A Theorem on Binomial Series

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Abstract: This paper presents a theorem on binomial series. These ideas can enable the scientific researchers to solve the real life problems.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea stimulated his mind to create a combinatorial geometric series [1-9]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r. In this article, binomial identities and multinomial theorem is provided using the binomial coefficients for combinatorial geometric series.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-9] is derived from the multiple summations of geometric series[10-19]. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r.

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \ \& \ \ V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1 \ \ and \ \ n,r \in N = \{0,1,2,3,\cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ refers to the combinatorial geometric series and

$V_n^r \ is$ the binomial coefficient for combinatorial geometric series.

Lemma 2.1: $V_{n-1}^{r+1} + V_n^r = V_n^{r+1}$.

Proof. Let us prove this lemma using the combinatorial geometric series.

By substituting $x = 1$ in the combinatorial geometric series $\sum_{i=0}^{n} V_i^r x^i$, we get

$$\sum_{i=0}^{n} V_i^r (1)^i = \sum_{i=0}^{n} V_i^r = V_0^r + V_1^r + V_2^n + V_3^r + \cdots + V_{n-1}^r + V_n^r = V_n^{r+1}.$$

This is one of the binomial identities based on the combinatorial geometric series.
From the above binomial identity, we get the following result:

$$V_{n-1}^{r+1} + V_n^r = V_n^{r+1}, \left(\because \sum_{i=0}^{n-1} V_i^r = V_{n-1}^{r+1} \right).$$

Let us prove the binomial equation $V_{n-1}^{r+1} + V_n^r = V_n^{r+1}$.)

$$V_{n-1}^{r+1} + V_n^r = \frac{n(n+1)(n+2)\cdots(n+r)}{(r+1)!} + \frac{(n+1)(n+2)\cdots(n+r)}{r!}$$

$$= \frac{(n+1)(n+2)\cdots(n+r)}{r!}\left(\frac{n}{r+1} + 1\right) =$$

$$= \frac{(n+1)(n+2)\cdots(n+r)}{r!}\left(\frac{n+r+1}{r+1}\right).$$

$$V_{n-1}^{r+1} + V_n^r = \frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!} = V_n^{r+1}.$$

Hence, the lemma is proved.

3. Theorem on Binomial Series

Theorem 3.1: $V_0^n \sum_{i=0}^{n-1} 2^i + V_1^{n-1} \sum_{i=0}^{n-2} 2^i + V_2^{n-2} \sum_{i=0}^{n-3} 2^i + \cdots + 3V_{n-2}^2 + V_{n-1}^1 = 3^n - 2^n.$

Proof. Let us prove this theorem using the following binomial series and its results.

If $x = 1$ and $y = 1$ in $\sum_{i=0}^{n} V_i^{n-i} x^i y^{n-1} = (x+y)^n$, then $\sum_{i=0}^{n} V_i^{n-i} = 2^n$. (1)

Also, if $x = 1$ and $y = 2$ in the binomial series, then $\sum_{i=0}^{n} V_i^{n-i} 2^{n-i} = 3^n$. (2)

From (1)and (2), we get $\sum_{i=0}^{n} V_i^{n-i}\left(2^{n-i} - 1\right) = 3^n - 2^n$, i.e. $V_0^n(2^n - 1) + V_1^{n-1}(2^{n-1} - 1) + V_2^{n-2}(2^{n-2} - 1) + V_2^{n-2}(2^{n-2} - 1) + V_3^{n-3}(2^{n-3} - 1) + \cdots + V_{n-3}^3(2^3 - 1) + V_{n-2}^2(2^2 - 1) + V_{n-1}^1(2^1 - 1) = 3^n - 2^n$.

From this expression, we conclude that

$$V_0^n \sum_{i=0}^{n-1} 2^i + V_1^{n-1} \sum_{i=0}^{n-2} 2^i + V_2^{n-2} \sum_{i=0}^{n-3} 2^i + \cdots + 7V_{n-3}^3 + 3V_{n-2}^2 + V_{n-1}^1 = 3^n - 2^n,$$

where $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Hence, theorem is proved.

Corollary 3.1: $V_1^{n-1} + 3V_2^{n-2} + \cdots + V_{n-2}^2 \sum_{i=0}^{n-3} 2^i + V_{n-1}^1 \sum_{i=0}^{n-2} 2^i + V_n^0 \sum_{i=0}^{n-1} 2^i = 3^n - 2^n.$

Proof. Let us prove this corollary using the following binomial series and its results.

If $x = 1$ and $y = 1$ in $\sum_{i=0}^{n} V_i^{n-i} x^i y^{n-1} = (x+y)^n$, then $\sum_{i=0}^{n} V_i^{n-i} = 2^n$. (1)

$$\text{Also, if } x = 2 \text{ and } y = 1 \text{ in the binomial series, then } \sum_{i=0}^{n} V_i^{n-i} 2^{n-i} = 3^n. \qquad (2)$$

Like theorem 3.1, by simplifying (1) and (2), we conclude that

$$V_1^{n-1} + 3V_2^{n-2} + 7V_3^{n-3} + \cdots + V_{n-2}^2 \sum_{i=0}^{n-3} 2^i + V_{n-1}^1 \sum_{i=0}^{n-2} 2^i + V_n^0 \sum_{i=0}^{n-1} 2^i = 3^n - 2^n$$

Hence, Corollary is proved.

4. Conclusion

In this article, a theorem on binomial series was proved with more details. This idea can enable the scientific researchers to solve the real life problems.

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