Non trivial zeros of the Zeta function using the differential equations

W. Oukil
Faculty of Mathematics.
University of Science and Technology Houari Boumediene.
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

April 15, 2023

Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

Keywords: Zeta function, Riemann hypothesis, Differential equations.
AMS subject classifications: 00A05

1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} \frac{\{t\}}{t^{1+s}}dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \qquad (1)$$

where $\{t\}$ is the fractional part of the real t. We prove the following Theorem.

Theorem 1. *Consider the function ζ defined by the Equation* (1)*. For every $\tau \in \mathbb{R}^*$ and $r \in (\frac{1}{2}, 1)$ we have*

$$|\zeta(r + i\tau)| \neq 0.$$

Thanks to the Riemann functional equation we deduce that any *non trivial zero* of the Zeta function has a real part equal to $\frac{1}{2}$, where the *non trivial zeros* are defined in the following sense

Definition 2. Consider the function ζ defined by the Equation (1). Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function ζ* if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*.$$

2 Main Proposition

For every $r \in (0,1)$ and $\tau \in \mathbb{R}^*$ the Equation (1) implies,

$$\frac{\zeta(r+i\tau)}{r+i\tau} = -\frac{1}{1-r-i\tau} - \int_1^{+\infty} u^{-i\tau-1-r}\{u\}du.$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_{\tau,r}(z,t) := t^{r+i\tau}\Big[z + \int_1^t u^{-i\tau-1-r}\{u\}du\Big], \quad z \in \mathbb{C},\ t \geq 1.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to prove that $\sup_{t\geq 1} |\psi_{\tau,r}(\frac{1}{1-r-i\tau}, t)| < +\infty$ implies $2r \leq 1$. In other words $|\frac{\zeta(r+i\tau)}{r+i\tau}| = 0$ implies $2r \leq 1$.

For every $\tau \in \mathbb{R}^*$ and $r \in (0,1)$ we consider the following differential equation

$$\frac{d}{dt}x = (r+i\tau)t^{-1}x + t^{-1}\{t\}, \tag{2}$$

$$t \in \mathbb{R}_+/\mathbb{N}, \quad x(1) = \frac{1}{1-r-i\tau}, \quad x : [1,+\infty) \to \mathbb{C}.$$

In this paper we derive the functions only on $\mathbb{R}_+/\mathbb{N}$.

Lemma 3. *For every $\tau \in \mathbb{R}^*$ and $r \in (0,1)$ there exists a unique continuous solution $\psi_{\tau,r}(t) : [1,+\infty) \to \mathbb{C}$ of the differential equation (2). Further,*

$$\psi_{\tau,r}(t) = t^{r+i\tau}\int_0^t u^{-i\tau-1-r}\{u\}du, \quad \forall t \geq 1.$$

Proof. Let be $r \in (0,1)$ and $\tau \in \mathbb{R}^*$ fixed. Since $\{u\} = u$ for every $u \in (0,1)$ then

$$t^{r+i\tau}\int_0^t u^{-i\tau-1-r}\{u\}du = \frac{t}{1-r-i\tau}, \quad \forall t \in [0,1].$$

Since $0 \leq \{u\} \leq 1$ for every $u \geq 1$ then the function

$$t \mapsto t^{r+i\tau}\int_0^t u^{-i\tau-1-r}\{u\}du,$$

is continuous and C^1 on $\mathbb{R}_+/\mathbb{N}$. The Equation (2) is a non-homogeneous linear differential equation. The unique continuous solution $\psi_{\tau,r}(t) : [1,+\infty) \to \mathbb{C}$ such that $\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau}$ is given by

$$\psi_{\tau,r}(t) = t^{r+i\tau}\int_0^t u^{-i\tau-1-r}\{u\}du,\ \forall t \geq 0.$$

□

Proposition 4. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0,1)$. Consider the continuous solution $\psi_{\tau,r}(t) : [1,+\infty) \to \mathbb{C}$ of the differential equation* (2)*. Suppose that $\sup_{t\geq 1} |\psi_{\tau,r}(t)| < +\infty$, then $2r \leq 1$.*

Notation 5. Denote

$$p(t) := \frac{1}{12} + \int_0^t \Big(\{u\} - \frac{1}{2}\Big) du, \quad \forall t \geq 0,$$

where we recall that $\{u\}$ is the fractional part of the real u.

Lemma 6. *The function p is a continuous 1-periodic function and satisfies*

$$p(t) = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \exp(ij2\pi t), \quad \forall t \geq 0.$$

Proof. The function $u \mapsto \{u\}$ is 1-periodic, then there exists a continuous 1-periodic function $p : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^t \{u\} du = t \int_0^1 \{u\} du - \frac{1}{12} + p(t), \quad \forall t \geq 0.$$

Since

$$\int_0^1 \{u\} du = \int_0^1 u \ du = \frac{1}{2},$$

we get

$$\int_0^t \{u\} du = t\frac{1}{2} - \frac{1}{12} + p(t), \quad \forall t \geq 0.$$

The function p is a piecewise C^∞, continuous on $\mathbb{R}$ and 1-periodic. By Dirichlet Theorem, the Fourier series

$$n \mapsto \sum_{k=-n}^{n} a_k \exp(ik2\pi t),$$

converge uniformly on $\mathbb{R}_+$ to the function $t \mapsto p(t)$, where $(a_k)_k \subset \mathbb{C}$ are the Fourier coefficients of the function p.

$$\int_0^t \Big(\{v\} - \frac{1}{2}\Big) dv + \frac{1}{12} = \frac{1}{12} + \sum_{j \in \mathbb{Z}} a_j \exp(ij2\pi t), \quad \forall t \geq 0.$$

By definition of the Fourier coefficients we have

$$\begin{aligned} a_j &= \int_0^1 \exp(-ij2\pi u)\Big(\int_0^u \Big(\{v\} - \frac{1}{2}\Big)dv\Big)du \\ &= \frac{1}{2}\int_0^1 \exp(-ij2\pi u)u(u-1)du = \frac{1}{(2j\pi)^2}, \quad \forall j \in \mathbb{Z}^*, \end{aligned}$$

and

$$a_0 = \frac{1}{2}\int_0^1 u(u-1)du = -\frac{1}{12}.$$

The function p satisfies

$$p(t) = \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^2}\exp(ij2\pi t), \quad \forall t \geq 0.$$

□

Lemma 7. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0,1)$. Let $\psi_{\tau,r}(t) : [1,+\infty) \to \mathbb{C}$ be the continuous solution of the differential equation (2). Suppose that $\sup_{t\geq 1}|\psi_{\tau,r}(t)| < +\infty$. Then*

$$\sup_{n\in\mathbb{N}^*}\Big|n^3\theta_{\tau,r}(n)\Big| < +\infty \quad \textit{and} \quad \sup_{\substack{k,n\in\mathbb{N}^*\\ k\geq n}}\Big|n^{2(1+r)}\int_n^k u^{-2r-1}\lfloor u\rfloor\theta_{\tau,r}(u)du\Big| < +\infty,$$

where

$$\theta_{\tau,r}(u) := \psi_{\tau,r}(u) + \frac{1}{2}\frac{1}{r+i\tau} - p(u)u^{-1}, \quad \forall u \geq 1,$$

and where $u \mapsto \lfloor u\rfloor$ is the floor function.

Proof. Let be $r \in (0,1)$ and $\tau \in \mathbb{R}^*$ fixed. By Lemma 3

$$\psi_{\tau,r}(t) = t^{r+i\tau}\int_0^t u^{-i\tau-1-r}\{u\}du, \quad \forall t \geq 1. \tag{3}$$

Suppose that

$$\sup_{t\geq 1}|\psi_{\tau,r}(t)| < +\infty.$$

Since

$$\sup_{t\geq 1}|\psi_{\tau,r}(t)| < +\infty \implies \Big|\int_0^{+\infty} u^{-i\tau-1-r}\{u\}du\Big| = 0,$$

then

$$\int_0^t u^{-i\tau-1-r}\{u\}du = -\int_t^{+\infty} u^{-i\tau-1-r}\{u\}du, \quad \forall t > 1.$$

Equation (3) can be written as

$$\psi_{\tau,r}(t) = -t^{r+i\tau}\int_t^{+\infty} u^{-i\tau-1-r}\{u\}du, \quad \forall t > 1. \tag{4}$$

Consider the function p given in the Notation 5. By Lemma 6, the function p is 1-periodic, then it is bounded. Use the integration part formula in Equation (4),

$$\begin{aligned}\psi_{\tau,r}(t) &= -\frac{1}{2}\frac{1}{r+i\tau} + p(t)t^{-1} \\ &\quad - \Big(\Pi_{j=1}^{2}(i\tau+j+r)\Big)t^{i\tau+r}\int_t^{+\infty} u^{-i\tau-3-r}\int_t^u p(v)dvdu,\end{aligned}$$

or by the notation of $\theta_{\tau,r}$ of the present Lemma, we have

$$\theta_{\tau,r}(t) = -\Big(\Pi_{j=1}^{2}(i\tau+j+r)\Big)t^{i\tau+r}\int_t^{+\infty} u^{-i\tau-3-r}\int_t^u p(v)dvdu. \tag{5}$$

By Lemma 6, we have

$$p(u) = \sum_{j\in\mathbb{Z}^*}\frac{1}{(j2\pi)^2}\exp(ij2\pi u), \quad \forall u \geq 1.$$

Since $\sum_{j\in\mathbb{Z}^*}\frac{1}{(j2\pi)^3} = 0$, then

$$\int_n^v p(\mu)d\mu = -i\sum_{j\in\mathbb{Z}^*}\frac{1}{(j2\pi)^3}\exp(ij2\pi v), \quad \forall v \geq 1,\ \forall n \in \mathbb{N},$$

and

$$\Big|\int_s^u\int_n^v p(\mu)d\mu dv\Big| \leq 2\sum_{j\in\mathbb{Z}^*}\frac{1}{(j2\pi)^4} < 1, \quad \forall u \geq s,\ \forall n \in \mathbb{N}. \tag{6}$$

Item 1.
Using the integration part formula in the Equation (5), for every $n \in \mathbb{N}$ we obtain

$$|\theta_{\tau,r}(n)| \leq \Pi_{j=1}^{3}|j+r+i\tau|\Big|n^{i\tau+r}\int_n^{+\infty} u^{-i\tau-4-r}\int_n^u\int_n^v p(\mu)d\mu dvdu\Big|.$$

By the Equation (6), we get

$$\frac{|\theta_{\tau,r}(n)|}{|3+r+i\tau|^3} < n^r \int_n^{+\infty} u^{-4-r} < n^{-3}, \quad \forall n \in \mathbb{N}^*.$$

Item 2.
In order to simplify the notation, for every $k, n \in \mathbb{N}^*$, denote

$$V_{\tau,r}(k,n) := \int_n^k u^{-1}\lfloor u\rfloor u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \int_u^v p(\mu)d\mu dv du.$$

For every $k, n \in \mathbb{N}^*$, we have

$$\begin{aligned} V_{\tau,r}(k,n) &= -i\int_n^k u^{-1}\lfloor u\rfloor u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi v)dv \\ &\quad + \frac{i}{i\tau+2+r}\int_n^k \lfloor u\rfloor u^{-3-2r} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi u)du. \end{aligned}$$

Implies

$$\begin{aligned} V_{\tau,r}(k,n) &= -i\int_n^k u^{-1}\lfloor u\rfloor u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-3-r} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi v)dv \\ &\quad - \frac{i}{i\tau+2+r}\int_n^k \{u\} u^{-3-2r} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi u)du \\ &\quad + \frac{i}{i\tau+2+r}\int_n^k u^{-2(1+r)} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi u)du. \end{aligned}$$

Use the fact

$$\sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3} = 0 \quad \text{and} \quad \int_n^k \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi v)dv = 0,$$

and use the integration part formula to obtain

$$\begin{aligned} &V_{\tau,r}(k,n) \\ &= (i\tau+3+r)\int_n^k u^{-1}\lfloor u\rfloor u^{i\tau-r} \int_u^{+\infty} v^{-i\tau-4-r} \int_u^v \int_n^s p(\mu)d\mu ds dv du \\ &- \frac{i}{i\tau+2+r}\int_n^k \{u\} u^{-3-2r} \sum_{j\in\mathbb{Z}^*} \frac{1}{(j2\pi)^3}\exp(ij2\pi u)du \\ &- 2\frac{1+r}{i\tau+2+r}\int_n^k u^{-3-2r} \int_n^u \int_n^v p(\mu)d\mu dv du. \end{aligned} \tag{7}$$

Using the Equation (6); the Equation (5) and (7) implies

$$\Big|\int_n^k u^{-2r-1}\lfloor u\rfloor\theta_{\tau,r}(u)du\Big| = \Pi_{j=1}^2|i\tau+j+r|\Big|V_{\tau,r}(k,n)\Big|$$
$$< |3+r+i\tau|^3\int_n^k u^{-r}\int_u^{+\infty} v^{-4-r}dvdu$$
$$+5|1+r+i\tau|\int_n^k u^{-3-2r}du < 6|3+r+i\tau|^3 n^{-2(r+1)}.$$

□

Proof of the Proposition 4. Let be $r\in(0,1)$ and $\tau\in\mathbb{R}^*$. Use the change of variable

$$\tilde{\psi}_{\tau,r}(t) := -\frac{t}{1-r-i\tau} + (r+i\tau)\psi_{\tau,r}(t) + \{t\},\quad t\geq 1.$$

The Equation (2) can be written as

$$\frac{d}{dt}\tilde{\psi}_{\tau,r}(t) = t^{-1}(r+i\tau)\tilde{\psi}_{\tau,r}(t),\quad \forall t\in\mathbb{R}_+/\mathbb{N},\ t\geq 1. \tag{8}$$

For the particular initial condition of $\psi_{\tau,r}$, we have

$$\psi_{\tau,r}(1) = \frac{1}{1-r-i\tau},$$

by the definition of $\tilde{\psi}_{\tau,r}$ we get

$$\tilde{\psi}_{\tau,r}(1) = -1.$$

The Equation (8) implies the following new differential equation,

$$t^{-1}\tilde{\psi}_{\tau,r}(t) = (r+i\tau)t^{-1}\int_1^t u^{-1}\tilde{\psi}_{\tau,r}(u)du - t^{-1}\lfloor t\rfloor,\quad \forall t\geq 1,$$

where we recall that $u\mapsto\lfloor u\rfloor$ is the floor function. The wronksien is given by

$$\frac{d}{dt}\Psi_{\tau,r}(t) = 2rt^{-1}\Psi_{\tau,r}(t) - 2t^{-1}\lfloor t\rfloor w_{\tau,r}(t), \tag{9}$$

where in order to simplify the notation, we denoted

$$\Psi_{\tau,r}(t) := \Big|\int_1^t u^{-1}\tilde{\psi}_{\tau,r}(u)du\Big|^2 \quad\text{and}\quad w_{\tau,r}(t) := \int_1^t u^{-1}\Re\Big(\tilde{\psi}_{\tau,r}(u)\Big)du.$$

By definition of $\tilde{\psi}_{\tau,r}$ we have

$$\begin{aligned}\int_1^t u^{-1}\tilde{\psi}_{\tau,r}(u)du &= -\frac{(t-1)}{1-r-i\tau} + \int_1^t u^{-1}\Big[(r+i\tau)\psi_{\tau,r}(u) + \{u\}\Big]du \\ &= -\frac{(t-1)}{1-r-i\tau} + \int_1^t \frac{d}{du}\psi_{\tau,r}(u)du \\ &= -\frac{t}{1-r-i\tau} + \psi_{\tau,r}(t)\end{aligned}$$

By consequence

$$\begin{cases} \Psi_{\tau,r}(t) &= \frac{1}{(1-r)^2+\tau^2}t^2 + \Big|\psi_{\tau,r}(t)\Big|^2 \\ & -2\frac{(1-r)\Re(\psi_{\tau,r}(t))+\tau\Im(\psi_{\tau,r}(t))}{(1-r)^2+\tau^2}t, \quad \forall t \geq 1, \\ w_{\tau,r}(t) &= -\frac{(1-r)}{(1-r)^2+\tau^2}t + \Re\Big(\psi_{\tau,r}(t)\Big), \quad \forall t \geq 1. \end{cases} \tag{10}$$

Prove the Proposition by contradiction. Suppose that $2r > 1$ and $\sup_{t\geq 1}|\psi_{\tau,r}(t)| < +\infty$. By The Lemma 7, we have

$$\sup_{n\in\mathbb{N}^*}\Big|n^3\psi_{\tau,r}(n) + \frac{1}{2}\frac{1}{r+i\tau}n^3 - n^2p(n)\Big| < +\infty. \tag{11}$$

Using the Equation (10), we get

$$\sup_{n\in\mathbb{N}^*} n^2\Big|\alpha_{\tau,r}\Psi_{\tau,r}(n) - \Phi_{\tau,r}(n)\Big| < +\infty, \tag{12}$$

where

$$\Phi_{\tau,r}(n) := n^2 - \frac{\beta_{\tau,r}-r}{\beta_{\tau,r}}n + \frac{\alpha_{\tau,r}}{4\beta_{\tau,r}} - 2(1-r)p(n),$$

and where

$$\alpha_{\tau,r} := (1-r)^2+\tau^2 \quad \text{and} \quad \beta_{\tau,r} := r^2+\tau^2.$$

Integrate the equation (9) as a non-homogeneous linear differential equation, we obtain

$$\Psi_{\tau,r}(t) = t^{2r}h_{\tau,r}(t,s), \quad \forall t \geq s \geq 1, \tag{13}$$

$$h_{\tau,r}(t,s) := s^{-2r}\Psi_{\tau,r}(s) - 2\int_s^t u^{-2r-1}\lfloor u\rfloor w_{\tau,r}(u)du.$$

By hypothesis $2r > 1$. From the Equation (12), remark that

$$\lim_{k\to+\infty} k^{-2r}\Big[k^2 - \alpha_{\tau,r}\Psi_{\tau,r}(k)\Big] = 0.$$

The strategy to prove the present Proposition is to find a constant time $n_{\tau,r} >> 1$ such that

$$\lim_{k\to+\infty}\left|k^{2(1-r)} - \alpha_{\tau,r}h_{\tau,r}(k,n_{\tau,r})\right| > 0,$$

which gives a contradiction. In other words, the aim is to studies the asymptotic behavior of the function (solution) $t \mapsto t^2 - \alpha_{\tau,r}\Psi_{\tau,r}(t)$ and find this function, when $2r > 1$, can not approximate a polynomial function of first order. More precisely, the polynomial approximated by this function is of $2r$ order. We have

$$\begin{aligned}k^{2(1-r)} - \alpha_{\tau,r}h_{\tau,r}(k,n) &= k^{2(1-r)} - \alpha_{\tau,r}n^{-2r}\Psi_{\tau,r}(n)\\ &\quad + 2\alpha_{\tau,r}\int_n^k u^{-2r-1}\lfloor u\rfloor w_{\tau,r}(u)du, \quad \forall k,n\in\mathbb{N}^*.\end{aligned}$$

Use the second Item of the Lemma 7 and the Equation (10). Use the Equations (12), we get

$$\sup_{\substack{k,n\in\mathbb{N}^*\\ k\geq n}}\left|n^{2(1+r)}\Big[k^{2(1-r)} - \alpha_{\tau,r}h_{\tau,r}(k,n) - \Gamma_{\tau,r}(k,n)\Big]\right| < +\infty, \tag{14}$$

where

$$\begin{aligned}\Gamma_{\tau,r}(k,n) &:= k^{2(1-r)} - n^{-2r}\Phi_{\tau,r}(n)\\ &\quad - \int_n^k u^{-2r-1}\lfloor u\rfloor\Big[2(1-r)u + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} - 2\alpha_{\tau,r}p(u)u^{-1}\Big]du,\end{aligned}$$

or by the notation of $\Phi_{\tau,r}$,

$$\begin{aligned}\Gamma_{\tau,r}(k,n) &= k^{2(1-r)} - n^{2(1-r)} + \frac{\beta_{\tau,r} - r}{\beta_{\tau,r}}n^{1-2r}\\ &\quad - \Big(\frac{\alpha_{\tau,r}}{4\beta_{\tau,r}} - 2(1-r)p(n)\Big)n^{-2r}\\ &\quad - \int_n^k u^{-2r-1}\lfloor u\rfloor\Big[2(1-r)u + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} - 2\alpha_{\tau,r}p(u)u^{-1}\Big]du.\end{aligned}$$

As in the Proof of the Lemma 7, we use the integration part formula. Using the fact

$$\frac{\beta_{\tau,r} - r}{\beta_{\tau,r}} + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}(1-2r)} - \frac{1-r}{1-2r} = 0,$$

Since $2r > 1$, we have

$$\lim_{k\to+\infty}\Gamma_{\tau,r}(k,n) = 2(1-r)p(n)n^{-2r} + 2\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-1}p(u)du$$
$$+ 2(1-r)\int_n^{+\infty} u^{-2r}(\{u\}-\frac{1}{2})du + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}\int_n^{+\infty} u^{-2r-1}(\{u\}-\frac{1}{2})du$$
$$- 2\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-2}\{u\}p(u)du.$$

Implies

$$\lim_{k\to+\infty}\Gamma_{\tau,r}(k,n) = 2\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-1}p(u)du$$
$$+ 4r(1-r)\int_n^{+\infty} u^{-2r-1}p(u)du + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}\int_n^{+\infty} u^{-2r-1}(\{u\}-\frac{1}{2})du$$
$$- \alpha_{\tau,r}\int_n^{+\infty} u^{-2r-2}p(u)du - 2\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-2}(\{u\}-\frac{1}{2})p(u)du.$$

$$\begin{aligned}\lim_{k\to+\infty}\Gamma_{\tau,r}(k,n) = -\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}n^{-2r-1}p(n)&\\
&+ 2(r+1)(2r+1)\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}\int_n^{+\infty} u^{-2r-3}\int_n^u p(v)dvdu\\
&+ 8r(1-r^2)(2r+1)\int_n^{+\infty} u^{-2r-3}\int_n^u\int_n^v p(\mu)d\mu dvu\\
&+ 4(2r+1)(r+1)\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-3}\int_n^u\int_n^v p(\mu)d\mu dvdu\\
&- 2(r+1)\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-3}\int_n^u p(v)dvdu\\
&- 2(r+1)\alpha_{\tau,r}\int_n^{+\infty} u^{-2r-3}\Big((p(u))^2-(p(n))^2\Big)du.\end{aligned}$$

Since

$$\Big|\int_n^u p(v)dv\Big| \le 1 \quad \text{and} \quad \Big|\int_n^u\int_n^v p(\mu)d\mu dv\Big| \le 1, \quad \forall u \ge n,\ n\in\mathbb{N},$$

Then, there exists $c_{\tau,r} > 0$ such that for all $n \in \mathbb{N}^*$ we have

$$\Big|\lim_{k\to+\infty}\Gamma_{\tau,r}(k,n) + \frac{r\alpha_{\tau,r}}{\beta_{\tau,r}}p(n)n^{-2r-1}\Big| < c_{\tau,r}n^{-2(1+r)}.$$

Since

$$p(n) = \frac{1}{12}, \quad \forall n \in \mathbb{N}$$

Then

$$\Big| \lim_{k \to +\infty} \Gamma_{\tau,r}(k,n) + \frac{1}{12}\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n^{-2r-1} \Big| < c_{\tau,r} n^{-2(1+r)}.$$

From the Equation (14), there exist $n_{\tau,r} \in \mathbb{N}^*$ s uch that

$$\lim_{k \to +\infty} \Big| k^{2(1-r)} - \alpha_{\tau,r} h_{\tau,r}(k, n_{\tau,r}) \Big| > \frac{1}{24}\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n_{\tau,r}^{-2r-1} > 0.$$

The Equation (13) implies

$$\lim_{k \to +\infty} k^{-2r} \Big| k^2 - \alpha_{\tau,r} \Psi_{\tau,r}(k) \Big| > \frac{1}{24}\frac{r\alpha_{\tau,r}}{\beta_{\tau,r}} n_{\tau,r}^{-2r-1}.$$

But since $r \in (\frac{1}{2}, 1)$, by the Equation (12) we have

$$\lim_{k \to +\infty} k^{-2r} \Big[k^2 - \alpha_{\tau,r} \Psi_{\tau,r}(k) \Big] = 0.$$

Contradiction. □

Remark 8. In the previous Proof, using the fact

$$\tilde{\psi}_{\tau,r}(1) = -1.$$

remark that the sequence $\left(n^{-r-i\tau} \tilde{\psi}(n) \right)_{n \in \mathbb{N}^*}$ is given by the Riemann series;

$$n^{-r-i\tau} \tilde{\psi}(n) = -\sum_{k=1}^{n} \frac{1}{k^{r+i\tau}}, \quad \forall n \in \mathbb{N}^*.$$

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $x \in (0,1)$ and $\tau \in \mathbb{R}^*$. Suppose that $|\zeta(x+i\tau)| = 0$. By the Equation (1) we have,

$$\frac{\zeta(x+i\tau)}{x+i\tau} = -\int_0^{+\infty} u^{-i\tau-1-x} \{u\} du.$$

Then

$$\Big| \int_0^{+\infty} u^{-i\tau-1-x} \{u\} du \Big| = 0.$$

Implies

$$\int_0^t u^{-i\tau-1-x}\{u\}du = -\int_t^{+\infty} u^{-i\tau-1-x}\{u\}du, \quad \forall t \geq 1.$$

$$\begin{aligned}\sup_{t\geq 1}\left|t^{x+i\tau}\int_0^t u^{-i\tau-1-x}\{u\}du\right| &= \sup_{t\geq 1}\left|t^{x+i\tau}\int_t^{+\infty} u^{-i\tau-1-x}\{u\}du\right| \\ &\leq \sup_{t\geq 1}\left[t^x\int_t^{+\infty} u^{-1-x}du\right] \leq \frac{1}{x}. \end{aligned} \tag{15}$$

Let $\psi_{\tau,x}(t) : [1, +\infty) \to \mathbb{C}$ be the unique continuous solution of the differential equation (2). By Lemma 3 and the Equation (15) we have,

$$\sup_{t\geq 1}|\psi_{\tau,x}(t)| < +\infty,$$

Thanks to the Main Proposition 4 we get $2x \leq 1$. □

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).