

An Alternative Method for the Gamma Function derived from Natural Logarithm and Pi Function

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Abstract: Several professors of mathematics from the renowned universities in Australia, Canada, Europe, India, USA, etc. argue with me that the gamma function was not derived from the factorial function. For them, this paper presents the derivation of gamma function from the natural logarithm and Euler's factorial function. Also, a novel factorial theorem, which is alternative to the gamma function, is introduced in this article and it computes the accurate values of factorial for positive real numbers.

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1. Introduction

In general, the factorial [1-6] for positive integer n , denoted by $n!$, is the product of all positive integers less than or equal to n . For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 720$. Note that $0! = 1$. In this article, the gamma function is proved using the natural logarithm and Pi function, also known as Euler's factorial function. Also, a novel factorial theorem, which is alternative to the gamma function, is introduced in this article and it computes the accurate values of factorial for positive real numbers.

2. Natural Logarithm

The natural logarithm of a real or complex number is a technique to solve the exponential function and vice-versa.

The natural logarithm is defined as follows:

$$\log_e u = \ln u.$$

The properties of natural logarithm are given below:

$$\ln u^v = v \ln u.$$

$$\ln uv = \ln u + \ln v.$$

$$\ln \frac{u}{v} = \ln u - \ln v.$$

$$\ln \frac{1}{u} = \ln 1 - \ln u = -\ln u \quad (\because \ln 1 = 0).$$

$$\ln e = 1.$$

$$\text{Let } e^m = n, \text{ then } \ln n = m. \text{ Also, } e^{\ln n} = n \text{ and } \ln e^m = m.$$

$$-\ln e^{-r} = -(-r) = r.$$

3. Gamma Function

The gamma function is proved here using Pi function and natural logarithm

The Pi (Π) function is given below:

$$\Pi(x) = x! = x(x-1)(x-2) \cdots 1, \forall x \in \mathbf{W}, \quad (1)$$

where $\mathbf{W}$ denotes the system of whole numbers.

Euler's factorial function, also known as Pi (Π) function, is the basis for gamma function [1-6].

Swiss mathematician Leonhard Euler has defined the pi function by integral as follows:

$$\Pi(x) = \int_0^1 (-\ln t)^n dt, \forall n \in \mathbf{W}, \quad (2)$$

where $\ln$ denotes the logarithm to the base of the mathematical constant e .

By substituting $t = e^{-x}$; $dt = -e^{-x}dx$; $t = 0 \Rightarrow e^{-x} = 0 \Rightarrow x = \infty$
and $t = 1 \Rightarrow e^{-x} = 1 \Rightarrow x = 0$ in (2), we obtain

$$\Pi(x) = \int_{\infty}^0 -x^n e^{-x} dx = \int_0^{\infty} x^n e^{-x} dx, \quad \forall n \in \mathbf{W}, \quad (3)$$

where $(-\ln t)^n = (-\ln e^{-x})^n = (-(-x))^n = x^n$.

By integrating (3), we obtain

$$\begin{aligned} &= [-x^n e^{-x}]_0^{\infty} - \int_0^{\infty} -nx^{n-1} e^{-x} dx. \\ &= [x^{\infty} e^{-\infty} - 0] - \int_0^{\infty} -nx^{n-1} e^{-x} dx. \end{aligned}$$

Here, $x^{\infty} e^{-\infty}$ is in indeterminate form. Let us apply L'Hospital's rule,

$$\begin{aligned} i. e. \lim_{x \rightarrow \infty} -\frac{x^n}{e^x} &= \lim_{x \rightarrow \infty} -\frac{nx^{n-1}}{e^x} = \cdots = \lim_{x \rightarrow \infty} -\frac{n(n-1) \cdots 1}{e^x} = 0. \\ &= 0 - \int_0^{\infty} -nx^{n-1} e^{-x} dx. \end{aligned}$$

$$\text{Now, } \Pi(x) = n \int_0^{\infty} x^{n-1} e^{-x} dx. \quad (4)$$

By substituting (3) in (4), we obtain

$$\Pi(x) = n\Pi(x-1). \quad (5)$$

The gamma function $\Gamma(x)$ is obtained as follows.

$$\Pi(x) = n\Pi(x-1) \Rightarrow \Gamma(x+1) = n\Gamma(x). \quad (6)$$

$$n\Gamma(x) = n \int_0^{\infty} x^{n-1} e^{-x} dx \Rightarrow \Gamma(x) = \int_0^{\infty} x^{n-1} e^{-x} dx. \quad (7)$$

The gamma function [7-8], therefore, is derived from the Euler's factorial function (Pi function) that uses the actual factorial function.

4. Error in Calculation using Gamma Function

The gamma function for integer values is given below:

$$\Gamma(x+1) = x(x-1)! = x\Gamma(x).$$

For examples,

$$\Gamma(1) = 0! = 1; \Gamma(2) = 1! = 1; \Gamma(3) = 2! = 2.$$

The gamma function [7-10] for real arguments or real numbers is shown below:

$$\Gamma(r) = (r-1)! = \int_0^{\infty} t^{r-1} e^{-t} dt.$$

For example,

$$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)! = \left(-\frac{1}{2}\right)! = \sqrt{\pi}.$$

It is understood that $\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi} \approx 1.77245385091$,

but $\Gamma(3) > \Gamma\left(\frac{1}{2}\right) > \Gamma(2)$, which is a contradiction.

$$\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 1 - 1\right)! = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}.$$

So, $\Gamma\left(\frac{1}{2}\right)$ is not true and also calculating the value of $\Gamma\left(\frac{3}{2}\right)$ using $\Gamma\left(\frac{1}{2}\right)$ can not be true.

From these results, it is concluded that the calculated values using $\Gamma\left(\frac{1}{2}\right)$ are not true.

5. Factorial Theorem

Theorem 2. 1: $(x+1)! = x! + \{(x+1)! - x!\}$, where $x \geq 0$.

Proof:

$$x! + \{(x+1)! - x!\} = x! + \{x!(x+1) - x!\} = x! + x!(x) = x!\{1+x\} = (x+1)!, (x \geq 0).$$

Hence, theorem is proved.

Let us verify the theorem with positive real numbers.

$$\begin{aligned} (x+1)! &= x! + \{(x+1)! - x!\} = x! + (0.64)\{(x+1)! - x!\} + (0.36)\{(x+1)! - x!\} \\ x! + (0.64 + 0.36)\{(x+1)! - x!\} &= x! + \{(x+1)! - x!\}. \end{aligned}$$

Theorem 2. 2: $y! = (y+1)! - \{(y+1)! - y!\}$, where $y \geq 1$.

Proof:

$$\begin{aligned} (y+1)! - \{(y+1)! - y!\} &= y!(y+1) - \{y!(y+1) - y!\} \\ &= y!(y+1) - y!(y) = y!(y+1-y) = y!, (y \geq 1). \end{aligned}$$

Hence, theorem is proved.

Let us verify the theorem with positive real numbers.

$$\begin{aligned} y! &= (y+1)! - \{(y+1)! - y!\} \\ &= (y+1)! - (1-0.64)\{(y+1)! - y!\} - (1-0.36)\{(y+1)! - y!\} \\ &= (y+1)! - (0.36)\{(y+1)! - y!\} - (0.64)\{(y+1)! - y!\} \\ &= (y+1)! - \{(y+1)! - y!\}. \end{aligned}$$

6. Factorial for Positive Real number

Let $z = i.f$ be a positive real number, where i is an integer part, f is the decimal or fractional part, and $z \geq 1$. The factorial function to positive real number is established from the perspective of the above theorems as follows:

$$z! = (i.f)! = i! + (0.f)\{(i+1)! - i!\}.$$

For examples,

$$(1.0)! = 1! + (0.0)\{(i+1)! - i!\} = 1 + 0 = 1.$$

$$(4.6)! = 4! + (0.6)(5! - 4!) = 24 + (0.6)(120 - 24) = 81.6.$$

The gamma function calculator [15] computes the factorial function $(4.6)!$ as follows:

$$\Gamma(5.6) = (5.6 - 1)! = (4.6)! = 61.5.$$

7. Factorial for Rational and Irrational numbers

If $\frac{p}{q}, (q \neq 0) = i\frac{f}{q}$ is a common rational number, where $\frac{p}{q} \geq 1, f < q$, & i is an integer.

$$\text{Then, } \left(\frac{p}{q}\right)! = i! + \left(\frac{f}{q}\right)\{(i+1)! - i!\}.$$

For examples,

$$\begin{aligned} 1! &= \left(1\frac{0}{1}\right)! = 1! + \left(\frac{0}{1}\right)\{(1+1)! - 1!\} = 1 + 0 = 1. \\ \left(\frac{14}{4}\right)! &= \left(3\frac{2}{4}\right)! = 3! + \left(\frac{2}{4}\right)(4! - 3!) = 6 + 9 = 15. \end{aligned}$$

Let $\sqrt{x}$ be a irrational number, where $\sqrt{x} \geq 1$. If i is the integer part on $\sqrt{x}$.

$$\text{Then, } (\sqrt{x})! = i! + \sqrt{x}\{(i+1)! - i!\} - i\{(i+1)! - i!\}.$$

For examples,

$$\begin{aligned} (\sqrt{1})! &= 1! + \sqrt{1}(2! - 1! - 1(2! - 1!)) = 1 + 1 - 1 = 1. \\ (\sqrt{15})! &= 3! + \sqrt{15}(4! - 3!) - 3(4! - 3!). \\ (119)! &= 10! + \sqrt{119}(11! - 10!) - 10(11! - 10!). \end{aligned}$$

8. Conclusion

In this article, factorial theorem for positive real numbers has been introduced and it is an alternative to the gamma function. This result can be used as an application in computing and mathematical sciences including probability and statistics.

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