A new form of initial curvature scalar for calculating f(R) gravity

Wen-Xiang Chen[a] and Jing-Yi Zhang*
Department of Astronomy, School of Physics and Materials Science, GuangZhou University, Guangzhou 510006, China

Yao-Guang Zheng
Department of Physics, College of Sciences, Northeastern University, Shenyang 110819, China

Previous literature applied the RVB method to various black holes[12–15] under general relativity. Therefore, whether the RVB complex coordinate system makes it difficult to calculate the temperature of many memorable black holes, it is found that the Hawking temperature of black holes under f(R) gravity can be easily obtained by the RVB method. In one of the works of literature[16], an attempt was made to apply the RVB method to calculate the Hawking temperature of a black hole under f(R) gravity. When calculating the Hawking temperature, we found a difference in the constant of integration between the RVB and the general method. In an article[19], a hypothesis was proposed. The form of the new gravitational constant concerning the curvature scalar is deduced in it. This paper concludes a new scalar form of initial curvature under f(R) gravity by combining those two pieces of literature.

Keywords: f(R) gravity, initial curvature scalar form, cosmological constant

1. INTRODUCTION

f(R) is an improved theory of gravity that generalizes Einstein's general theory of relativity. f(R) Gravity is a family of theories, each defined by a different function of the Ricci scalar R. The simplest case is when the function is equal to a scalar; this is general relativity. As a result of introducing arbitrary functions, the universe's accelerated expansion and structure formation can be freely explained without adding unknown forms of dark energy or dark matter. Revisions to the theory of quantum gravity may inspire some functional forms. The f(R) gravity was first proposed by Hans Adolph Buchdahl in 1970 (albeit using ϕ instead of f as the name of an arbitrary function). It has become an active area of research following Starobinsky's work on expanding the universe. By employing different functions, a wide range of phenomena can arise from this theory.

In $f(R)$ gravity, one tries to generalize the Lagrangian of the Einstein-Hilbert interaction: [1–18]

$$S[g]=\int \frac{1}{2\kappa} R\sqrt{-g}\ \mathrm{d}^4x \tag{1}$$

arrive to

$$S[g]=\int \frac{1}{2\kappa} f(R)\sqrt{-g}\ \mathrm{d}^4x, \tag{2}$$

where $\kappa=\frac{8\pi G}{c^4}, g=\det g_{\mu\nu}$ is the determinant of the metric tensor, and $f(R)$ is the Ricci Some functions of scalars. There are two ways to track the effect of changing R to $f(R)$, i.e., obtaining the theoretical field equation. While these two forms lead to the same field equation of general relativity, when $f(R)\neq R$, the field equation may be different.

Previous literature applied the RVB method to various black holes[12–15] under general relativity. Therefore, whether the RVB complex coordinate system makes it difficult to calculate the temperature of many special black holes, it is found that the Hawking temperature of black holes under f(R) gravity can be easily obtained by the RVB method. One of the papers [16] tried to use the RVB method to calculate the Hawking temperature of a black hole under f(R) gravity. When calculating the Hawking temperature, we found a difference in the constant of integration between the RVB and the general method. In an article[19], a hypothesis was proposed. The form of the new gravitational constant concerning the curvature scalar is deduced in it.

This paper combines those two kinds of literature [16, 19] to derive a new initial curvature scalar form under f(R) gravity. The organization of this paper is as follows. The second part introduces the main formulas studied in this paper. In the third part, a new scalar form of initial curvature is obtained by analyzing the metric under the gravitational force of f(R), whose integral constant is not 0. The fourth part is the conclusion and discussion.

*Electronic address: zhangjy@gzhu.edu.cn

2. NEW ACTION AND FIELD EQUATIONS

In one of the articles[16],the Einstein-Hilbert effect $\sqrt{-g}R$ is the total derivative, where $\sqrt{-g}$ is 1. So the Ricci scalar of measure must be the total derivative, and the only one we can write must be $R \sim -g''$. To be precise

$$R \sim -g''(r). \tag{3}$$

therefore

$$\int drR = -g'(r), \quad \rightarrow \quad \left(\int drR\right)\Big|_{r\rightarrow r_+} \sim -g'\left(r_+\right) + C. \tag{4}$$

This leads to a relationship with the Hawking temperature. The more general case also works, with

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{n(r)}, \quad \rightarrow \quad \sqrt{-g}R \sim -\left(\sqrt{\frac{n}{g}}g'\right)'. \tag{5}$$

In another article[19], this theory aims to discover that the modified Einstein's gravity equations have a vacuum solution. First, we can consider the following equation (modified Einstein's gravitational equation). The eigentime in spherical coordinates is

$$\begin{aligned} ds^2 &= G(t,r)dt^2 - \frac{1}{G(t,r)}dr^2 + \left[r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right], \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &+ \Lambda\left(\left(g^{\theta\theta}\right)^2\right)g_{\mu\nu} = -\frac{8\pi G}{C^4}T_{\mu v}. \end{aligned} \tag{6}$$

In this work, the action is given by the following relation, which reduces to the Einstein-Maxwell expansion gravity in a special case:

$$S = \int d^4x \frac{1}{16\pi}\sqrt{-g}\left[R - \nabla_\mu\phi\nabla^\mu\phi - 2\Lambda\left(\left(g^{\theta\theta}\right)^2\right) - e^{-2\Phi}F_{\mu\nu}F^{\mu\nu}\right], \tag{7}$$

where Λ is a function of the Ricci scalar R and Φ is the representation of the dilatonic field, also $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (we set $8G = c = 1$).

We also have $\nabla^\nu\nabla_\mu\Lambda_R = g^{\alpha\nu}\left[\left(\Lambda_R\right)_{,\mu,\alpha} - \Gamma^m_{\mu\alpha}\left(\Lambda_R\right)_{,m}\right]$. we got

$$\begin{aligned} \nabla_\mu\nabla^\mu\Lambda_R &= \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}\partial^r\right)\Lambda_R = \left(G'\Lambda_R' + G\Lambda_R'' + \frac{G}{r}\Lambda_R'\right) \\ \nabla^t\nabla_t\Lambda_R &= g^{tt}\left[\left(\Lambda_R\right)_{,t,t} - \Gamma^m_{tt}\left(\Lambda_R\right)_{,m}\right] = \frac{1}{2}G'\Lambda_R' \\ \nabla^r\nabla_r\Lambda_R &= g^{rr}\left[\left(\Lambda_R\right)_{,r,r} - \Gamma^m_{rr}\left(\Lambda_R\right)_{,m}\right] = \left(G\Lambda_R'' + \frac{G'}{2}\Lambda_R'\right) \\ \nabla^\theta\nabla_\theta\Lambda_R &= g^{\theta\theta}\left[\left(\Lambda_R\right)_{,\theta,\theta} - \Gamma^m_{\theta\theta}\left(\Lambda_R\right)_{,m}\right] = \frac{G}{r}\Lambda_R', \end{aligned} \tag{8}$$

where $\Lambda_R = \frac{d\Lambda(R)}{dR}$ and $\nabla_\mu\nabla^\mu\Lambda_R = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\right)\Lambda_R$.

From the tt and rr components of the field equation, the following relationship can be easily derived:

$$\nabla^r\nabla_r\Lambda_R = \nabla^t\nabla_t\Lambda_R \Longrightarrow G\Lambda_R'' = 0 \Longrightarrow \Lambda_R'' = 0. \tag{9}$$

This will result in the following:

$$\Lambda_R = z + yr. \tag{10}$$

In this relation, y and z are two integration constants assumed to be favorable to avoid non-physical ambiguities.

3. NEW INITIAL CURVATURE SCALAR FORM

The f(R) static black hole solution and its thermodynamics (for constant Ricci curvature not equal to 0) are briefly reviewed. Its general form of action is

$$I = \frac{1}{2}\int d^4x\sqrt{-g}f(R) + S_{\text{mat}}. \tag{11}$$

3.1. In the case of constant Ricci curvature (initial conditions)

3.1.1. Schwarzschild-de Sitter-f(R) black holes

A spherically symmetric solution with constant curvature (initial curvature scalar) R_0 is first considered a simple but important example. Schwarzschild solution ($R_0 = 0$) or the solution of Schwarzschild -de Sitter is[17–36]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \tag{12}$$

$$g(r) = 1 - \frac{2M}{r} - \frac{R_0 r^2}{12}. \tag{13}$$

and we see that R_0 can be regarded as a remainder coefficient under the metric.

3.1.2. $f(R) = R - qR^{\beta+1}\frac{\alpha\beta+\alpha+\epsilon}{\beta+1} + q\epsilon R^{\beta+1}\ln\left(\frac{a_0^\beta R^\beta}{c}\right)$

One form of f(R) gravity is [17–26]

$$f(R) = R - qR^{\beta+1}\frac{\alpha\beta+\alpha+\epsilon}{\beta+1} + q\epsilon R^{\beta+1}\ln\left(\frac{a_0^\beta R^\beta}{c}\right), \tag{14}$$

where $0 \leq \epsilon \leq \frac{e}{4}\left(1 + \frac{4}{e}\alpha\right), q = 4a_0^\beta/c(\beta+1), \alpha \geq 0, \beta \geq 0$ and $a_0 = l_p^2$, a and c are constants. Since $R \neq 0$, this $f(R)$ theory has no Schwarzschild solution. Its metric form is

$$ds^2 = -g(r)dt^2 + h(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{15}$$

and

$$g(r) = h(r)^{-1} = 1 - \frac{2m}{r} + \beta_1 r, \tag{16}$$

where m is related to the mass of the black hole, and β_1 is a model parameter. By computing the Ricci scalar as

$$R = -\frac{6\beta_1}{r}, \tag{17}$$

for

$$\Lambda_R = z + yr, \tag{18}$$

so

$$R_0 = 6\beta_1(-\frac{z}{r} + y\ln[r]) \tag{19}$$

3.1.3. $df(R)/dR = 1 + \alpha r$

The following line elements form [16]

$$ds^2 = -g(r)dt^2 + 1/g(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{20}$$

$$g(r) = C_2 r^2 + \frac{1}{2} + \frac{1}{3\alpha r} + \frac{C_1}{r}[3\alpha r - 2 - 6\alpha^2 r^2 + 6\alpha^3 r^3 \ln[1 + \frac{1}{\alpha r}]], \tag{21}$$

C1 and C2 are constants. The Ricci scalars are as follows:

$$R = \frac{1}{r^2(1+ar)^2}\left(1+108a^3C_1r^2-12C_2r^2+72a^4C_1r^3+a\left(-6C_1+2r-24C_2r^3\right)+\right.$$
$$\left.a^2r\left(24C_1+r-12C_2r^3\right)-72a^3C_1r^2(1+ar)^2\ln[1+\frac{1}{ar}]\right), \tag{22}$$

for

$$\Lambda_R = z + yr, \tag{23}$$

so

$$R_0 = \frac{6a^2C_1(-y+az)}{(1+ar)^2} + \frac{12a^2C_1(-2y+3az)}{1+ar} + \frac{z-6aC_1z}{r^2} + \frac{2\left(y-6aC_1y+18a^2C_1z\right)}{r} +$$
$$36a^2C_1(-y+2az)\ln[r] - 36a^2C_1(-y+2az)\ln[1+ar]. \tag{24}$$

3.1.4. $f(R) = R + \Lambda + \frac{R+\Lambda}{R/R_0+2/\alpha}\ln\frac{R+\Lambda}{R_c}$

The form of the line element is as follows[37]

$$ds^2 = -g(r)dt^2 + 1/g(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2, \tag{25}$$

$$g(r) = 1 - \frac{2M}{r} + \beta r - \frac{\Lambda r^2}{3}. \tag{26}$$

$\beta > 0$. By calculating the Ricci scalar as

$$R = -\frac{6\beta_1}{r} + 4\Lambda, \tag{27}$$

for

$$\Lambda_R = z + yr, \tag{28}$$

so

$$R_0 = 6\beta_1(-\frac{z}{r} + y\ln[r]) \tag{29}$$

3.2. $f(R) = -4\eta^2M\ln(-6\Lambda - R) + \xi R + R_0$

We get [27, 28, 37–39].

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega^2, \tag{30}$$

$$g(r) = -\Lambda r^2 - M(2\eta r + \xi). \tag{31}$$

The Ricci scalar is

$$R = 6\Lambda + \frac{4M\eta}{r}. \tag{32}$$

$$R_0 = \frac{4M\eta(z - ry\ln[r])}{r} \tag{33}$$

3.3. $f(R) = -2\eta M \ln(6\Lambda + R) + R_0$

With $\Phi(r) = 0$, the charged $(2+1)$ dimensional solution under pure $f(R)$ gravity is [27, 28]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \tag{34}$$

$$g(r) = -\Lambda r^2 - Mr - \frac{2Q^2}{3\eta r}. \tag{35}$$

The curvature scalar is

$$R = \frac{2M}{r} + 6\Lambda. \tag{36}$$

$$R_0 = \frac{2M(z - ry\ln[r])}{r}. \tag{37}$$

4. CONCLUSION AND DISCUSSION

Previous literature applied the RVB method to various black holes[12–16] under general relativity. Therefore, whether the RVB complex coordinate system makes it difficult to calculate the temperature of many memorable black holes, it is found that the Hawking temperature of black holes under f(R) gravity can be easily obtained by the RVB method. In one of the papers, an attempt was made to apply the RVB method to calculate the Hawking temperature of a black hole under f(R) gravity. When calculating the Hawking temperature, we found a difference in the constant of integration between the RVB and the general method. In an article, a hypothesis was proposed. The form of the new gravitational constant concerning the curvature scalar is deduced in it. Combining those two papers, this paper concludes a new scalar form of initial curvature under f(R) gravity. That way, we construct new metric structures.

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