Gamma Function derived from Natural Logarithm and Pi Function

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Abstract: Several professors of mathematics from the renowned universities in Australia, Canada, Europe, India, USA, etc. argue with me that the gamma function was not derived from the factorial function. For them, this paper presents the derivation of gamma function from the natural logarithm and Euler's factorial function, also known as Pi function.

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1. Introduction

In general, the factorial [1-6] for positive integer n, denoted by n!, is the product of all positive integers less than or equal to n. For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 720$. Note that 0!=1. In this article, the gamma function is proved using the natural logarithm and Pi function, also known as Euler's factorial function.

2. Natural Logarithm

The natural logarithm of a real or complex number is a technique to solve the exponential function and vice-versa.

The natural logarithm is defined as follows:

$$\log_e u = \ln u.$$

The properties of natural logarithm are given below:

$$\ln u^v = v \ln u.$$

$$\ln uv = \ln u + \ln v.$$

$$\ln \frac{u}{v} = \ln u - \ln v.$$

$$\ln \frac{1}{u} = \ln 1 - \ln u = -\ln u \ (\because \ln 1 = 0).$$

$$\ln e = 1.$$

$$\text{Let } e^m = n, \text{ then } \ln n = m. \text{ Also, } e^{\ln n} = n \text{ and } \ln e^m = m.$$

$$-\ln e^{-r} = -(-r) = r.$$

3. Gamma Function

The gamma function is proved here using Pi function and natural logarithm

The Pi ($\boldsymbol{\Pi}$) function is given below:

$$\mathbf{\Pi}(x) = x! = x(x-1)(x-2)\cdots 1, \forall\, x \in \boldsymbol{W}, \tag{1}$$

where $\boldsymbol{W}$ denotes the system of whole numbers.

Euler's factorial function, also known as Pi (**Π**) function, is the basis for gamma function [1-6].

Swiss mathematician Leonhard Euler was defined the pi function by integral as follows:

$$\mathbf{\Pi}(x) = \int_0^1 (-\mathbf{ln}\, t)^n\, dt, \forall\, n \in \boldsymbol{W}, \tag{2}$$

where **ln** denotes the logarithm to the base of the mathematical constant e.

By substituting $t = e^{-x}$ and $dt = e^{-x}dx$ in (3)

$$\mathbf{\Pi}(x) = \int_0^\infty x^n\, e^{-x} dx, \qquad \forall\, n \in \boldsymbol{W}, \tag{3}$$

where $(-\mathbf{ln}\, t)^n = (-\mathbf{ln}\, e^{-x})^n = (-(-x))^n = x^n$.

By integrating (3), we obtain

$$= [-x^n e^{-x}]_0^\infty - \int_0^\infty -nx^{n-1}\, e^{-x} dx.$$

$$= 0 - \int_0^\infty -nx^{n-1}\, e^{-x} dx.$$

$$\text{Now}, \mathbf{\Pi}(x) = n \int_0^\infty x^{n-1}\, e^{-x} dx. \tag{4}$$

By substituting (3) in (4), we obtain

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1). \tag{5}$$

The gamma function $\Gamma(x)$ is obtained as follows.

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1) \Longrightarrow \Gamma(x+1) = n\Gamma(x). \tag{6}$$

$$n\Gamma(x) = n \int_0^\infty x^{n-1}\, e^{-x} dx \Longrightarrow \Gamma(x) = \int_0^\infty x^{n-1}\, e^{-x} dx. \tag{7}$$

The gamma function [7-8], therefore, is derived from the Euler's factorial function (Pi function) that uses the actual factorial function. Conclusion

4. Conclusion

In this article, the gamma function was proved using the natural logarithm and Euler's factorial function, also known as Pi function.

References

[1] Annamalai, C. (2023) Review on the Gamma Function and Error Correction. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4397585.

[2] Annamalai, C. (2023) Factorial Theorem: An Alternative to Gamma Function. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4376857.

[3] Annamalai, C. (2023) Analysis of Factorial Function for Non-Negative Real Numbers. OSF Preprints. https://dx.doi.org/10.31219/osf.io/4acyu.

[4] Annamalai, C. (2023) Factorials: Difference between 0! and 1!. OSF Preprints. https://dx.doi.org/10.31219/osf.io/bm7uw.

[5] Annamalai, C. (2022) Factorials, Integers, and Factorial Theorems for Computing and Cryptography. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4189241.

[6] Annamalai, C. (2023) Factorial Theorem for Computing the Factorial of Positive Real Number. *COE*, *Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-7p472.

[7] Annamalai, C. (2023) Proof of Gamma Function using Natural Logarithm and Pi Function. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc-v2.

[8] Annamalai, C. (2023) Gamma Function derived from Factorial Function based-Pi Function. *COE*, *Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc.