On The Elementary and Numerical Approach to *k*-tuple Conjecture

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Abstract: In 1922 Hardy and Littlewood proposed a conjecture on the asymptotic density of admissible prime k-tuples. Here we have used a sieve method and shown an elementary process to calculate the approximate number of admissible prime k-tuples and compared with Hardy-Littlewood conjecture and real values. Then we have combined our elementary formula with the results obtained from numerical data of real values and generated a new formula which gives almost same results as Hardy-Littlewood conjecture. We have also proposed an easy form of this conjecture which gives us a new perspective to think about it.

Keywords: Prime k-tuples, asymptotic density, Hardy-Littlewood conjecture, elementary approach, prime constellations, admissible k-tuples, twin prime.

1. Introduction

In 1922 Hardy and Littlewood proposed a conjecture on the asymptotic density of admissible prime k-tuples (G. H. Hardy, 1922). This conjecture is generally believed to be true, but has not been proven. In this article, we have used a sieve method and shown an elementary and numerical approach to calculate the approximate density of admissible prime k-tuples. Our main results may stated as follows: by using a concept of calculating the approximate number of outputs of linear functions and a new idea of constellation table (see definition 3.1) we have calculated the approximate density of admissible prime k-tuples. We have also given a new kind of definition of admissible prime k-tuples (see definition 4.1). Let an admissible form of prime k-tuples is $(p, p+m_1, p+m_2, \dots, p+m_n)$. If at least one number among $m_1, m_2, \dots, m_n$ is not divisible by 6 then the number of prime constellations of this form less than or equal to k will be approximately,

$$P_K(p, p+m_1, \dots, p+m_n) \sim \frac{k-m_n}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{s_i}{p_i}\right)$$

Again, if $m_1, m_2, \dots, m_n$ all the numbers are multiples of 6 then the number of prime constellations of this form less than or equal to k will be approximately,

$$P_K(p, p+m_1, \dots, p+m_n) \sim \frac{2(k-m_n)}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{s_i}{p_i}\right)$$

$$\text{where} \quad s_i = n(p_i: 0, m_1, m_2, \dots, m_n)$$

Here, $n(p_i: m_1, m_2, \dots, m_n)$ gives the number of unique reminders after dividing $m_1, m_2, \dots, m_n$ by p_i (see notation 3.2) and the function $\pi(\sqrt{k})$ gives the number of prime numbers less than or equal to $\sqrt{k}$. Now, according to the Hardy-Littlewood conjecture the number of prime constellations of this form less than or equal to k will be,

$$P_K(\, p, p + m_1, \dots, p + m_n) \sim C(m_1, m_2, \dots, m_n) \int_2^k \frac{dx}{ln^{n+1}x}$$

Where $C(m_1, m_2, \dots, m_n)$ is a constant (see conjecture 5.1). Here our elementary formulas give better approximation than Hardy-Littlewood conjecture when k is less than or equal to 10^6. Again, our elementary formulas give the same ratio as Hardy-Littlewood conjecture of the densities of prime k-tuples of different forms for a fixed k. Then we have combined our elementary formulas with the numerical data of real values and generated new formulas which give almost same results as Hardy-Littlewood conjecture. Here are the formulas respectively,

$$P_K(\, p, p + m_1, \dots, p + m_n) \sim \frac{3^{n+1}(k - m_n)}{6k} \prod_{i=3}^{\pi(\sqrt{k})} \frac{\left(1 - \frac{s_i}{p_i}\right)}{\left(1 - \frac{1}{p}\right)^{n+1}} \int_2^k \frac{dx}{ln^{n+1}x}$$

$$P_K(\, p, p + m_1, \dots, p + m_n) \sim 2\frac{3^{n+1}(k - m_n)}{6k} \prod_{i=3}^{\pi(\sqrt{k})} \frac{\left(1 - \frac{s_i}{p_i}\right)}{\left(1 - \frac{1}{p}\right)^{n+1}} \int_2^k \frac{dx}{ln^{n+1}x}$$

These formulas are very difficult and time consuming to use. So Hardy-Littlewood conjecture is still the best approximation for the density of admissible prime k-tuples. Again, we have proposed an easy version of Hardy-Littlewood conjecture by using the concept of constellation table (see conjecture 4.1).

2. The number of outputs of linear function

First of all we will define a special kind of linear function and it will be called a prime function. The definition is as follows,

Definition 2.1 (Prime function). The function of the form $f(n) = pn + a$ is called a prime function where the coefficient p is a prime number, $n \in N$, the constant $a \in Z$ and $-p < a < p$.

Let, $f_1(n) = p_1 n + a_1$ is a prime function and it will be called prime function for p_1 and $f_2(n) = p_2 n + a_2$ is another prime function which will be called prime function for p_2. In case of the prime function for p_1, the first output is $p_1 + a_1$, second output is $2p_1 + a_1$, third output is $3p_1 + a_1$ and so on because $n \in N$. So the number of outputs of the prime function $f_1(n)$ less than or equal to a natural number x will be approximately,

$$\frac{x - a_1}{p_1} = \frac{x}{p_1} - \frac{a_1}{p_1}$$

Here, $-1 < \frac{a_1}{p_1} < 1$ because $-p_1 < a_1 < p_1$. So we can write,

$$\frac{x-a_1}{p_1} \approx \frac{x}{p_1}$$

Similarly, the number of outputs of the prime function $f_2(n)$ less than or equal to a natural number x will be approximately,

$$\frac{x}{p_2}$$

According to the definition, every output of a prime function will be a positive integer. In case of $f_1(n)$ and $f_2(n)$, a_1 and a_2 are constant, p_1 and p_2 are prime numbers and $p_1 \neq p_2$. So the difference between the common outputs of $f_1(n)$ and $f_2(n)$ will be the least common multiple of p_1 and p_2 and that is p_1p_2. Let, the first common output of these two prime functions is m. So, the second common output will be $m + p_1p_2$, third common output will be $m + 2p_1p_2$ and so on. If $a_1 = a_2 = 0$ then $m = p_1p_2$. But if $a_1 \neq a_2$ then m must be less than p_1p_2 because if $m > p_1p_2$ then there will be positive integer $m - p_1p_2$ which will also be a common output of $f_1(n)$ and $f_2(n)$. But that is not possible because m is first common output of these two prime functions. That's why, $m < p_1p_2$. Now the function for common outputs of the functions $f_1(n)$ and $f_2(n)$ will be,

$$g(n) = np_1p_2 + m = np_1p_2 - (p_1p_2 - m) \; n \in N$$

Here $(p_1p_2 - m) \geq 0$ and $(p_1p_2 - m) < p_1p_2$. So, the number of outputs of the function $g(n)$ less than or equal to a natural number x will be approximately,

$$\frac{x + (p_1p_2 - m)}{p_1p_2} = \frac{x}{p_1p_2} + \frac{(p_1p_2 - m)}{p_1p_2}$$

Here, $0 \leq \frac{(p_1p_2 - m)}{p_1p_2} < 1$ because $0 \leq (p_1p_2 - m) < p_1p_2$. So we can write,

$$\frac{x}{p_1p_2} + \frac{(p_1p_2 - m)}{p_1p_2} \approx \frac{x}{p_1p_2}$$

Now if there is another prime function $f_3(n) = p_3n + a_3$ then by using the similar process we will get the number of common outputs of the functions $g(n)$ and $f_3(n)$ less than or equal to a natural number x will be approximately,

$$\frac{x}{p_1p_2p_3}$$

So, if the number of the prime functions of different prime number is r then the number of common outputs of those functions less than or equal to a natural number x will be approximately,

$$\frac{x}{p_1p_2 \cdots p_r}$$

These approximations are very important for the further calculation. Now, let $f_{11}(n) = p_1n + a_{11}$ and $f_{12}(n) = p_1n + a_{12}$ are two prime functions for the prime number p_1 and $a_{11} < a_{12}$. If $|a_{11} - a_{12}| = p_1$ then all the outputs of these two functions will be same except the first output of $f_{11}(n)$. So the difference between the number of outputs of these two functions less than or equal to a natural number x will be 1. But according to the above approximations, the number of outputs of these functions less than or equal to a natural number x will be approximately same and that is,

$$\frac{x}{p_1}$$

In this case we are avoiding the difference of 1 because when the value of x is so big it can be avoided. We will consider these two functions as same and we will call them same prime functions for p_1.This approximation will give a very interesting result. Again, if $|a_{11} - a_{12}| \neq p_1$, where $a_{11} \neq a_{12}$ then there will be no common output of these two functions. In this case we will call them unique prime functions for p_1. If $a_{11} = a_{12}$ then these two functions are definitely same function for p_1.

Example 2.1. Let $f_{11}(n) = p_1 n + a_{11}$ and $f_{12}(n) = p_1 n + a_{12}$ are two prime functions for the prime number p_1 and $a_{11} < a_{12}$. If $p_1 = 5$, $a_{11} = -4$ and $a_{12} = 1$ then $f_{11}(n) = 5n - 4$ and $f_{12}(n) = 5n + 1$ and $|-4 - 1| = 5$. Now, first few outputs of $f_{11}(n)$ are 1,6,11,16,21 etc. and first few outputs of $f_{12}(n)$ are 6,11,16,21 etc. All the outputs of these two functions will be same except the first output of $f_{11}(n)$ and that is 1. Now, the exact number of outputs of these functions less than or equal to 100 will be respectively 20 and 19. But according to the above approximations, the number of outputs of these functions less than or equal to 100 will be approximately same and that is $\frac{100}{5} = 20$. Now, the ratios are $\frac{20}{20} = 1$, $\frac{19}{20} = 0.95$. Again, the exact number of outputs of these functions less than or equal to 1000 will be respectively 200 and 199. But according to the above approximations, the number of outputs of these functions less than or equal to 1000 will be approximately same and that is $\frac{1000}{5} = 200$. Now, the ratios are $\frac{200}{200} = 1$, $\frac{199}{200} = 0.995$. As the value of x is getting bigger, the approximation is being more accurate.

Let, $p_1, p_2, p_{3,} \dots, p_r$ are prime numbers and the number of these prime numbers is r. The number of unique prime functions for the prime number p_1 is s_1, the number of unique prime functions for the prime number p_2 is s_2, the number of unique prime functions for the prime number p_3 is s_3,…, the number of unique prime functions for the prime number p_r is s_r. Again, $s_1 \leq p_1$, $s_s \leq p_2$,…, , $s_r \leq p_r$. The number of unique prime functions for a prime number cannot be greater than that prime number because if the number of unique prime functions for a prime number becomes greater than that prime number then there will be two or more prime functions which will be same. In other words, there will be two or more prime functions whose difference of the constants will be equal to that prime number. As we are taking only the unique prime functions for a prime number, that's why $s_1 \leq p_1$, $s_s \leq p_2$,…, , $s_r \leq p_r$.

Let, x is a natural number and $x > p_r$. The unique prime functions for the prime number p_1 are $f_{1(1)}(n) = p_1 n + a_{1(1)}$, $f_{1(2)}(n) = p_1 n + a_{1(2)}$,…, $f_{1(s_1)}(n) = p_1 n + a_{1(s_1)}$. There will be no common outputs of these functions because these functions are unique prime functions for the same prime number p_1. Now, the number of outputs of these functions less than or equal to x will be approximately,

$$\frac{x}{p_1} + \frac{x}{p_1} + \frac{x}{p_1} + \cdots \text{upto } s_1 \tag{2.1}$$

$$= \frac{xs_1}{p_1}$$

So, the quantity of the numbers less than or equal to x which are not the outputs of these prime functions for p_1 will be approximately,

$$x - \frac{xs_1}{p_1} = x\left(1 - \frac{s_1}{p_1}\right) \tag{2.2}$$

Again, let the unique prime functions for the prime number p_2 are $f_{2(1)}(n) = p_2 n + a_{2(1)}$, $f_{2(2)}(n) = p_2 n + a_{2(2)}$,…, $f_{2(s_2)}(n) = p_2 n + a_{2(s_2)}$. Now by using the same process of (2.1), the number of outputs of these functions less than or equal to x will be approximately,

$$\frac{xs_2}{p_2}$$

But there will be some outputs which will also be the outputs of the prime functions for p_1. Now in case of the first prime function $f_{2(1)}(n)$ for p_2, the number of common outputs less than or equal to x between $f_{2(1)}(n)$ and any of the prime function for p_1 will be approximately,

$$\frac{x}{p_1 p_2}$$

So the number of common outputs of all the prime functions for p_1 and any of the prime function for p_2 will be approximately,

$$\frac{xs_1}{p_1 p_2}$$

So the number of common outputs of all the prime functions for p_1 and p_2 will be approximately,

$$\frac{xs_1}{p_1 p_2} + \frac{xs_1}{p_1 p_2} + \frac{xs_1}{p_1 p_2} + \cdots \text{ upto } s_2$$

$$= \frac{xs_1 s_2}{p_1 p_2} \qquad (2.3)$$

Now, the number of outputs less than or equal to x of all the prime functions for p_2 which are not the outputs of any prime function for p_1 will be approximately,

$$\frac{xs_2}{p_2} - \frac{xs_1 s_2}{p_1 p_2}$$

Now, the quantity of the numbers less than or equal to x which are not the outputs of the prime functions for p_1 or p_2 will be approximately,

$$x - \frac{xs_1}{p_1} - \left(\frac{xs_2}{p_2} - \frac{xs_1 s_2}{p_1 p_2}\right)$$

$$= x\left(1 - \frac{s_1}{p_1}\right) - \frac{xs_2}{p_2}\left(1 - \frac{s_1}{p_1}\right)$$

$$= x\left(1 - \frac{s_1}{p_1}\right)\left(1 - \frac{s_2}{p_2}\right) \qquad (2.4)$$

Again, let the unique prime functions for the prime number p_3 are $f_{3(1)}(n) = p_3n + a_{3(1)}$, $f_{3(2)}(n) = p_3n + a_{3(2)}$,…, $f_{3(s_3)}(n) = p_3n + a_{3(s_3)}$. Now by using the same process of (2.1), the number of outputs of these functions less than or equal to x will be approximately,

$$\frac{xs_3}{p_3}$$

But there will be some outputs which will also be the outputs of the prime functions for p_1 and p_2. Now by using the process of (2.3), the number of common outputs of all the prime functions for p_1 and p_3 will be approximately,

$$\frac{xs_1s_3}{p_1p_3}$$

Again, the number of common outputs of all the prime functions for p_2 and p_3 will be approximately,

$$\frac{xs_2s_3}{p_2p_3}$$

Now, the number of common outputs of all the prime functions for p_1, p_2 and p_3 will be approximately,

$$\frac{xs_1s_2s_3}{p_1\,p_2p_3}$$

So, the number of outputs less than or equal to x of all the prime functions for p_3 which are not the outputs of any prime function for p_1 or p_2 will be approximately,

$$\frac{\mathrm{xs_3}}{\mathrm{p_3}} - \frac{\mathrm{xs_1s_3}}{\mathrm{p_1p_3}} - \frac{\mathrm{xs_2s_3}}{\mathrm{p_2p_3}} + \frac{\mathrm{xs_1s_2s_3}}{\mathrm{p_1\,p_2p_3}}$$

Now, the quantity of the numbers less than or equal to x which are not the outputs of the prime functions for p_1 or p_2 or p_3 will be approximately,

$$x - \left(\frac{xs_1}{p_1}\right) - \left(\frac{xs_2}{p_2} - \frac{xs_1s_2}{p_1p_2}\right) - \left(\frac{xs_3}{p_3} - \frac{xs_1s_3}{p_1p_3} - \frac{xs_2s_3}{p_2p_3} + \frac{xs_1s_2s_3}{p_1\,p_2p_3}\right)$$

$$= x\left(1 - \frac{s_1}{p_1}\right) - \frac{xs_2}{p_2}\left(1 - \frac{s_1}{p_1}\right) - \frac{xs_3}{p_3}\left(1 - \frac{s_1}{p_1}\right) + \frac{xs_2s_3}{p_2p_3}\left(1 - \frac{s_1}{p_1}\right)$$

$$= x\left(1 - \frac{s_1}{p_1}\right)\left(1 - \frac{s_2}{p_2} - \frac{s_3}{p_3} + \frac{s_2s_3}{p_2p_3}\right)$$

$$= x\left(1 - \frac{s_1}{p_1}\right)\left\{1 - \frac{s_2}{p_2} - \frac{s_3}{p_3}\left(1 - \frac{s_2}{p_2}\right)\right\}$$

$$= x\left(1 - \frac{s_1}{p_1}\right)\left(1 - \frac{s_2}{p_2}\right)\left(1 - \frac{s_3}{p_3}\right) \tag{2.5}$$

By using the similar procedure we get, the quantity of the numbers less than or equal to x which are not the outputs of the prime functions for p_1 or p_2 or p_3 or p_4 will be approximately,

$$x\left(1-\frac{s_1}{p_1}\right)\left(1-\frac{s_2}{p_2}\right)\left(1-\frac{s_3}{p_3}\right)\left(1-\frac{s_4}{p_4}\right)$$

Again, the quantity of the numbers less than or equal to x which are not the outputs of the prime functions for p_1 or p_2 or p_3 or p_4 will be approximately,

$$x\left(1-\frac{s_1}{p_1}\right)\left(1-\frac{s_2}{p_2}\right)\left(1-\frac{s_3}{p_3}\right)\left(1-\frac{s_4}{p_4}\right)\left(1-\frac{s_5}{p_5}\right)$$

Now look at the above results and (2.2),(2.4),(2.5). Here we can see the beautiful pattern. According to this pattern we can write, the quantity of the numbers less than or equal to x which are not the outputs of the prime functions for p_1 or p_2 or p_3 or,…, or p_r will be approximately,

$$x\left(1-\frac{s_1}{p_1}\right)\left(1-\frac{s_2}{p_2}\right)\left(1-\frac{s_3}{p_3}\right)\dots\left(1-\frac{s_r}{p_r}\right)$$

$$= x\prod_{i=1}^{r}\left(1-\frac{s_i}{p_i}\right)$$

We call it "Key formula" because this is the key for the main results of this paper. Here we can see, if $s_i = p_i$ then the whole expression becomes 0. It means if the number of unique prime functions for p_i is equal to p_i then all the numbers less than or equal to x will be the outputs of those prime functions for p_i. But it will be possible if the constants of those prime functions are less than or equal to 0. Let a general form of the prime functions for a prime number p is

$$f(n) = pn + a$$

Again, let the number of unique prime functions for p is equal to p. Now, if $a \le 0$ then all the numbers less than or equal to x will be the outputs of those prime functions for p. In this case our key formula gives the accurate result. Again, if $a \ge 0$ then there will be first $p-1$ numbers which will not be the outputs of those prime functions for p. But this time our key formula will give the result 0. This time the approximation is not so accurate. So the better approximation of our key formula depends on how many constants of the prime functions for a prime number is less than or equal to 0. In other words, if we use key formula or main the algorithm of key formula then the better approximation will depend on how many constants of the prime functions for a prime number is less than or equal to 0.

3. Approximate density of prime constellations

In this section we will calculate the approximate density of prime constellations or prime k-tuple by using our key formula. Now look at the sieve below,

1	2	3	4	5	6
7	8	9	10	11	12

13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
⋮	⋮	⋮	⋮	⋮	⋮

Here we can see, the column 2,4 and 6 contain all the even numbers and the column 3 contains all the multiples of 3. Column 1 and 5 contain all the prime numbers greater than 3. The general form of the numbers (except 1) which are on the column 1 is $6n + 1$ where $n \in N$ and the general form of the numbers which are on the column 5 is $6n - 1$ where $n \in N$. Now we will analyse this two columns.

5
11
17
23
29
⋮

7
13
19
25
31
⋮

Here the left column contains all the prime numbers of the form $6n - 1$ where $n \in N$. Let a prime number of the form $6n - 1$ is p. As the difference between two consecutive rows is 6, so the first composite multiple of p on the left column will be $p + 6p$. Similarly, the second composite multiple will be $p + 12p$, the third composite multiple will be $p + 18p$ and so on. Now, the row number of the row that contains the prime number p will be,

$$\frac{p+1}{6} \text{ which is less than } p$$

Similarly, the row numbers of those rows that contain the first, second and third composite multiple of p on the left column will be respectively,

$$\frac{p+6p+1}{6} = p + \frac{p+1}{6}, \frac{p+12p+1}{6} = 2p + \frac{p+1}{6}, \frac{p+18p+1}{6} = 3p + \frac{p+1}{6}$$

So, the row number of the row that contains the nth composite multiple of p will be,

$$pn + \frac{p+1}{6} \tag{3.1}$$

Here $0 < \frac{p+1}{6} < p$, so this is a prime function. The row numbers of those rows that contain the composite multiples of a prime number of the form $6n - 1$ on the left column can be expressed as the outputs of a prime function for that prime number. Again, let a prime number of the form $6n + 1$ is $6d + 1$ where $n = d$. Now, a composite multiple of this prime will be,

$$(6d + 1)(6d - 1) = 6(6d^2) - 1$$

So, there are some composite multiples of the prime numbers of the form $6n + 1$ that can be expressed as the form of $6n - 1$. That's why, the left column also contains some composite multiples of the prime numbers of the form $6n + 1$. Let, a prime number of the form $6n + 1$ is q and the row number of the row that contains the first composite multiple of q on the left column is m. Here m can be represented as $q - (q - m)$. As the difference between two consecutive rows is 6, so the row number of the row that contains the second composite multiple of q will be,

$$m + \frac{6q}{6} = m + q = q - (q - m) + q = 2q - (q - m)$$

Similarly, the row number of the row that contains the nth composite multiple of q on the left column will be,

$$qn - (q - m) \tag{3.2}$$

Here we can see, the shortest difference between two row numbers of the rows which contain the composite multiples of q on the left column is q. So m cannot be greater than q. If $m > q$, then the $(m - q)$th row will contain a composite multiples of q. But that is impossible because m is the first composite multiple of q on the left column. So, m must be less than or equal to q.

Now, $0 \leq (q - m) < q$ because $m \leq q$. The row numbers of those rows on the left column that contain the composite multiples of a prime number of the form $6n + 1$ can be expressed as the outputs of a prime function for that prime number. So it is proved that The row numbers of those rows on the left column that contain the composite multiples of a prime number greater than 3 can be expressed as the outputs of a prime function for that prime number.

Again, if we consider any of the row on the left column as the first row and start from that then the row numbers of those rows on the left column which will contain the composite multiples of a prime number greater than 3 can also be expressed as the outputs of a prime function for that prime number. Because,

$i.$ According to our above calculation, the row number of the row that contains the first composite multiple of a prime number p of the form $6n - 1$ is less than $2p$.

$ii.$ The shortest difference between two row numbers of the rows which contain the composite multiples of a prime number on the left column is equal to that prime number.

As a result (see 3.1 and 3.2), whatever the row we choose as the first row, the row numbers of those rows which will contain the composite multiples of a prime number p greater than 3 can be expressed as the outputs of a prime function of the following form,

$$f(n) = pn + m \quad \text{where } -p < m < p$$

Example 3.1.

5	11	17	23
11	17	23	29
17	23	29	35
23	29	35	41
29	35	41	47
⋮	⋮	⋮	⋮

For these four columns we have considered different rows of the upper left column as the first row. The prime function for a prime number greater than 3 which generate the row numbers of the rows that contain the composite multiples of that prime number will respectively,

Prime function for 5: $f_{11}(n) = 5n + 1$, $f_{21}(n) = 5n + 0$, $f_{31}(n) = 5n - 1$, $f_{41}(n) = 5n - 2$.

Prime function for 7: $f_{12}(n) = 7n - 1$, $f_{22}(n) = 7n - 2$, $f_{32}(n) = 7n - 3$, $f_{42}(n) = 7n - 4$.

Prime function for 11: $f_{13}(n) = 11n + 2$, $f_{23}(n) = 11n + 1$, $f_{33}(n) = 11n + 0$, $f_{43}(n) = 11n - 1$.

So, whatever the row we choose as the first row, the row numbers of those rows which will contain the composite multiples of a prime number p greater than 3 can be expressed as the outputs of a prime function.

Again, the upper right column contains all the prime numbers of the form $6n + 1$ where $n \in N$. Let a prime number of the form $6n + 1$ is r. As the difference between two consecutive rows is 6, so the first composite multiple of r on the upper right column will be $r + 6r$. Similarly, the second composite multiple will be $r + 12r$, the third composite multiple will be $r + 18r$ and so on. Now, the row number of the row that contains the prime number r will be,

$$\frac{r-1}{6} \text{ which is less than } r$$

Similarly, the row numbers of those rows that contain the first, second and third composite multiple of r on the right column will be respectively,

$$\frac{r+6r-1}{6} = r + \frac{r-1}{6}, \frac{r+12r-1}{6} = 2r + \frac{r-1}{6}, \frac{r+18r-1}{6} = 3r + \frac{p-1}{6}$$

So, the row number of the row that contains the nth composite multiple of r will be,

$$rn + \frac{r-1}{6} \tag{3.3}$$

Here $0 < \frac{r-1}{6} < r$, so this is a prime function. The row numbers of those rows that contain the composite multiples of a prime number of the form $6n + 1$ on the right column can be expressed as the outputs of a prime function for that prime number. Again, let a prime number of the form $6n - 1$ is $6e - 1$ where $n = e$. Now, a composite multiple of this prime will be,

$$(6e - 1)(6e - 1) = 36e^2 - 6e - 6e + 1 = 6(6e^2 - 2e) + 1$$

So, there are some composite multiples of the prime numbers of the form $6n - 1$ that can be expressed as the form of $6n + 1$. That's why, the right column also contains some composite multiples of the prime numbers of the form $6n - 1$. Let, a prime number of the form $6n - 1$ is s and the row number of the row that contains the first composite multiple of s on the left column is u. Here u can be represented as $s - (s - u)$. As the difference between two consecutive rows is 6, so the row number of the row that contains the second composite multiple of s will be,

$$u + \frac{6s}{6} = u + s = s - (s - u) + s = 2s - (s - u)$$

Similarly, the row number of the row that contains the nth composite multiple of s on the right column will be,

$$sn - (s - u) \tag{3.4}$$

Here we can see, the shortest difference between two row numbers of the rows which contain the composite multiples of s on the right column is s. So u cannot be greater than s. If $u > s$, then the $(u - s)$th row will contain a composite multiples of s. But that is impossible because u is the first composite multiple of s on the right column. So, u must be less than or equal to s.

Now, $0 \leq (s - u) < s$ because $u \leq s$. The row numbers of those rows on the right column that contain the composite multiples of a prime number of the form $6n - 1$ can be expressed as the outputs of a prime function for that prime number. So it is proved that the row numbers of those rows on the right column that contain the composite multiples of a prime number greater than 3 can be expressed as the outputs of a prime function for that prime number.

Again, if we consider any of the row on the right column as the first row and start from that then the row numbers of those rows on the right column which will contain the composite multiples of a prime number greater than 3 can also be expressed as the outputs of a prime function for that prime number. Because,

$i.$ According to our above calculation, the row number of the row that contains the first composite multiple of a prime number p of the form $6n + 1$ is less than $2r$.

$ii.$ The shortest difference between two row numbers of the rows which contain the composite multiples of a prime number on the right column is equal to that prime number.

As a result (see 3.3 and 3.4), whatever the row we choose as the first row, the row numbers of those rows which will contain the composite multiples of a prime number p greater than 3 can be expressed as the outputs of a prime function of the following form,

$$f(n) = pn + m \quad \text{where } -p < m < p$$

Now, again look at the following columns,

Col. A
5
11
17
23
29
. . .

Col. B
7
13
19
25
31
. . .

In case of the column A, the number of rows less than or equal to a number k will be approximately,

$$\frac{k+1}{6}$$

It is well known that if we just remove the composite multiples of the prime numbers less than or equal to $\sqrt{n}$ then the remaining numbers less than or equal to n will be primes.

Here the column A does not contain any composite multiple of 2 and 3. So, if we subtract the number of rows which contain the composite multiples of the prime numbers greater than 3 and less than or equal to $\sqrt{k}$ from $\frac{k+1}{6}$ then the number of remaining rows will be the number of prime numbers less than or equal to k on the column A. As we know, the row numbers of those rows on the column A that contain the composite multiples of a prime number greater than 3 can be expressed as the outputs of a prime function for that prime number. As there are one prime function for every prime number, so by using our key formula the number of the rows less than or equal to $\frac{k+1}{6}$ which are not the outputs of the prime functions for the prime numbers greater than 3 and less than or equal to $\sqrt{k}$ will be approximately,

$$\frac{k+1}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{1}{p_i}\right)$$

Notation 3.1. The function $\pi(n)$ gives the number of prime numbers less than or equal to n.

Similarly, the number of rows on the column B which contain the prime numbers greater than 3 and less than or equal to k will be approximately,

$$\frac{k-1}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{1}{p_i}\right)$$

Here these two columns do not contain 2 and 3. So the number of prime numbers less than or equal to k will be approximately,

$$\pi(k) \sim 2 + \frac{k+1}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{1}{p_i}\right) + \frac{k-1}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{1}{p_i}\right)$$

$$\pi(k) \sim 2 + \frac{k}{3} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{1}{p_i}\right)$$

Now, look at the following representations by using column A and column B,

Representation 1	
5	7
11	13
17	19
23	25
29	31
⋮	⋮

Representation 2	
7	11
13	17
19	23
25	29
31	35
⋮	⋮

Representation 3	
5	13
11	19
17	25
23	31
29	37
⋮	⋮

Representation 4	
7	17
13	23
19	29
25	35
31	41
⋮	⋮

As we know, simply a prime constellation is a set of consecutive prime numbers. Here we can see, the representation 1 will contain all the prime constellations of the form $(p, p+2)$ by using the prime numbers greater than 3. Similarly, the representation 2,3 and 4 will contain all the prime constellations of the form $(p, p+4)$, $(p, p+8)$ and $(p, p+10)$ respectively by using the prime numbers greater than 3. Now, in case of the representation 1 if we remove the rows which contain at least one composite multiple of a prime number less than or equal to $\sqrt{k}$ then the remaining rows will contain only the prime constellations of the form $(p, p+2)$ less than or equal to k. In case of the representation 1, the number of rows less than or equal to a number k will be approximately,

$$\frac{k-1}{6}$$

Now, if we subtract the number of rows which contain the composite multiples of the prime numbers greater than 3 and less than or equal to $\sqrt{k}$ from $\frac{k-1}{6}$ then the number of remaining rows will be the number of prime constellations less than or equal to k. As we know, for both columns of the representation 1 the row numbers of those rows that contain the composite multiples of a prime number greater than 3 can be expressed as the outputs of a prime function for that prime number. So there will two prime functions for every prime number. Let a prime number less than or equal to $\sqrt{k}$ is p. If the two prime functions for p are same then the multiples of p will be on the same rows for both columns. Again, the difference between two consecutive rows is 6 for every column. So if we divide the two numbers of any row by p then reminders will be same. Similarly if the two prime functions for p are not same then the multiples of p will not be on the same rows. In that case, if we divide the two numbers of any row by p then reminders will not be same. Again, let x and y are multiples of p and $x < y$. Now if we subtract a same number m from x and y where $m \leq x$ and divide by p then the reminders will be same because x and y are the multiples of p. But if x and y are not the multiples of p then the reminders will be different. Now, the two numbers of the first row of the representation 1 are 5 and 7. If we subtract 5 from both numbers we get 0,2. Again, if we divide 0 and 2 by the prime numbers greater than 3 we will get the reminders 0 and 2 respectively. So in case of the representation 1, the two prime functions for every prime number will be different. We can express this procedure by the following way,

$$n_i = n(p_i: 0,2)$$

Notation 3.2. $n(p_i: m_1, m_2, \ldots, m_n)$ gives the number of unique reminders after dividing $m_1, m_2, \ldots, m_n$ by p_i.

Now in case of the representation 1, there are two unique prime functions for every prime number greater than 3 and less than or equal to $\sqrt{k}$. Here the number of prime constellations on the representation 1 less than or equal to k will be equal to the number of rows less than or equal to $\frac{k-1}{6}$ which are not the outputs of any prime functions for these prime numbers. In this case we will use our key formula because the key formula will calculate the approximate number of rows less than or equal to $\frac{k-1}{6}$ which are not the outputs of any prime functions for the prime numbers greater than 3 and less than or equal to $\sqrt{k}$. In our key formula s_i is the number of unique prime functions for a prime number p_i. So we can write,

$$s_i = n_i = n(p_i: 0,2)$$

So according to our key formula, the number of rows less than or equal to $\frac{k-1}{6}$ which are not the outputs of any prime functions for the prime numbers greater than 3 and less than or equal to $\sqrt{k}$ will be approximately,

$$\frac{k-1}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{2}{p_i}\right)$$

The representation 1 does not contain the prime constellation (3,5). So the number of prime constellations of the form $(p, p+2)$ less than or equal to k will be approximately,

$$P_K(p, p+2) \sim 1 + \frac{k-1}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{2}{p_i}\right) \quad (3.5)$$

Now we will use this same method to calculate the number of prime constellations from any representation. In that case we will use the following two steps,

$i.$ Firstly we will calculate the number of unique prime functions for a prime number greater than 3 and less than or equal to $\sqrt{k}$.

$ii.$ Then by using our key formula, we will calculate the number of rows which contain the numbers less than or equal to k and which are not the outputs of the prime functions for the prime numbers greater than 3 and less than or equal to $\sqrt{k}$ because these number of rows are the number of prime constellations less than or equal to k.

Now look at the representation 2 which contains all the prime constellations of the form $(p, p+4)$ except $(3,7)$. In this case we can write,

$$s_i = n(p_i: 0,4)$$

Here, $s_3 = n(5: 0,4) = 2$, $s_4 = n(7: 0,4) = 2$ and so on. We can see, if we divide 0 and 4 by the prime numbers greater than 3 we will get the reminders 0 and 4 respectively. So in case of the representation 2, the two prime functions for every prime number will be different. Again the number of rows on the representation 2 less than or equal to k is approximately,

$$\frac{k-5}{6}$$

So the number of prime constellations of the form $(p, p+4)$ less than or equal to k will be approximately,

$$P_K(p, p+4) \sim 1 + \frac{k-5}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{2}{p_i}\right) \quad (3.6)$$

Again, the representation 3 contains all the prime constellations of the form $(p, p+8)$ except $(3,11)$. In this case we can write,

$$s_i = n(p_i: 0,8)$$

Here, $s_3 = n(5: 0,8) = 2$, $s_4 = n(7: 0,8) = 2$ and so on. We can see, if we divide 0 and 8 by the prime numbers greater than 3 we will get different reminders. So in case of the representation 3, the two prime functions for every prime number will be unique. So the number of prime constellations of the form $(p, p+8)$ less than or equal to k will be approximately,

$$P_K(p, p+8) \sim 1 + \frac{k-7}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{2}{p_i}\right)$$

Here $\frac{k-7}{6}$ is the approximate number of rows less than or equal to k on the representation 3. Again, the representation 4 contains all the prime constellations of the form $(p, p+10)$ except $(3,13)$. In this case we can write,

$$s_i = n(p_i: 0,10)$$

Here, $s_3 = n(5: 0,10) = 1$, $s_4 = n(7: 0,10) = 2$, $s_5 = n(11: 0,10) = 2$ and so on. We can see, if we divide 0 and 10 by 5 then we get the same reminder. So the prime functions for 5 for both columns on the representation 4 will be same. In other word, there is one prime function for 5. But there are two unique prime functions for the prime numbers greater than 5. So the number of prime constellations of the form $(p, p+10)$ less than or equal to k will be approximately,

$$1 + \frac{k-11}{6}\left(1-\frac{1}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{11}\right)\dots\left(1-\frac{2}{p_i}\right)$$

$$P_K(p, p+10) \sim 1 + \frac{k-11}{6}\left(1-\frac{1}{5}\right)\prod_{i=4}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)$$

Here $\frac{k-11}{6}$ is the approximate number of rows less than or equal to k on the representation 4. Again look at the representations below,

Representation 5		
5	7	11
11	13	17
17	19	23
23	25	29
29	31	35
⋮	⋮	⋮

Representation 6			
7	11	13	19
13	17	19	25
19	23	25	31
25	29	31	37
31	35	37	43
⋮	⋮	⋮	⋮

Representation 7				
5	11	17	19	29
11	17	23	25	35
17	23	29	31	41
23	29	35	37	47
29	35	41	43	53
⋮	⋮	⋮	⋮	⋮

Now look at the representation 5 which contains all the prime constellations of the form $(p, p+2, p+6)$. There are three columns in this representation and in every column there is a prime function for a prime number greater than 3. So there are three prime functions for every prime number greater than 3. Now the first row contains 5,7,11. If we subtract 5 from them we get, 0,2,6. Now we can write,

$$s_i = n(p_i: 0,2,6)$$

Here, $s_3 = n(5: 0,2,6) = 3$, $s_4 = n(7: 0,2,6) = 3$, $s_5 = n(11: 0,2,6) = 3$ and so on. As every time we are getting 3 so there are three unique prime functions for every prime numbers on the representation 5. Again the number of rows on the representation 5 less than or equal to k is approximately,

$$\frac{k-5}{6}$$

So the number of prime constellations of the form $(p, p+2, p+6)$ less than or equal to k will be approximately,

$$P_K(p, p+2, p+6) \sim \frac{k-5}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{3}{p_i}\right)$$

Again, look at the representation 6 which contains all the prime constellations of the form $(p, p+4, p+6, p+12)$. There are four columns in this representation. So there are four prime functions for every prime number greater than 3. Now the first row contains 7,11,13,19. If we subtract 7 from them we get, 0,4,6,12. Now we can write,

$$s_i = n(p_i : 0,4,6,12)$$

Here, $s_3 = n(5: 0,4,6,12) = 4$, $s_4 = n(7: 0,4,6,12) = 4$, $s_5 = n(11: 0,4,6,12) = 4$ and so on. As every time we are getting 4 so there are four unique prime functions for every prime numbers on the representation 6. Again the number of rows on the representation 6 less than or equal to k is approximately,

$$\frac{k-13}{6}$$

So the number of prime constellations of the form $(p, p+4, p+6, p+12)$ less than or equal to k will be approximately,

$$P_K(p, p+4, p+6, p+12) \sim \frac{k-13}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{4}{p_i}\right)$$

Again the representation 7 contains the prime constellations of the form $(p, p+6, p+12, p+14, p+24)$. In this case we can write,

$$s_i = n(p_i : 0,6,12,14,24)$$

Here, $s_3 = n(5: 0,6,12,14,24) = 4$, $s_4 = n(7: 0,6,12,14,24) = 4$, $s_5 = n(11: 0,6,12,14,24) = 5$, $s_6 = n(13: 0,6,12,14,24) = 5$, $s_7 = n(17: 0,6,12,14,24) = 5$ and so on. So there are four unique prime functions for 5 and 7 and five unique prime functions for the prime numbers greater than 7. Again the number of rows on the representation 7 less than or equal to k is approximately,

$$\frac{k-23}{6}$$

So the number of prime constellations of the form $(p, p+6, p+12, p+14, p+24)$ less than or equal to k will be approximately,

$$P_K(p, p+6, p+12, p+14, p+24) \sim \frac{k-23}{6}\left(1 - \frac{4}{5}\right)\left(1 - \frac{4}{7}\right) \prod_{i=5}^{\pi(\sqrt{k})} \left(1 - \frac{5}{p_i}\right)$$

Again look at the representations below,

Representation 8.1	
5	11

Representation 8.2	
7	13

Representation 9.1		
5	11	17

Representation 9.2		
7	13	19

11	17
17	23
23	29
29	35
⋮	⋮

13	19
19	25
25	31
31	37
⋮	⋮

11	17	23
17	23	29
23	29	35
29	35	41
⋮	⋮	⋮

13	19	25
19	25	31
25	31	37
31	37	43
⋮	⋮	⋮

Here the representation 8.1 and 8.2 contain all the prime constellations of the form $(p, p+6)$. In case of the representation 8.1, there are two columns. So there are two prime functions for every prime number greater than 3. Now we can write,

$$s_i = n(p_i: 0{,}6)$$

Here, $s_3 = n(5: 0{,}6) = 2$, $s_4 = n(7: 0{,}6) = 2$ and so on. So in case of the representation 8.1, the two prime functions for every prime number greater than 3 will be unique. Again the number of rows on the representation 8.1 less than or equal to k is approximately,

$$\frac{k-5}{6}$$

So the number of prime constellations on the representation 8.1 of the form $(p, p+6)$ less than or equal to k will be approximately,

$$\frac{k-5}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)$$

Similarly, the number of prime constellations on the representation 8.2 of the form $(p, p+6)$ less than or equal to k will be approximately,

$$\frac{k-7}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)$$

Here,

$$\frac{k-5}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)+\frac{k-7}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)=\frac{2(k-6)}{6}\prod_{i=3}^{\pi(\sqrt{k})}\left(1-\frac{2}{p_i}\right)$$

So the number of prime constellations of the form $(p, p+6)$ less than or equal to k will be approximately,

$$P_K(p, p+6) \sim \frac{2(k-6)}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{2}{p_i}\right) \tag{3.7}$$

Again, the representation 9.1 and 9.2 contain all the prime constellations of the form $(p, p+6, p+12)$. In case of the representation 9.1, there are three columns. So there are three prime functions for every prime number greater than 3. Now we can write,

$$s_i = n(p_i: 0,6,12)$$

Here, $s_3 = n(5: 0,6,12) = 3$, $s_4 = n(7: 0,6,12) = 3$, $s_5 = n(11: 0,6,12) = 3$ and so on. So in case of the representation 9.1, the three prime functions for every prime number greater than 3 will be unique. Again the number of rows on the representation 9.1 less than or equal to k is approximately,

$$\frac{k-11}{6}$$

So the number of prime constellations on the representation 9.1 of the form $(p, p+6, p+12)$ less than or equal to k will be approximately,

$$\frac{k-11}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right)$$

Similarly, the number of prime constellations on the representation 9.2 of the form $(p, p+6, p+12)$ less than or equal to k will be approximately,

$$\frac{k-13}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right)$$

Here,

$$\frac{k-11}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right) + \frac{k-13}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right) = \frac{2(k-12)}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right)$$

So the number of prime constellations of the form $(p, p+6, p+12)$ less than or equal to k will be approximately,

$$P_K(p, p+6, p+12) \sim \frac{2(k-12)}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1 - \frac{3}{p_i}\right) \tag{3.8}$$

Now, the above representations are called constellation table. To make those representations we have used column A and B. The definition of a constellation table is as follows,

Definition 3.1 (Constellation table). A constellation table is a representation by using column A and B or only column A or only column B where column A contains the numbers of the form $6n - 1$, column B contains the numbers of the form $6n+1$, the number of columns is finite, the number of rows is infinite, the first number of the first row is 5 or 7 and every row contains the numbers of the form $(x, x+m_1, x+m_2, \dots, x+m_n)$.

Definition 3.2 (Constellation table for the prime constellations of a form). A constellation table where every row contains the numbers of the form $(x, x+m_1, x+m_2, \dots, x+m_n)$ is called a constellation table for the prime constellations of the form $(p, p+m_1, p+m_2, \dots, p+m_n)$.

Let, $(p, p+m_1, p+m_2, \dots, p+m_n)$ is a general form of some prime constellations. Now look at the above representations. If at least one number among $m_1, m_2, \dots, m_n$ is not divisible by 6 then there will be one possible constellation table for the prime constellations of this form. Again, if $m_1, m_2, \dots, m_n$ all are the multiples of 6 then there will be two possible constellation tables. At First, let at least one number among $m_1, m_2, \dots, m_n$ is not divisible by 6 then there will be one possible constellation table. The first number of the first row can be 5 or 7. So the number of rows of the constellation table less than or equal to k will be approximately,

$$\frac{k-(5+m_n-6)}{6} = \frac{k-m_n+1}{6} \approx \frac{k-m_n}{6}$$

Or,

$$\frac{k-(7+m_n-6)}{6} = \frac{k-m_n-1}{6} \approx \frac{k-m_n}{6}$$

Again, look at the approximations such as (3.5) and (3.6). There is an additional 1. We get this 1 because in case of some prime constellations the number 3 is used. But a constellation table contains the prime constellations where the prime numbers are greater than 3. As we are calculating the approximate number of prime constellations less than or equal to k, so we can avoid the additional 1 for a better form of approximation.

Now, look at the above calculations. By using the similar procedure of those calculations, the number of prime constellations less than or equal to k of the form $(p, p+m_1, p+m_2, \dots, p+m_n)$ where at least one number among $m_1, m_2, \dots, m_n$ is not divisible by 6 will be approximately,

$$P_K(p, p+m_1, \dots, p+m_n) \sim \frac{k-m_n}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1-\frac{s_i}{p_i}\right)$$

$$\text{where} \quad s_i = n(p_i\colon 0, m_1, m_2, \dots, m_n)$$

We call it master formula 1. Again, let $m_1, m_2, \dots, m_n$ all the numbers are multiples of 6. So there will be two possible constellation tables. One constellation tale will be obtained by using only the column A and other one will be obtained by using only the column B. Now we will use the same procedure what we have used to get the approximations (3.7) and (3.8). So, the number of prime constellations less than or equal to k of the form $(p, p+m_1, p+m_2, \dots, p+m_n)$ where $m_1, m_2, \dots, m_n$ all the numbers are multiples of 6 will be approximately,

$$P_K(p, p+m_1, \dots, p+m_n) \sim \frac{2(k-m_n)}{6} \prod_{i=3}^{\pi(\sqrt{k})} \left(1-\frac{s_i}{p_i}\right)$$

$$\text{where} \quad s_i = n(p_i: 0, m_1, m_2, \dots, m_n)$$

We call it master formula 2. Now we have shown that (see 3.2) whatever the row on column A we choose as the first row, the row numbers of those rows which will contain the composite multiples of a prime number q of the form $6n + 1$ can be expressed as the outputs of a prime function of the following form,

$$qn - (q - m)$$

Where $0 \leq (q - m) < q$. In case of a constellation table, we use column A and start from different rows. For every use of column A there is a prime function for q. As $0 \leq (q - m) < q$ so all the constants of the unique prime functions for q for the use of column A will be less than or equal to 0. Again, we have shown that (see 3.1) whatever the row on column A we choose as the first row, the row numbers of those rows which will contain the composite multiples of a prime number p of the form $6n - 1$ can be expressed as the outputs of a prime function of the following form,

$$pn + \frac{p+1}{6}$$

Now, all possible prime functions for p for the use of column A will be,

$$pn + \frac{p+1}{6}, pn + \left(\frac{p+1}{6} - 1\right), pn + \left(\frac{p+1}{6} - 2\right), \dots, pn + 0, pn - 1, \dots, pn - (p - 1)$$

The number of prime functions where the constant is positive is $\frac{p+1}{6}$. Here, $\frac{p+1}{6}$ is almost 17% of p. So almost 83% constants of the unique prime functions for p for the use of column A will be less than or equal to 0. We can show the similar thing for column B. So in case of a constellation table, most of the constants of the unique prime functions for a prime number will be less than or equal to 0. It increases the chance of accuracy of our above approximations because the better approximation of our key formula depends on how many constants of the prime functions for a prime number is less than or equal to 0.

4. Admissible form of prime constellations

If there is a possibility of getting infinitely many prime constellations of a specific form then that will be an admissible form of prime constellations. So if there is finite number of prime constellations of a specific form then that will not be an admissible form. For example, there is only one prime constellation of the form $(p, p + 2, p + 4)$ and that is $(3,5,7)$. So this is not an admissible form. Now, let $(p, p + m_1, p + m_2, \dots, p + m_n)$ is an admissible form. So there must be one or two constellation tables for the prime constellations of this form. If there is no constellation table then it cannot be an admissible form. For example, we cannot make any constellation table for the prime constellations of the form $(p, p + 2, p + 4)$. Again, according to our master formulas, s_i must be less than p_i where $i \geq 3$. In other words, the number of unique prime functions for p_i must be less than p_i. If $s_i = p_i$ then every row of the constellation table will contain at least one composite multiple of p_i. But first few rows may not contain any composite multiple of p_i and the row numbers of those rows will be less than p_i. Let some of those rows contain prime constellations. But number will be finite. So, s_i must be less than p_i. Now we can define an admissible form of prime constellations as following,

Definition 4.1 (Admissible form of prime constellations). A specific form of prime constellations $(p, p + m_1, p + m_2, \dots, p + m_n)$ is admissible if there is one or two constellation tables for the prime constellations of this form and $s_i < p_i$ where $i \geq 3$ and $s_i = n(p_i\colon 0, m_1, m_2, \dots, m_n)$.

There is an easy method to check if there is any constellation table or not. If $x, x + m_1, x + m_2, \dots, x + m_n$ all the numbers are of the form $6n + 1$ or $6n - 1$ where x is equal to 5 or 7 then there will be one or two prime constellation tables.

The well known definition of prime constellation is as follows,

Definition 4.2 (prime constellation). A prime constellation is a sequence of consecutive primes $(p_1, p_2, \dots, p_k)$ with $p_k - p_1 = s(k)$ where $s(k)$, is the smallest number s for which there exist k integers $b_1 < b_2 < \cdots < b_k$, $b_k - b_1 = s$ and for every prime q, not all the residues modulo q are represented by $b_1, b_2, \dots, b_k$.

According to this definition, an admissible pattern of prime constellations is also admissible by definition 4.1. So, definition 4.1 is a new kind of definition of admissible pattern of prime constellations. Now, the well known prime k-tuple conjecture states that every admissible pattern for a prime constellation occurs infinitely often. According to the above discussions we get, the k-tuple conjecture would be true if the following conjecture is true.

Conjecture 4.1. Let every row of a constellation table contains the numbers of the form $(x, x + m_1, x + m_2, \dots, x + m_n)$. Now there will be infinitely many rows which will contain only the prime numbers of the form $(p, p + m_1, p + m_2, \dots, p + m_n)$ if $s_i < p_i$ where $i \geq 3$ and $s_i = n(p_i\colon 0, m_1, m_2, \dots, m_n)$.

This conjecture is an easy form of the prime k-tuple conjecture. It opens a new perspective to think differently about k-tuple conjecture.

5. Comparing our master formulas with the first Hardy-Littlewood conjecture

The first Hardy-Littlewood conjecture or the k-tuple conjecture states that the asymptotic number of prime constellations can be computed explicitly.

Conjecture 5.1 (Asymptotic density of prime k-tuples). Let $P = (p, p + m_1, p + m_2, \dots, p + m_n)$ denote an admissible prime k-tuple and let $\pi_P(k)$ denote the number of primes p less then a positive integer k such that for all $1 \leq i \leq n$, $p + m_i$ is prime. Then

$$\pi_P(k) \sim C(m_1, m_2, \dots, m_n) \int_2^k \frac{dx}{ln^{n+1} x}$$

Where $C(m_1, m_2, \dots, m_n)$ is a constant obtained through a product over all primes q greater than 2 and the amount of distinct residues of $m_1, m_2, \dots, m_n$ modulo q, denoted by $w(q; m_1, m_2, \dots, m_n)$, as follows,

$$C(m_1, m_2, \dots, m_n) = 2^n \prod_q \frac{1 - \frac{w(q; m_1, m_2, \dots, m_n)}{q}}{\left(1 - \frac{1}{q}\right)^{n+1}}$$

For example, when $k = 1$ and $m_1 = 2$ (i.e., the twin primes), we have $C_2 = 1.320323632 \ldots$, called the twin prime constant.

Now we will compare the results given by our master formulas and k-tuple conjecture. We will calculate the approximate number of prime constellations of different forms less than or equal to k. We will take the value of k upto 10^8. Then we will compare the results with real values. We will shortly call our master formulas M, Hardy-Littlewood conjecture HL and the real values R. Here the following table shows the comparison,

Patterns		10^2	10^3	10^5	10^6	10^7	10^8
$(p, p+2)$	R	8	35	1224	8169	58980	440312
	M	7	31	1231	8656	63963	489442
	HL	14	46	1249	8248	58754	440368
$(p, p+6)$	R	15	74	2447	16386	117207	879908
	M	13	62	2461	17313	127927	978883
	HL	27	92	2497	16496	117508	880736
$(p, p+2, p+6)$	R	4	15	259	1393	8543	55600
	M	4	10	257	1518	9638	64511
	HL	14	26	279	1446	8591	55491
$(p, p+2, p+6, p+8)$	R	2	5	38	166	899	4768
	M	1	2	36	179	974	5705
	HL	14	16	53	184	863	4735

Here we can see, our master formula gives the better result than the Hardy-Littlewood conjecture when k is less than or equal to 10^5. Again, in case of the prime constellations of the form $(p, p+2, p+6, p+8)$, our master formula give the better approximation when k is less than or equal to 10^6. But for large values of k Hardy-Littlewood conjecture gives the best results. But the most important result is our master formulas give the same ratio as Hardy-Littlewood conjecture of the densities of prime k-tuples of different forms for a fixed k. For example, let $k = 2$, so the density of prime 2-tuples of the form $(p, p+4)$ is equal to the density of prime 2-tuples of the form $(p, p+2)$. Again, the density of prime 2-tuples of the form $(p, p+6)$ is twice of the density of prime 2-tuples of the form $(p, p+2)$ and so on. Now, depending on numerical experiment if we multiply a function with our master formula then we will get almost same results as Hardy-Littlewood conjecture. The function is as follows,

$$\frac{k^n \int_2^k \frac{dx}{ln^{n+1}x}}{\left\{\frac{k}{3}\prod_{i=3}^{\sqrt{k}}\left(1-\frac{1}{p}\right)\right\}^{n+1}}$$

So, we get our modified master formula 1 and master formula 2 from numerical experiment as follows,

$$P_K(p, p+m_1, \ldots, p+m_n) \sim \frac{3^{n+1}(k-m_n)}{6k}\prod_{i=3}^{\pi(\sqrt{k})}\frac{\left(1-\frac{s_i}{p_i}\right)}{\left(1-\frac{1}{p}\right)^{n+1}}\int_2^k \frac{dx}{ln^{n+1}x}$$

$$P_K(p, p+m_1, \dots, p+m_n) \sim 2\frac{3^{n+1}(k-m_n)}{6k}\prod_{i=3}^{\pi(\sqrt{k})}\frac{\left(1-\frac{s_i}{p_i}\right)}{\left(1-\frac{1}{p}\right)^{n+1}}\int_2^k \frac{dx}{ln^{n+1}x}$$

Here,

$$\frac{3^{n+1}(k-m_n)}{6k}\prod_{i=3}^{\pi(\sqrt{k})}\frac{\left(1-\frac{s_i}{p_i}\right)}{\left(1-\frac{1}{p}\right)^{n+1}} \tag{5.1}$$

$$2\frac{3^{n+1}(k-m_n)}{6k}\prod_{i=3}^{\pi(\sqrt{k})}\frac{\left(1-\frac{s_i}{p_i}\right)}{\left(1-\frac{1}{p}\right)^{n+1}} \tag{5.2}$$

$$C(m_1, m_2, \dots, m_n) = 2^n\prod_q \frac{1-\frac{w(q;m_1,m_2,\dots,m_n)}{q}}{\left(1-\frac{1}{q}\right)^{n+1}} \tag{5.3}$$

If we increase the value of k then (5.1) or (5.2) starts to give closest value of (5.3). According to modified master formulas,

$$P_K(p, p+2) \sim \frac{3^2(k-2)}{6k}\prod_{i=3}^{\pi(\sqrt{k})}\frac{\left(1-\frac{2}{p_i}\right)}{\left(1-\frac{1}{p}\right)^2}\int_2^k \frac{dx}{ln^2x}$$

Again, according to Hardy-Littlewood conjecture,

$$P_K(p, p+2) \sim 2\prod_{p\geq 3}\frac{p(p-2)}{(p-1)^2}\int_2^k \frac{dx}{ln^2x}$$

Here we will call our modified master formula as MM. Now the following table shows the comparison of these two approximations,

Pattern		10^2	10^3	10^4	10^5	10^6	10^7	10^8
$(p, p+2)$	R	8	35	205	1224	8169	58980	440312
	HL	14	46	214	1249	8248	58754	440368
	MM	14	46	215	1249	8249	58756	440372
	Ratio of HL and MM	1	1	0.99535	1	0.99989	0.99997	0.99999

So, we can see our MM gives the better approximations as HL. But the main problem is the use of MM is very hard and time consuming. So, still now HL is the best approximation for prime constellations.

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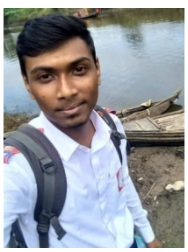
Author's Biography



This is Salman Mahmud. I live in Bogura, Bangladesh. My core research interest is in modern physics and pure mathematics. But I am interested in all the sectors of science. I have contributed in so many researches. I believe mathematics is the ornament of every subject. Email: salmanmahmud6675@gmail.com.