The Root of a Binomial Coefficient is equal to the Sum of its Leaves

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Abstract: This paper focuses on the successive partitions of a binomial coefficient in combinatorial geometric series such as root, predecessor, successor, and leaf of a binomial coefficient. The coefficient for each term in combinatorial geometric series refers to a binomial coefficient. These ideas can enable the scientific researchers to solve the real life problems.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea stimulated his mind to create a combinatorial geometric series [1-17]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r. In this article, binomial identities and multinomial theorem is provided using the binomial coefficients for combinatorial geometric series.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-17] is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r.

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \& \ V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1$ and $n, r \in N = \{0, 1, 2, 3, \cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ refers to the combinatorial geometric series and

V_n^r is the binomial coefficient for combinatorial geometric series.

Here, $V_0^1 = 1;\ V_1^1 = 2;\ V_2^1 = 3;\ V_3^1 = 4;\ V_4^1 = 5;\ V_5^1 = 6;\ \cdots$

$N = \{V_0^1, V_1^1, V_2^1, V_3^1, V_4^1, V_4^1, \cdots\} = \{1,2,3,\cdots\}$ is a set of natural numbers .

$V_n^r = V_r^n, (n, r > 0$ and $n, r \in N)$ is an important binomial identity.

$$V_n^0 = V_0^n \ for \ n = 1, 2, 3, \cdots$$

Theorem 2.1: $V_{k-1}^{n+1} = V_k^{n+1} - V_k^n$.

Proof. Let us prove this theorem using the binomial expansion for coefficients of combinatorial geometric series.

$$V_k^{n+1} - V_k^n = \frac{(k+1)(k+2)\cdots(k+n)(k+n+1)}{(n+1)!} - \frac{(k+1)(k+2)\cdots(k+n)}{(n+1)!}$$
$$= \frac{(k+1)(k+2)\cdots(k+n)}{n!}\left(\frac{k+n+1)-n-1}{n+1}\right)$$
$$= \frac{k(k+1)(k+2)\cdots(k+n)}{(n+1)!} = V_{k-1}^{n+1}.$$

Hence, the theorem is proved.

For examples,

$$V_0^{n+1} = V_1^{n+1} - V_1^n;\ V_1^{n+1} = V_2^{n+1} - V_2^n,\ V_2^{n+1} = V_3^{n+1} - V_3^n, \text{and so on.}$$

3. Partition of Binomial Coefficient

There are some terms such as root, predecessor, successor, and leaf [16] under the successive partitions of a binomial coefficient in combinatorial geometric series. The Unique binomial coefficient with no predecessor under successive partition is called the root of partition. A binomial coefficient of the root with no successors is called a leaf.

Theorem 3.1: The sum of partition of V_n^r is $V_n^{r-1} + V_r^{n-1}$.

Proof. Let us prove that $V_n^r = V_n^{r-1} + V_r^{n-1}$ using the binomial identity $V_n^r = V_r^n$.

$$V_n^{r-1} + V_r^{n-1} = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)}{(r-1)!} + \frac{n(n+1)(n+2)\cdots(n+r-1)}{r!}$$
$$= (n+1)(n+2)(n+3)\cdots(n+r-1)\left(\frac{1}{(r-1)!} + \frac{n}{r!}\right)$$
$$= (n+1)(n+2)(n+3)\cdots(n+r-1)\left(\frac{r}{r!} + \frac{n}{r!}\right)$$
$$= \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!} = V_n^r.$$

$\therefore\ V_n^r = V_n^{r-1} + V_r^{n-1}$.

Hence, theorem is proved.

The successive partitions of a binomial coefficient (root) V_n^r are given below:

$V_n^r = V_n^{r-1} + V_r^{n-1}$.

$V_n^r = (V_n^{r-2} + V_{r-1}^{n-1}) + (V_r^{n-2} + V_{n-1}^{r-1})$.

$V_n^r = (V_n^{r-3} + V_{r-2}^{n-1}) + (V_{r-1}^{n-2} + V_{n-1}^{r-2}) + (V_r^{n-3} + V_{n-2}^{r-1}) + (V_{n-1}^{r-2} + V_{r-1}^{n-2})$.

Here, V_n^r is named as root. V_n^{r-1} is predecessor of the successors $V_n^{r-2}$and V_{r-1}^{n-1}.

Theorem 3.2: The root of a binomial coefficient of combinatorial geometric series is equal to the sum of its leaves under successive partitions.

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Proof. This theorem is proved by sum of successive partitions of a binomial coefficient in combinatorics geometric series.

Step 1: $V_1^1 = V_1^0 + V_0^1$.

Step 2: $V_1^2 = V_1^1 + V_2^0 = V_1^0 + V_0^1 + V_2^0$.

Step 3: $V_2^2 = V_2^1 + V_2^1 = V_1^0 + V_0^1 + V_2^0 + V_1^0 + V_0^1 + V_2^0$.

$Step\ 4$: $V_2^3 = V_2^2 + V_3^1 = V_2^1 + V_2^1 + V_3^0 + V_1^2$
$= V_1^0 + V_0^1 + V_2^0 + V_1^0 + V_0^1 + V_2^0 + V_1^0 + V_0^1 + V_2^0 + V_3^0.$

We can continue the same process up to the binomial coefficient V_n^r.
The leaves of the root V_n^r are either $V_{n-k}^0\ or\ V_0^{r-k}, (n-k \geq 1\ \&\ \ r-k \geq 1)$.

Let S=$\{V_0^n \mid V_n^0 = V_0^n; n \geq 1\ \&\ n \epsilon N\}$ be a singleton set,

$i.e. S$ is a subset of $N = \{V_0^1, V_1^1, V_2^1, V_3^1,\ V_4^1, V_4^1, \cdots\}$.

Then, the leaf $V_{n-k}^0\ \ or\ \ V_0^{r-k} \epsilon\ S$, where $V_{n-k}^0\ \ or\ \ V_0^{r-k}$ $is\ a\ leaf\ of\ the\ root\ V_n^r$.

From the above results, we conclude that the root of a binomial coefficient of combinatorial geometric series is equal to the sum of its leaves under successive partitions.

4. Conclusion

In this article, successive partitions of a binomial coefficient have been introduced and a theorem on the successive partitions of a binomial coefficient in combinatorial geometric series provided with detailed proof. This idea can enable the scientific researchers to solve the real life problems.

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