1 + \overline{\det({}^1A)}e_2][\det({}^1A)e_1 + \det({}^2A)e_2] = 1e_1 + 1e_2$
$\Rightarrow \overline{\det({}^2A)}\det({}^1A)e_1 + \overline{\det({}^1A)}\det({}^2A)e_2 = 1e_1 + 1e_2$
$\Rightarrow \overline{\det({}^2A)}\det({}^1A) = 1$ and $\overline{\det({}^1A)}\det({}^2A) = 1$
$\Rightarrow \overline{\det({}^2A)}\det({}^1A) = 1$ or equivalently $\overline{\det({}^1A)}\det({}^2A) = 1$
(ii) From **(i)**
$\overline{\det({}^2A)}\det({}^1A) = 1$ and $\overline{\det({}^1A)}\det({}^2A) = 1$
$\Rightarrow \det({}^1A) = \frac{1}{\overline{\det({}^2A)}}$ and $\det({}^2A) = \frac{1}{\overline{\det({}^1A)}}$
$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \frac{1}{\overline{\det({}^2A)}}e_1 + \frac{1}{\overline{\det({}^1A)}}e_1$
$\Rightarrow \det(A) = \frac{1}{\overline{\det({}^2A)}}e_1 + \frac{1}{\overline{\det({}^1A)}}e_1$
(iii) Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1 - Unitary Matrix
$\Rightarrow A^{*t}A = AA^{*t} = I$

$\Rightarrow \det(A^{*t}A) = \det(I)$
$\Rightarrow \det(A^{*t})\det(A) = 1$
$\Rightarrow [\det(A)]^{*}\det(A) = 1$ $(\because \det(A^{*t}) = [\det(A)]^{*})$
$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{*}[\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow [\det(A_1)e_2 + \det(A_2)e_1][\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow [\det(A_2)e_1 + \det(A_1)e_1][\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow \det(A_2)\det(A_1)e_1 + \det(A_1)\det(A_2)e_2 = 1e_1 + 1e_2$
$\Rightarrow \det(A_2)\det(A_1) = 1$ and $\det(A_1)\det(A_2) = 1$
$\Rightarrow \det(A_1)\det(A_2) = 1$

(iv) From(iii)
$\det(A_1)\det(A_2) = 1$
$\Rightarrow \det(A_1) = \frac{1}{\det(A_2)}$ and $\det(A_2) = \frac{1}{\det(A_1)}$
$\Rightarrow \det(A_1)e_1 + \det(A_2)e_2 = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2 \Rightarrow \det(A) = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2.$ □

Note 3.10. i_1-unitary matrix is invertible and $A^{-1} = A^{*t}$.

Proposition 3.16. *Let* $A = {}^1Ae_1 + {}^2Ae_2 \in \mathbb{C}_2^{n\times n}$ *be a* i_2*-unitary Matrix. Then*

(i) $\det({}^1A)\det({}^2A) = 1.$
(ii) $\det(A) = \frac{1}{\det({}^2A)}e_1 + \frac{1}{\det({}^1A)}e_2.$
(iii) $\overline{\det(A_2)}\det(A_1) = 1$ *or equivalently* $\overline{\det(A_1)}\det(A_2) = 1.$
(iv) $\det(A) = \frac{1}{\det(A_2)}e_1 + \frac{1}{\det(A_1)}e_2.$

Proof. Let $A \in C_2^{n\times n}$ be a i_2-unitary matrix. Then
$A^{\#t}A = AA^{\#t} = I$
$\Rightarrow \det(A^{\#t}A) = \det(I)$
$\Rightarrow \det(A^{\#t})\det(A) = 1$
$\Rightarrow [\det(A)]^{\#}\det(A) = 1$ $(\because \det(A^{\#t}) = [\det(A)]^{\#})$
$\Rightarrow [\det({}^1A)e_1 + \det({}^2A)e_2]^{\#}[\det({}^1A)e_1 + \det({}^2A)e_2] = 1$
$\Rightarrow [\det({}^1A)e_2 + \det({}^2A)e_1][\det({}^1A)e_1 + \det({}^2A)e_2] = 1$
$\Rightarrow [\det({}^2A)e_1 + \det({}^1A)e_2][\det({}^1A)e_1 + \det({}^2A)e_2] = 1$
$\Rightarrow \det({}^2A)\det({}^1A)e_1 + \det({}^1A)\det({}^2A)e_2 = 1e_1 + 1e_2$
$\Rightarrow \det({}^2A)\det({}^1A) = 1$ and $\det({}^1A)\det({}^2A) = 1$
$\Rightarrow \det({}^2A)\det({}^1A) = 1$

(ii) From **(i)**
$\det({}^2A)\det({}^1A) = 1$
$\Rightarrow \det({}^1A) = \frac{1}{\det({}^2A)}$ and $\det({}^2A) = \frac{1}{\det({}^1A)}$
$\Rightarrow \det({}^1A)e_1 + \det({}^2A)e_2 = \frac{1}{\det({}^2A)}e_1 + \frac{1}{\det({}^1A)}e_2$
$\Rightarrow \det(A) = \frac{1}{\det({}^2A)}e_1 + \frac{1}{\det({}^1A)}e_2$

(iii) $A \in \mathbb{C}_2^{n\times n}$ be a i_2 - Unitary Matrix
$\Rightarrow A^{\#t}A = AA^{\#t} = I$

$\Rightarrow [\det(A)]^{\#} \det(A) = 1$
$\Rightarrow [\det(A_1)e_1 + \det(A_2)e_2]^{\#}[\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow [\overline{\det(A_1)}e_2 + \overline{\det(A_2)}e_1][\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow [\overline{\det(A_2)}e_1 + \overline{\det(A_1)}e_2][\det(A_1)e_1 + \det(A_2)e_2] = 1e_1 + 1e_2$
$\Rightarrow \overline{\det(A_2)} \det(A_1)e_1 + \overline{\det(A_1)} \det(A_2)e_2 = 1e_1 + 1e_2$
$\Rightarrow \overline{\det(A_2)} \det(A_1) = 1$ and $\overline{\det(A_1)} \det(A_2) = 1$
$\Rightarrow \overline{\det(A_2)} \det(A_1) = 1$ or equivalently $\overline{\det(A_1)} \det(A_2) = 1$

(iv) From (iii)
$\overline{\det(A_2)} \det(A_1) = 1$ or equivalently $\overline{\det(A_1)} \det(A_2) = 1$
$\Rightarrow \det(A_1) = \frac{1}{\overline{\det(A_2)}}$ and $\det(A_2) = \frac{1}{\overline{\det(A_1)}}$
$\Rightarrow \det(A_1)e_1 + \det(A_2)e_2 = \frac{1}{\overline{\det(A_2)}}e_1 + \frac{1}{\overline{\det(A_1)}}e_2$
$\Rightarrow \det(A) = \frac{1}{\overline{\det(A_2)}}e_1 + \frac{1}{\overline{\det(A_1)}}e_2.$ □

Note 3.11. i_2 - Unitary Matrix is invertible and $A^{-1} = A^{\#t}$.

Proposition 3.17. *Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2-unitary matrix. Then $||\det(A)|| = 1$.*

Proof. Let $A \in \mathbb{C}_2^{n\times n}$ be a i_1i_2-unitary matrix. Then
$A'^tA = AA'^t = I$
$\Rightarrow \det(A'^tA) = \det(I)$
$\Rightarrow \det(A'^t) \det(A) = 1$
$\Rightarrow [\det(A)]' \det(A) = 1$ ($\because \det(A'^t) = [\det(A)]'$)
$\Rightarrow [\det(^1A)e_1 + \det(^2A)e_2]'[\det(^1A)e_1 + \det(^2A)e_2] = 1$
$\Rightarrow [\overline{\det(^1A)}e_1 + \overline{\det(^2A)}e_2][\det(^1A)e_1 + \det(^2A)e_2] = 1$
$\Rightarrow \overline{\det(^1A)} \det(^1A)e_1 + \overline{\det(^2A)} \det(^2A)e_2 = 1e_1 + 1e_2$
$\Rightarrow \overline{\det(^1A)} \det(^1A) = 1$ and $\overline{\det(^2A)} \det(^2A) = 1$
$\Rightarrow |\det(^1A)|^2 = 1$ and $|\det(^2A)|^2 = 1$
$\Rightarrow ||\det(A)|| = \sqrt{\frac{|\det(^1A)|^2+|\det(^2A)|^2}{2}} = 1.$ □

Note 3.12. The i_1i_2-unitary matrix, A is invertible and $A^{-1} = A'^t$.

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