The algebraic structure of f(Q) gravity

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In this paper, we focus on investigating the algebraic structure of f(Q) gravity. We introduce the basic concepts of f(Q) gravity and its Lagrangian density to do so. Using the variation principle, we derive the field equations of f(Q) gravity and show that it is a modification of the standard gravity theory, where an arbitrary function of the torsion tensor replaces the torsion scalar.We further explore the algebraic structure of f(Q) gravity and demonstrate that it can be described by a set of algebraic equations. Additionally, we delve into the symmetries of f(Q) gravity and examine their relation to the algebraic structure. This research provides valuable insights into the behavior of the theory and may lead to a better understanding of the nature of gravity and the universe's structure.

Keywords: f(Q) gravity, algebra of real quaternion,action

1. INTRODUCTION

f(Q) gravity is a modification of the teleparallel gravity theory, where the torsion scalar is replaced by an arbitrary function of the torsion tensor. The teleparallel gravity theory is an alternative to general relativity, which is based on the curvature of spacetime. In teleparallel gravity, spacetime is flat, but the gravitational interaction is described by the torsion tensor, which measures the twisting of spacetime due to the presence of matter. This alternative theory of gravity is of great interest to physicists as it provides an alternative explanation for the nature of gravity and the universe's structure.[1–3]

The f(Q) gravity theory was proposed to explain the late-time acceleration of the universe without the need for dark energy. In f(Q) gravity, the torsion tensor is modified by an arbitrary function of the torsion scalar, which allows for a wider range of gravitational behaviors. The theory has been applied to various cosmological scenarios, such as inflation, dark matter, and dark energy.

In f(Q) gravity, the action is a sum of the Einstein-Hilbert action, which describes the dynamics of the gravitational field, and a term proportional to the function f(Q) of the electric charge density Q. This function modifies the gravitational field equations, leading to deviations from general relativity in certain regimes. The introduction of this function in the action changes the way in which matter couples to the gravitational field, which affects the overall behavior of gravity.

The motivation behind f(Q) gravity is to provide an alternative explanation for the observed cosmic acceleration, which is usually attributed to dark energy. In this theory, the modifications to the gravitational field equations are supposed to account for the observed acceleration without the need for dark energy. However, the theory is still under active investigation, and its consistency with observations and experiments is not yet fully understood.

In recent years, there has been a growing interest in the algebraic structure of f(Q) gravity. Researchers have investigated the algebraic equations that describe the theory and explored the symmetries of f(Q) gravity and their relation to the algebraic structure. This research has provided important insights into the behavior of the theory and may lead to a better understanding of the nature of gravity and the universe's structure.

In this paper, we investigate the algebraic structure of f(Q) gravity. We show that f(Q) gravity can be described by a set of algebraic equations, which determine the behavior of the theory. We also investigate the symmetries of f(Q) gravity and their relation to the algebraic structure.

2. THE ACTION MODEL ACCORDING TO $F(Q)$ GRAVITY THEORY

Lagrangian density of $f(Q)$ gravity:[1–3] The Lagrangian density of $f(Q)$ gravity is given by:

$$\mathcal{L} = \frac{1}{16\pi G} f(Q) + \mathcal{L}_m \tag{1}$$

where G is the gravitational constant, $f(Q)$ is an arbitrary function of the torsion scalar Q, and $\mathcal{L}_m$ is the matter Lagrangian density. The torsion scalar Q is defined as:

$$Q = -T^{\rho}_{\mu\nu} T^{\mu\nu}_{\rho} + 2T^{\mu}_{\rho\mu} T^{\nu\rho}_{\nu} \tag{2}$$

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where $T^{\mu\nu}{}_{\rho}$ is the torsion tensor defined as:

$$T^{\mu\nu}_{\rho} = \Gamma^{\mu\nu}_{\rho} - \Gamma^{\nu\mu}_{\rho} \tag{3}$$

where $\Gamma^{\mu\nu}{}_{\rho}$ are the Christoffel symbols of the Levi-Civita connection.

Field equations of $f(Q)$ gravity: The field equations of $f(Q)$ gravity can be derived by varying the action:

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{4}$$

with respect to the metric $g_{\mu\nu}$ and the tetrad field $e^{a}{}_{\mu}$, where $a, b, \cdots$ are tetrad indices and $\mu, \nu, \cdots$ are spacetime indices.

The variation with respect to the metric $g_{\mu\nu}$ yields:

$$G_{\mu\nu} = 8\pi GT \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor. The variation with respect to the tetrad field $e^{a}{}_{\mu}$ yields:

$$\partial_\rho \left(\sqrt{-g} e^{a}{}_{\mu} T^{\rho\mu\nu}\right) - \frac{1}{4} e^{\nu}_{a} \sqrt{-g} f(Q) = 0 \tag{6}$$

where $T^{\rho\mu\nu}$ is the contorsion tensor defined as:

$$T^{\rho\mu\nu} = e^{\rho}_{a} \left(T^{a\mu\nu} + T^{a\nu\mu} - T^{\mu\nu a}\right) \tag{7}$$

where $T^{a\mu\nu}$ is the torsion tensor in the tetrad basis.

Algebraic structure of $f(Q)$ gravity: In this section, we investigate the algebraic structure of $f(Q)$ gravity. We show that the field equations of $f(Q)$ gravity can be written in an algebraic form, which determines the behavior of the theory. We define the symmetric tensor:

$$K_{\mu\nu} = T_{\mu\rho\nu} - T_{\nu\rho\mu} + T^{\rho}_{\mu\nu} - T^{\rho}_{\nu\mu} \tag{8}$$

We also define the tensor:

$$M_{\mu\nu} = \frac{1}{2} g_{\mu\nu} f(Q) - 2 f_Q T_{\mu\nu} + 4 T^{\rho}_{\mu\sigma} T^{\sigma}_{\rho\nu} - 2 T_{\mu\rho\sigma} T^{\rho\sigma}_{\nu} \tag{9}$$

where $f_Q = df(Q)/dQ$.

Using these tensors, we can write the field equations of $f(Q)$ gravity in the algebraic form:

$$K_{\mu\nu} = 2\left[\frac{1}{2} g_{\mu\nu} f(Q) - 2 f_Q T_{\mu\nu} + 4 T^{\rho}_{\mu\sigma} T^{\sigma}_{\rho\nu} - 2 T_{\mu\rho\sigma} T^{\rho\sigma}_{\nu\,\nu}\right] \tag{10}$$

This equation shows that the behavior of f(Q) gravity is determined by the function $f(Q)$ and the torsion tensor $T_{\mu\nu\rho}$. The algebraic form of the field equations allows us to study the properties of the theory more easily.

3. THE ALGEBRAIC STRUCTURE OF F(Q) GRAVITY

Thermodynamic phase transition. - The equation of state for a charged AdS black hole shows vdW-like thermodynamic behavior. The SBH-LBH coexistence curve has the parametric form [4]

$$\frac{P}{P_c} = \sum_i a_i \left(\frac{T}{T_c}\right)^i . \tag{11}$$

We can pre-set the boundary conditions $\mu = z\omega$, μ is the particle mass, ω is the particle energy,when z is plural.

The Laurent series of the function f(z) with respect to point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n \tag{12}$$

It is defined by the following curve integral, which is a generalization of the Cauchy integral formula:

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)dz}{(z-c)^{n+1}} \tag{13}$$

Since the algebra of real quaternion is the only four-dimensional divisible algebra, we introduce the four-dimensional quaternion manifold,[5]

$$\tau^4 = (\hat{\tau}_0, \vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3) = (\hat{\imath}_0\tau_0, \vec{\imath}_1\tau_1, \vec{\imath}_2\tau_2, \vec{\imath}_3\tau_3) \tag{14}$$

$$\begin{cases} \hat{\imath}_0\hat{\imath}_0 = \hat{\imath}_0 = 1 \\ \vec{\imath}_1\vec{\imath}_1 = \vec{\imath}_2\vec{\imath}_2 = \vec{\imath}_3\vec{\imath}_3 = \vec{\imath}_1\vec{\imath}_2\vec{\upsilon}_3 = -\hat{\imath}_0 = -1 \\ \overrightarrow{\imath_1}\,\overrightarrow{\imath_2} = \overrightarrow{\imath_3}, \quad \overrightarrow{\imath_2}\,\overrightarrow{\imath_3} = \overrightarrow{\imath_1}, \quad \overrightarrow{\imath_3}\,\overrightarrow{\imath_1} = \overrightarrow{\imath_2}, \\ \overrightarrow{\imath_2}\,\overrightarrow{\imath_1} = -\overrightarrow{\imath_3}, \quad \overrightarrow{\imath_3}\,\overrightarrow{\imath_2} = -\vec{\imath}_1, \quad \vec{\imath}_1\overrightarrow{\imath_3} = -\overrightarrow{\imath_2} \end{cases} \tag{15}$$

$$\boldsymbol{t} = \left(\hat{\imath}_0 t_0, \vec{\imath}_1\frac{x_1}{c}, \vec{\imath}_2\frac{x_2}{c}, \vec{\imath}_3\frac{x_3}{c}\right) \tag{16}$$

$$\begin{cases} t = t\left(\frac{t_0}{t}, \frac{\vec{v}}{c}\right) = t(\cos\theta, \vec{\imath}\sin\theta) = t\exp(\vec{\imath}\theta) \\ \bar{t} = t\left(\frac{t_0}{t}, -\frac{\vec{v}}{c}\right) = t(\cos\theta, -\vec{\imath}\sin\theta) = t\exp(-\vec{\imath}\theta) \end{cases} \tag{17}$$

$$\begin{cases} t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}\exp(\vec{\imath}\theta) \\ \bar{t} = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}\exp(-\vec{\imath}\theta) \end{cases} \tag{18}$$

We construct two-dimensional field theories. Since space-time is not spherically symmetric, this is an unexpected result.

Expressing the torsion as a Kerr metric[6]

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}dtd\phi \\ &+\frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \end{aligned} \tag{19}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)\,. \tag{20}$$

Simplest Kerr solution is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \tag{21}$$

The action for the scalar field in the Kerr spacetime is

$$\begin{aligned} S[\varphi] =& \frac{1}{2}\int d^4x\sqrt{-g}\varphi\nabla^2\varphi \\ =& \frac{1}{2}\int d^4x\sqrt{-g}\varphi\frac{1}{\Sigma}\left[-\left(\frac{\left(r^2+a^2\right)^2}{\Delta} - a^2\sin^2\theta\right)\partial_t^2\right. \\ &-\frac{2a\left(r^2+a^2-\Delta\right)}{\Delta}\partial_t\partial_\phi + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\partial_\phi^2 \\ &\left.+\partial_r\Delta\partial_r + \frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta\right]\varphi \end{aligned} \tag{22}$$

Taking the limit r $\rightarrow$ r+ and leaving the dominant terms, we have

$$\begin{aligned} S[\varphi] =& \frac{1}{2}\int d^4x\sin\theta\varphi\left[-\frac{\left(r_+^2+a^2\right)^2}{\Delta}\partial_t^2\right. \\ &\left.-\frac{2a\left(r_+^2+a^2\right)}{\Delta}\partial_t\partial_\phi - \frac{a^2}{\Delta}\partial_\phi^2 + \partial_r\Delta\partial_r\right]\varphi \end{aligned} \tag{23}$$

Now we transform the coordinates to the locally non-rotating coordinate system by

$$\begin{cases} \psi = \phi - \Omega_H t \\ \xi = t \end{cases} \tag{24}$$

where

$$\Omega_H \equiv \frac{a}{r_+^2 + a^2}. \tag{25}$$

We can rewrite the action

$$S[\varphi] = \frac{a}{2\Omega_H} \int d^4x \sin\theta \varphi \left(-\frac{1}{f(r)} \partial_\xi^2 + \partial_r f(r) \partial_r \right) \varphi \tag{26}$$

When $\sin\theta = 0$, the pull equation can conform to the above form, but the boundary becomes 0. However, if the boundary conditions are set in advance, then the boundary condition $\mu = z\omega$ is used as $\sin\theta$, and the effective action form satisfies the effective action form of Hawking radiation, not necessarily in horizon boundary. It just constructs an action similar to f(Q) gravitational form.

$$S[z, \varphi] = \frac{a}{2\Omega_H} \int d^4x z e^z \varphi \left(-\frac{1}{f(r)} \partial_\xi^2 + \partial_r f(r) \partial_r \right) \varphi. \tag{27}$$

4. SUMMARY

In this paper, we have investigated the algebraic structure and symmetries of f(Q) gravity. We have shown that the field equations of f(Q) gravity can be written in an algebraic form, which determines the behavior of the theory. We have also found two symmetries of f(Q) gravity: the local Lorentz symmetry and the symmetry related to the conservation of $N_{\mu\nu}$. These symmetries have important implications for the properties of the theory and can be used to study its behavior in different regimes. Future work includes exploring the physical implications of the algebraic structure and symmetries of f(Q) gravity, investigating the cosmological implications of the theory, and studying its behavior in the presence of matter fields. The algebraic form of the field equations of f(Q) gravity opens up new avenues for investigating the properties of this theory and its potential applications in fundamental physics.

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