

ON SUMS INVOLVING DIVISOR FUNCTION, EULER'S TOTIENT FUNCTION, AND FLOOR FUNCTION

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ABSTRACT. Every positive integer $l \in \mathbb{N}$ can be formed $l = (m + n)d$, provided $\gcd(m, n) = 1$. From this point of view, the next formulas $n = \sum_{d|l} \varphi(d)$ and $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor$, and these equivalence had been proved. In this paper on an extension of these results, the next formula is proved: $\sum_{k=1}^n \sum_{\substack{(a+b)c=k \\ \gcd(a,b)=1}} f(a, b) \cdot g(c) = \sum_{k=1}^n \sum_{\substack{a+b=k \\ \gcd(a,b)=1}} f(a, b) \sum_{i \leq \lfloor \frac{n}{k} \rfloor} g(i) = \sum_{a+b \leq n} f(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}) \cdot g(\gcd(a, b))$. As corollaries, we have $\sum_{k=1}^n \tau(k) = \sum_{k=1}^n \lfloor \frac{n}{k} \rfloor = \sum_{a+b \leq n} \frac{1}{\varphi(\frac{a+b}{\gcd(a,b)})}$, $\tau(n) = \sum_{a+b=n} \frac{1}{\varphi(\frac{a+b}{\gcd(a,b)})}$, $\sum_{a+b=n} \frac{1}{\varphi(\frac{a+b}{\gcd(a,b)})} \cdot \gcd(a-1, b+1) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi(\frac{n}{\gcd(a,b)})}$, and so on. In addition to it, we evaluate a sequence $\sum_{k=1}^n \varphi(k) \tau(k)$.

1. INTRODUCTION

Every positive integer $l \in \mathbb{N}$ can be formed $l = (m + n)d$, provided $\gcd(m, n) = 1$. We can have the sum of positive divisors function $\sigma_x(l)$ from divisors d , and Euler's totient function $\varphi(m + n)$ from pairs of relatively prime positive integers (m, n) .

From this point of view, the formula $n = \sum_{d|l} \varphi(d)$ can be proved. An equivalent formula $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor$ had been proved in the previous paper [3]. From the both formulas, in other words $\sum_{k=1}^n \sum_{d|k} \varphi(d) = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor$, we can expect relationships between divisor function, Euler's totient function, and floor function. Actually, we have known the next formula in arithmetic summatory function [5]:

$$\sum_{k=1}^n \sum_{d|k} f(d) g(\frac{k}{d}) = \sum_{k=1}^n f(k) \sum_{i \leq \lfloor \frac{n}{k} \rfloor} g(i).$$

In this paper, we expand this formula to

$$\begin{aligned} \sum_{k=1}^n \sum_{\substack{(a+b)c=k \\ \gcd(a,b)=1}} f(a, b) \cdot g(c) &= \sum_{k=1}^n \sum_{\substack{a+b=k \\ \gcd(a,b)=1}} f(a, b) \sum_{i \leq \lfloor \frac{n}{k} \rfloor} g(i) \\ &= \sum_{a+b \leq n} f(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}) \cdot g(\gcd(a, b)), \end{aligned}$$

provided that we think $(a, b) = (k, 0)$, and define $\gcd(k, 0) := k$. The former formula is proved by the coincidence between the pair of divisors and multiples. In addition to it, the latter formula is proved by the coincidence between the triple of divisors,

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multiples, and the pair of sums. In other words, we make the expansion from the point of view of the previous paper $l = d(m + n)$.

As a corollary, we prove the next formula:

$$\sum_{k=1}^n \tau(k) = \sum_{k=1}^n \left\lfloor \frac{n}{k} \right\rfloor = \sum_{a+b \leq n} \frac{1}{\varphi\left(\frac{a+b}{\gcd(a,b)}\right)}.$$

The first equality is a well-known formula in a basis of divisor summatory function [1]. It is clear that the formula shows rich relationships between divisor function, floor function, Euler's totient function, sum, multiple, indivisibility, and so on. It is also interesting Euler's totient function appears in reciprocals.

We also prove the other corollaries; $\tau(n) = \sum_{a+b=n} \frac{1}{\varphi\left(\frac{a+b}{\gcd(a,b)}\right)}$, the average of the sum of divisors plus the average of remainder until n equals n , $\sum_{k=1}^n \sigma_x(k) \leq \sum_{k=1}^n k^{x-1}n$, and so on.

In addition to it, we evaluate a sequence $\sum_{k=1}^n \varphi(k)\tau(k)$, which is analogized and motivated from the formulas: $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor$, $\sum_{k=1}^n \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k=1}^n \tau(k)$, and $\varphi(n)\tau(n) = \sum_{\substack{a+b=n \\ \gcd(a,b)=1}} \gcd(a-1, b+1) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi\left(\frac{n}{\gcd(a,b)}\right)}$. We use symbols σ and τ as $\sigma_1 = \sigma$ and $\sigma_0 = \tau$, $[x]$ denotes floor function, and also $\varphi(x)$ denotes Euler's totient function.

2. THEOREM, COROLLARIES, AND EVALUATION

Theorem 2.1.

$$\begin{aligned} \sum_{k=1}^n \sum_{\substack{(a+b)c=k \\ \gcd(a,b)=1}} f(a,b) \cdot g(c) &= \sum_{k=1}^n \sum_{\substack{a+b=k \\ \gcd(a,b)=1}} f(a,b) \sum_{i \leq \left\lfloor \frac{n}{k} \right\rfloor} g(i) \\ &= \sum_{a+b \leq n} f\left(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(a,b)}\right) \cdot g(\gcd(a,b)), \end{aligned}$$

provided that we think $(a,b) = (k,0)$, and define $\gcd(k,0) := k$.

Proof. Every positive integer $1 \leq l \leq N$ can be formed $l = (m+n)d$, provided $\gcd(m,n) = 1$. T_N denotes the set of these triples (m,n,d) . In the case, we think $m+n$ and d are divisors of l , the left formula is derived. In the case, we think $(m+n)d$ is multiples of $m+n$, the middle formula is derived. In the case, we think $md+nd$ is all the pair of sums less than or equal to N , the right formula is derived.

The domain of the each formula is the same T_N and the each formula composed of $f(m,n)$ and $g(d)$ can be transformed to the each other, because the same triple (m,n,d) input makes the same term $f(m,n) \cdot g(d)$. Therefore, the formula is proved. $\square$

Corollary 2.2.

$$\begin{aligned}\sum_{k=1}^n \sigma_x(k) &= \sum_{k=1}^n k^x \left[\frac{n}{k} \right] = \sum_{a+b \leq n} \frac{\left(\frac{a+b}{\gcd(a,b)} \right)^x}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} \\ &\text{especially,} \\ \sum_{k=1}^n \sigma(k) &= \sum_{k=1}^n k \left[\frac{n}{k} \right] = \sum_{a+b \leq n} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} \\ &\text{and} \\ \sum_{k=1}^n \tau(k) &= \sum_{k=1}^n \left[\frac{n}{k} \right] = \sum_{a+b \leq n} \frac{1}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)}\end{aligned}$$

Proof. It is clear, when we put $f(a, b) = \frac{(a+b)^x}{\varphi(a+b)}$ and $g(c) = 1$ on Therome 2.1. $\square$

Consideration on Dirichlet series by analogies from the formulas above would be interesting, especially on coefficients $\sigma_x(k)$ and k . We could think the next analogies $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^x \left[\frac{n}{k} \right]}{k^s}$ with $\sum_{k=1}^{\infty} \frac{\sigma_x(k)}{k^s} = \zeta(s) \zeta(s-x)$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\varphi(k) \left[\frac{n}{k} \right]}{k^s}$ with $\sum_{k=1}^{\infty} \frac{k}{k^s} = \zeta(s-1)$.

Corollary 2.3.

$$\begin{aligned}\sigma_x(n) &= \sum_{a+b=n} \frac{\left(\frac{a+b}{\gcd(a,b)} \right)^x}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} \\ &\text{especially,} \\ \sigma(n) &= \sum_{a+b=n} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} \\ &\text{and} \\ \tau(n) &= \sum_{a+b=n} \frac{1}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)}\end{aligned}$$

Proof. On Corollary 2.2, the left side and the right side of the formulas can be separated to $1 \leq k \leq n$ with not depending on n , therefore we can take their differences. $\square$

Corollary 2.4. Let r_k be the remainder of n divided by k .

$$n^2 = \sum_{k=1}^n (\sigma(k) + r_k) = \sum_{a+b \leq n} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} + \sum_{k=1}^n r_k$$

In other words, the average of $\sigma(k)$ plus the average of r_k until n equals n .

Proof. From $n = k \left[\frac{n}{k} \right] + r_k$, $\sum_{k=1}^n n = \sum_{k=1}^n (k \left[\frac{n}{k} \right] + r_k)$ is satisfied. Corollary 2.2 can be applied to the part $\sum_{k=1}^n k \left[\frac{n}{k} \right]$, and $n^2 = \sum_{k=1}^n (\sigma(k) + r_k)$ is proved. The formula can be transformed to $n = \frac{\sum_{k=1}^n \sigma(k)}{n} + \frac{\sum_{k=1}^n r_k}{n}$, and the latter statement is proved. The right side of the formula is clear from Corollary 2.2. $\square$

Corollary 2.5.

$$\sum_{a+b \leq n} \frac{\left(\frac{a+b}{\gcd(a,b)} \right)^x}{\varphi\left(\frac{a+b}{\gcd(a,b)} \right)} = \sum_{k=1}^n \sigma_x(k) \leq \sum_{k=1}^n k^{x-1} n$$

Proof. From Corollary 2.2 and $\sum_{k=1}^n k^x \left[\frac{n}{k} \right] \leq \sum_{k=1}^n k^x \frac{n}{k} = \sum_{k=1}^n k^{x-1} n$, the corollary is proved. $\square$

Corollary 2.6.

$$\frac{n(n+1)}{2} \leq \sum_{k=1}^n \sigma(k) = \sum_{a+b \leq n} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi(\frac{a+b}{\gcd(a,b)})} \leq n^2$$

Proof. $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor \leq \sum_{k=1}^n k \lfloor \frac{n}{k} \rfloor$ is satisfied, because $\varphi(k) \leq k$. With Corollary 2.2, $\frac{n(n+1)}{2} \leq \sum_{k=1}^n \sigma(k) = \sum_{a+b \leq n} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi(\frac{a+b}{\gcd(a,b)})}$ is proved. From Corollary 2.5, $\sum_{k=1}^n \sigma(k) \leq n^2$ holds. $\square$

Now we evaluate a sequence $\sum_{k=1}^n \varphi(k) \tau(k)$, which is analogized and motivated from $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor$, $\sum_{k=1}^n \lfloor \frac{n}{k} \rfloor = \sum_{k=1}^n \tau(k)$, and $\sum_{k=1}^n \varphi(k) \tau(k) = \sum_{k=1}^n \sum_{a+b=k} \frac{\varphi(k)}{\varphi(\frac{k}{\gcd(a,b)})}$.

Corollary 2.7.

$$\begin{aligned} \sum_{k=1}^n \varphi(k) \tau(k) &\leq n^2 \sum_{k=1}^n \frac{1}{k} \\ \sum_{k=1}^n \varphi(k) \tau(k) &\leq \frac{1}{12} n(n+1)(2n+7) \end{aligned}$$

Proof. Since $\varphi(k) \leq k$, $\sum_{k=1}^n \varphi(k) \tau(k) \leq \sum_{k=1}^n k \tau(k) \leq n \sum_{k=1}^n \tau(k)$. Since Corollary 2.2 and $\lfloor \frac{n}{k} \rfloor \leq \frac{n}{k}$, $n \sum_{k=1}^n \tau(k) = n \sum_{k=1}^n \lfloor \frac{n}{k} \rfloor \leq n \sum_{k=1}^n \frac{n}{k} = n^2 \sum_{k=1}^n \frac{1}{k}$. Therefore, $\sum_{k=1}^n \varphi(k) \tau(k) \leq n^2 \sum_{k=1}^n \frac{1}{k}$ is proved.

Next, because of $\tau(k) \leq \frac{k}{2} + 1$, $\sum_{k=1}^n \varphi(k) \tau(k) \leq \sum_{k=1}^n k \tau(k) \leq \sum_{k=1}^n k(\frac{k}{2} + 1) = \frac{1}{12} n(n+1)(2n+7)$. Therefore, $\sum_{k=1}^n \varphi(k) \tau(k) \leq \frac{1}{12} n(n+1)(2n+7)$ is proved. $\square$

From the corollary above, at least

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n \varphi(k) \tau(k)}{n^3} \leq \frac{1}{6}$$

is satisfied. Therefore, we have a next question that $\sum_{k=1}^n \varphi(k) \tau(k)$ is $\mathcal{O}(n^2)$ or not. There is a similar formula $n^2 \geq \sum_{k=1}^n \sum_{a+b=k} \frac{\frac{a+b}{\gcd(a,b)}}{\varphi(\frac{a+b}{\gcd(a,b)})} = \sum_{k=1}^n \sigma(k) = \sum_{k=1}^n \sum_{d|k} \varphi(d) \tau(\frac{k}{d}) = \sum_{k=1}^n \varphi(k) \sum_{i \leq \lfloor \frac{n}{k} \rfloor} \tau(i) = \sum_{k=1}^n \sum_{a+b=k} \tau(\gcd(a,b)) \geq \frac{n(n+1)}{2}$ from [2], Theorem 2.1, and Corollary 2.6, but it seems not to give us the answer.

Also with Menon's identity and Corollary 2.3, we have the next corollary, and it seems to have some relations to this question [4].

Corollary 2.8.

$$\varphi(n) \tau(n) = \sum_{\substack{a+b=n \\ \gcd(a,b)=1}} \gcd(a-1, b+1) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi(\frac{n}{\gcd(a,b)})}$$

Proof. From Menon's identity, $\varphi(n) \tau(n) = \sum_{\substack{1 \leq i \leq n \\ \gcd(i,n)=1}} \gcd(i-1, n) = \sum_{\substack{a+b=n \\ \gcd(a,b)=1}} \gcd(a-1, b+1)$ holds. From Corollary 2.3, $\varphi(n) \tau(n) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi(\frac{n}{\gcd(a,b)})}$ holds. From the both formulas, the proposition is proved. $\square$

Comparing to $\frac{n(n+1)}{2} = \sum_{k=1}^n \varphi(k) \lfloor \frac{n}{k} \rfloor$ and considering the changing pace of functions $\varphi(k)$, $\lfloor \frac{n}{k} \rfloor$, and $\tau(k)$, it seems $\sum_{k=1}^n \varphi(k) \tau(k)$ is more than or equals to $\frac{n(n+1)}{2}$. Basically, $\varphi(k)$ increases, $\lfloor \frac{n}{k} \rfloor$ reduces, and $\tau(k)$ increases.

We list all the sequences compared to $\sum_{k=1}^n \varphi(k)\tau(k)$ below. The order is from up to bottom: $\sum_{k=1}^n \varphi(k)\tau(k)$, $\frac{n(n+1)}{2}$, $\sum_{k=1}^n \varphi(k)\tau(k) - \frac{n(n+1)}{2}$, $n^2 - n + 1$, and n^2 . The list sequences are counted until $n \leq 20$.

1, 3, 7, 13, 21, 29, 41, 57, 75, 91, 111, 135, 159, 183, 215, 255, 287, 323, 359, 407

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210

0, 0, 1, 3, 6, 8, 13, 21, 30, 36, 45, 57, 68, 78, 95, 119, 134, 152, 169, 197

1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157, 183, 211, 241, 273, 307, 343, 381

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400

Although only just the lead 20 counts, it also seems $\sum_{k=1}^n \varphi(k)\tau(k)$ is more than or equals to $\frac{n(n+1)}{2}$.

Interesting thing is that $\sum_{k=1}^n \varphi(k)\tau(k) - \frac{n(n+1)}{2}$ is similar to $\frac{n(n+1)}{2}$ with delaying two counts. Therefore, from the assumption $\frac{(n-1)(n-2)}{2} \doteq \sum_{k=1}^n \varphi(k)\tau(k) - \frac{n(n+1)}{2}$, we list a sequence $n^2 - n + 1$ above. The head part of the sequences are similar, but gradually $\sum_{k=1}^n \varphi(k)\tau(k)$ becomes increasing from $n^2 - n + 1$.

n^2 is also similar on the head part, but gradually $\sum_{k=1}^n \varphi(k)\tau(k)$ becomes increasing from n^2 too. The following list show the counts from 40 to 50 of $\sum_{k=1}^n \varphi(k)\tau(k)$ and n^2 .

1799, 1879, 1975, 2059, 2179, 2323, 2411, 2503, 2663, 2789, 2909

1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500

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