Gamma Function derived from Factorial Function based-Pi Function

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: Several professors of mathematics from the renowned universities in Australia, Canada, Europe, India, USA, etc. argue with me that the gamma function is not related to the factorial function. For them, this paper describes the derivation of gamma function from the factorial function.

MSC Classification codes: 05A10, 33B15

Keywords: computation, Euler integral, derivation

Euler's Factorial Function and Gamma Function

In general, the factorial for positive integer n is the product of all positive integers less than or equal to n. Symbolically, n!. For example, 5!=1×2×3×4×5=720 and 0!=1.

The Pi (**Π**) function is given below:

$$\mathbf{\Pi}(x) = x! = x(x-1)(x-2)\cdots 1, \forall\, x \,\epsilon\, \boldsymbol{W}, \tag{1}$$

where $\boldsymbol{W}$ denotes the system of whole numbers.

Euler's factorial function, also known as Pi (**Π**) function, is the basis for gamma function [1-6].

Swiss mathematician Leonhard Euler was defined the pi function by integral as follows:

$$\mathbf{\Pi}(x) = \int_0^1 (-\mathbf{ln}\, t)^n\, dt, \forall\, n \,\epsilon\, \boldsymbol{W}, \tag{2}$$

where **ln** denotes the logarithm to the base of the mathematical constant e.

By substituting $t = e^{-x}$ in (3), we get

$$\mathbf{\Pi}(x) = \int_0^\infty x^n\, e^{-x} dx, \qquad \forall\, n \,\epsilon\, \boldsymbol{W}. \tag{3}$$

By integrating (3), we obtain

$$= [-x^n e^{-x}]_0^\infty - \int_0^\infty -nx^{n-1}\, e^{-x} dx.$$

$$= 0 - \int_0^\infty -nx^{n-1}\, e^{-x} dx.$$

$$\text{Now}, \mathbf{\Pi}(x) = n\int_0^\infty x^{n-1}\, e^{-x} dx. \tag{4}$$

By substituting (3) in (4), we obtain

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1). \tag{5}$$

The gamma function $\Gamma(x)$ is obtained as follows.

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1) \Longrightarrow \Gamma(x+1) = n\Gamma(x). \qquad (6)$$

$$n\Gamma(x) = n\int_0^\infty x^{n-1}\, e^{-x} dx \Longrightarrow \Gamma(x) = \int_0^\infty x^{n-1}\, e^{-x} dx. \quad (7)$$

The gamma function [1-6], therefore, is derived from the Euler's factorial function that uses the actual factorial function.

References

[1] Annamalai, C. (2023) Review on the Gamma Function and Error Correction. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4397585.

[2] Annamalai, C. (2023) Factorial Theorem: An Alternative to Gamma Function. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4376857.

[3] Annamalai, C. (2023) Analysis of Factorial Function for Non-Negative Real Numbers. OSF Preprints. https://dx.doi.org/10.31219/osf.io/4acyu.

[4] Annamalai, C. (2023) Factorials: Difference between 0! and 1!. OSF Preprints. https://dx.doi.org/10.31219/osf.io/bm7uw.

[5] Annamalai, C. (2022) Factorials, Integers, and Factorial Theorems for Computing and Cryptography. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4189241.

[6] Annamalai, C. (2023) Factorial Theorem for Computing the Factorial of Positive Real Number. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-7p472.