ON SUMS INVOLVING DIVISOR FUNCTION, EULER'S TOTIENT FUNCTION, AND FLOOR FUNCTION

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Abstract. Every positive integer $l \in \mathbb{N}$ can be formed $l = (m+n)d$, provided $gcd(m,n) = 1$. From this point of view, the next formulas $n = \sum_{d|l} \varphi(d)$ and $\frac{n(n+1)}{2} = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}]$, and these equivalence had been proved. In this paper on an extension of these results, the next identity is proved: $\sum_{k=1}^{n} \sum_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a,b) \cdot g(c) = \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=1}} f(a,b) \sum_{i \leq [\frac{n}{k}]} g(i) = \sum_{a+b \leq n} f(\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)}) \cdot g(gcd(a,b))$. We also show the next formulas are corollaries of it: $\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} [\frac{n}{k}] = \sum_{a+b \leq n} \frac{1}{\varphi(\frac{a+b}{gcd(a,b)})}$, $\sum_{d|n} f(d) \cdot g(\frac{n}{d}) = \sum_{k=1}^{n} f(gcd(k,n)) \cdot \frac{g(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}$, $\sigma(n) = \sum_{k=1}^{n} \frac{\frac{n}{gcd(k,n)}}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{gcd(k,n)}{\varphi(\frac{n}{gcd(k,n)})}$, $n = \sum_{k=1}^{n} \frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}$, $\sum_{\substack{a+b=n \\ gcd(a,b)=1}} gcd(a-1, b+1) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi(\frac{n}{gcd(a,b)})}$, and so on. In addition to it, we evaluate a sequence $\sum_{k=1}^{n} \varphi(k)\tau(k)$.

1. Introduction

Every positive integer $l \in \mathbb{N}$ can be formed $l = (m+n)d$, provided $gcd(m,n) = 1$. We can have the sum of positive divisors function $\sigma_x(l)$ from divisors d, and Euler's totient function $\varphi(m+n)$ from pairs of relatively prime positive integers (m,n).

From this point of view, the formula $n = \sum_{d|l} \varphi(d)$ can be proved. An equivalent formula $\frac{n(n+1)}{2} = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}]$ had been proved in the previous paper [5]. From the both formulas, in other words $\sum_{k=1}^{n} \sum_{d|k} \varphi(d) = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}]$, we can expect relationships between divisor function, Euler's totient function, and floor function. Actually, we have known the next identity in arithmetic summatory function:

$$\sum_{k=1}^{n} \sum_{d|k} f(d) g(\frac{k}{d}) = \sum_{k=1}^{n} f(k) \sum_{i \leq [\frac{n}{k}]} g(i).$$

In this paper, we expand this identity to

$$\begin{aligned} \sum_{k=1}^{n} \sum_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a,b) \cdot g(c) &= \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=1}} f(a,b) \sum_{i \leq [\frac{n}{k}]} g(i) \\ &= \sum_{a+b \leq n} f(\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)}) \cdot g(gcd(a,b)), \end{aligned}$$

provided that we think $(a,b) = (k,0)$, and define $gcd(k,0) := k$. The former formula is proved by the coincidence between the pair of divisors and multiples. In addition

2020 AMS Subject Classification: 11A25, 11N56, 11B13.

Key Words and Phrases. divisor function, Euler's totient function, floor function, divisor summatory function, arithmetic function.

to it, the latter formula is proved by the coincidence between the triple of divisors, multiples, and the pair of sums. In other words, we make the expansion from the point of view of the previous paper $l = (m+n)d$.

As a corollary, we prove the next formula:

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} [\frac{n}{k}] = \sum_{a+b \leq n} \frac{1}{\varphi(\frac{a+b}{gcd(a,b)})}.$$

The first equality is a well-known formula in a basis of divisor summatory function [1]. It is clear that the formula shows rich relationships between divisor function, floor function, Euler's totient function, sum, multiple, indivisibility, and so on. It is interesting Euler's totient function appears in reciprocals.

We also prove the other corollaries;

$$\sum_{d|n} f(d) \cdot g(\frac{n}{d}) = \sum_{k=1}^{n} f(gcd(k,n)) \cdot \frac{g(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})},$$

$\tau(n) = \sum_{a+b=n} \frac{1}{\varphi(\frac{a+b}{gcd(a,b)})}$, $\sigma(n) = \sum_{k=1}^{n} \frac{\frac{n}{gcd(k,n)}}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{gcd(k,n)}{\varphi(\frac{n}{gcd(k,n)})}$, $n = \sum_{k=1}^{n} \frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}$, $\sum_{k=1}^{n} \sigma_x(k) \leq \sum_{k=1}^{n} k^{x-1}n$, and so on. The left formula of the first identity is Dirichlet convolution, and the identity is expansion of gcd-sum function. It is expansion of the next known identity in gcd-sum function [6, 4, 3]: $\sum_{k=1}^{n} f(gcd(k,n)) = \sum_{d|n} f(d)\varphi(\frac{n}{d})$.

In addition to it, we evaluate a sequence $\sum_{k=1}^{n} \varphi(k)\tau(k)$, which is analogized and motivated from the formulas: $\frac{n(n+1)}{2} = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}]$, $\sum_{k=1}^{n}[\frac{n}{k}] = \sum_{k=1}^{n} \tau(k)$, and $\varphi(n)\tau(n) = \sum_{\substack{a+b=n \\ gcd(a,b)=1}} gcd(a-1, b+1) = \sum_{a+b=n} \frac{\varphi(n)}{\varphi(\frac{n}{gcd(a,b)})}$. We use symbols σ and τ as $\sigma_1 = \sigma$ and $\sigma_0 = \tau$, $[x]$ denotes floor function, and also $\varphi(x)$ denotes Euler's totient function.

2. Theorem, corollaries, and Evaluation

Theorem 2.1.

$$\begin{aligned}\sum_{k=1}^{n} \sum_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a,b) \cdot g(c) &= \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=1}} f(a,b) \sum_{i \leq [\frac{n}{k}]} g(i) \\ &= \sum_{a+b \leq n} f(\frac{a}{gcd(a,b)}, \frac{b}{gcd(a,b)}) \cdot g(gcd(a,b)),\end{aligned}$$

provided that we think $(a,b) = (k,0)$, and define $gcd(k,0) := k$.

Proof. Every positive integer $1 \leq l \leq N$ can be formed $l = (m+n)d$, provided $gcd(m,n) = 1$. T_N denotes the set of these triples (m,n,d). In the case, we think $m+n$ and d are divisors of l, the left formula is derived. In the case, we think $(m+n)d$ is multiples of $m+n$, the middle formula is derived. In the case, we think $md + nd$ is all the pair of sums less than or equal to N, the right formula is derived.

The domain of each formula is the same T_N and each formula composed of $f(m,n)$ and $g(d)$ can be transformed to each other, because the same triple (m,n,d) input makes the same term $f(m,n) \cdot g(d)$. Therefore, the formula is proved. □

We can make Theorem2.1 a little more general and arranged:

$$\sum_{k=1}^{n} \sum_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a,b,c) = \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=1}} \sum_{c \leq [\frac{n}{k}]} f(a,b,c) = \sum_{k=1}^{n} \sum_{\substack{a+b=k \\ gcd(a,b)=c}} f(\frac{a}{c}, \frac{b}{c}, c).$$

Corollary 2.2.

$$\begin{aligned} \sum_{k=1}^{n} \sigma_x(k) & = \textstyle\sum_{k=1}^{n} k^x [\frac{n}{k}] & = \sum_{a+b \leq n} \frac{(\frac{a+b}{gcd(a,b)})^x}{\varphi(\frac{a+b}{gcd(a,b)})} \\ & especially, & \\ \sum_{k=1}^{n} \sigma(k) & = \textstyle\sum_{k=1}^{n} k [\frac{n}{k}] & = \sum_{a+b \leq n} \frac{\frac{a+b}{gcd(a,b)}}{\varphi(\frac{a+b}{gcd(a,b)})} \\ & and & \\ \sum_{k=1}^{n} \tau(k) & = \textstyle\sum_{k=1}^{n} [\frac{n}{k}] & = \sum_{a+b \leq n} \frac{1}{\varphi(\frac{a+b}{gcd(a,b)})} \end{aligned}$$

Proof. It is clear, when we put $f(a,b) = \frac{(a+b)^x}{\varphi(a+b)}$ and $g(c) = 1$ on Therome2.1. □

Consideration on Dirichlet series by Theorem2.1, Corollary2.2, and other formulas in this paper would be interesting. For examples on coefficients $\sigma_x(k)$ and k, we could think the next analogies $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^x [\frac{n}{k}]}{k^s}$ with $\sum_{k=1}^{\infty} \frac{\sigma_x(k)}{k^s} = \zeta(s)\zeta(s-x)$ and $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{\varphi(k)[\frac{n}{k}]}{k^s}$ with $\sum_{k=1}^{\infty} \frac{k}{k^s} = \zeta(s-1)$.

Corollary 2.3.

$$\begin{aligned} \sigma_x(n) & = \textstyle\sum_{a+b=n} \frac{(\frac{a+b}{gcd(a,b)})^x}{\varphi(\frac{a+b}{gcd(a,b)})} \\ & especially, \\ \sigma(n) & = \textstyle\sum_{a+b=n} \frac{\frac{a+b}{gcd(a,b)}}{\varphi(\frac{a+b}{gcd(a,b)})} \\ & and \\ \tau(n) & = \textstyle\sum_{a+b=n} \frac{1}{\varphi(\frac{a+b}{gcd(a,b)})} \end{aligned}$$

Proof. On Corollary2.2, the left side and the right side of the formulas can be separated to $1 \leq k \leq n$ with not depending on n, therefore we can take their differences. □

When we define $\varphi_x(n) := \frac{\varphi(n)}{n^x}$, we can describe the formula above as

$$\sigma_x(n) = \sum_{k=1}^{n} \frac{1}{\varphi_x(\frac{n}{gcd(k,n)})}.$$

We have known the next identity in gcd-sum function [6, 4, 3]: $\sum_{k=1}^{n} f(gcd(k,n)) = \sum_{d|n} f(d)\varphi(\frac{n}{d})$. With setting $f(i) = \frac{1}{\varphi_x(\frac{n}{i})}$, then we can also have Corollary2.3. It is natural, because we can derive the identity from Theorem2.1 as following.

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Corollary 2.4.

$$\begin{aligned}\sum_{d|n} f(d)\cdot g(\frac{n}{d}) &= \sum_{k=1}^{n} f(gcd(k,n))\cdot\frac{g(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}\\ &especially,\\ \sum_{d|n} f(d)\varphi(\frac{n}{d}) &= \sum_{k=1}^{n} f(gcd(k,n))\end{aligned}$$

Proof. From Theorem2.1, $\sum_{\substack{(a+b)c=n\\ gcd(a,b)=1}} f(a,b)\cdot g(c) = \sum_{a+b=n} f(\frac{a}{gcd(a,b)},\frac{b}{gcd(a,b)})\cdot g(gcd(a,b))$ holds. When $f(a,b)=\frac{h(a+b)}{\varphi(a+b)}$,

$$\sum_{\substack{(a+b)c=n\\ gcd(a,b)=1}} \frac{h(a+b)}{\varphi(a+b)}\cdot g(c) = \sum_{a+b=n}\frac{h(\frac{a+b}{gcd(a,b)})}{\varphi(\frac{a+b}{gcd(a,b)})}\cdot g(gcd(a,b))$$

holds. Next, we think $a+b=d$ in the left formula and $a=k$ in the right formula, and we can replace h to f again:

$$\sum_{d|n} f(d)\cdot g(\frac{n}{d}) = \sum_{k=1}^{n}\frac{f(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}\cdot g(gcd(k,n)).$$

In the left formula, we can exchange f and g, therefore

$$\sum_{d|n} f(d)\cdot g(\frac{n}{d}) = \sum_{k=1}^{n}\frac{g(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})}\cdot f(gcd(k,n))$$

holds. Now, we think $g(n)=\varphi(n)$, and the identities are proved. □

The left formula of Corollary2.4 is Dirichlet convolution, and the identity is also expansion of gcd-sum function.

Corollary 2.5.

$$\sum_{d|n} f(d) = \sum_{k=1}^{n}\frac{f(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n}\frac{f(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}$$

Proof. We can exchange f and g in Corollary2.4, and set $g(n)=1$. □

Corollary 2.6.

$$\begin{aligned}\sigma(n) &= \sum_{k=1}^{n}\frac{\frac{n}{gcd(k,n)}}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n}\frac{gcd(k,n)}{\varphi(\frac{n}{gcd(k,n)})}\\ &= \sum_{a+b=n}\frac{\frac{a+b}{gcd(a,b)}}{\varphi(\frac{a+b}{gcd(a,b)})} = \sum_{a+b=n}\frac{gcd(a,b)}{\varphi(\frac{a+b}{gcd(a,b)})}\end{aligned}$$

Proof. On Corollary2.5, we think $f(n)=n$. □

We should note the difference between Corollary2.6 and Corollary2.3, and $\sum_{k=1}^{n}\frac{n}{gcd(k,n)}$ does not necessarily equal $\sum_{k=1}^{n} gcd(k,n)$.

Corollary 2.7.

$$n = \sum_{k=1}^{n}\frac{\varphi(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{a+b=n}\frac{\varphi(gcd(a,b))}{\varphi(\frac{a+b}{gcd(a,b)})}$$

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Proof. On Corollary2.5, we think $f(n) = \varphi(n)$. $\square$

Corollary2.6 and Corollary2.7 would be on an extension of the previous paper [5].

Corollary 2.8. *Let r_k be the remainder of n divided by k.*

$$n^2 = \sum_{k=1}^{n}(\sigma(k) + r_k) = \sum_{a+b\leq n} \frac{gcd(a,b)}{\varphi(\frac{a+b}{gcd(a,b)})} + \sum_{k=1}^{n} r_k$$

In other words, the average of $\sigma(k)$ plus the average of r_k until n equals n.

Proof. From $n = k[\frac{n}{k}] + r_k$, $\sum_{k=1}^{n} n = \sum_{k=1}^{n}(k[\frac{n}{k}] + r_k)$ is satisfied. Corollary2.2 can be applied to the part $\sum_{k=1}^{n} k[\frac{n}{k}]$, and $n^2 = \sum_{k=1}^{n}(\sigma(k)+r_k)$ is proved. The formula can be transformed to $n = \frac{\sum_{k=1}^{n}\sigma(k)}{n} + \frac{\sum_{k=1}^{n} r_k}{n}$, and the latter statement is proved. The right side of the formula is clear from Corollary2.2 and Corollary2.6. $\square$

Corollary 2.9.

$$\sum_{a+b\leq n} \frac{(\frac{a+b}{gcd(a,b)})^x}{\varphi(\frac{a+b}{gcd(a,b)})} = \sum_{k=1}^{n} \sigma_x(k) \leq \sum_{k=1}^{n} k^{x-1}n$$

Proof. From Corollary2.2 and $\sum_{k=1}^{n} k^x[\frac{n}{k}] \leq \sum_{k=1}^{n} k^x \frac{n}{k} = \sum_{k=1}^{n} k^{x-1}n$, the corollary is proved. $\square$

Corollary 2.10.

$$\frac{n(n+1)}{2} \leq \sum_{k=1}^{n} \sigma(k) = \sum_{a+b\leq n} \frac{\frac{a+b}{gcd(a,b)}}{\varphi(\frac{a+b}{gcd(a,b)})} \leq n^2$$

Proof. $\frac{n(n+1)}{2} = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}] \leq \sum_{k=1}^{n} k[\frac{n}{k}]$ is satisfied, because $\varphi(k) \leq k$. With Corollary2.2, $\frac{n(n+1)}{2} \leq \sum_{k=1}^{n} \sigma(k) = \sum_{a+b\leq n} \frac{\frac{a+b}{gcd(a,b)}}{\varphi(\frac{a+b}{gcd(a,b)})}$ is proved. From Corollary2.9, $\sum_{k=1}^{n} \sigma(k) \leq n^2$ holds. $\square$

Now we evaluate a sequence $\sum_{k=1}^{n} \varphi(k)\tau(k)$, which is analogized and motivated from $\frac{n(n+1)}{2} = \sum_{k=1}^{n} \varphi(k)[\frac{n}{k}]$, $\sum_{k=1}^{n}[\frac{n}{k}] = \sum_{k=1}^{n} \tau(k)$, and $\sum_{k=1}^{n} \varphi(k)\tau(k) = \sum_{k=1}^{n} \sum_{a+b=k} \frac{\varphi(k)}{\varphi(\frac{k}{gcd(a,b)})}$.

Corollary 2.11.

$$\sum_{k=1}^{n} \varphi(k)\tau(k) \leq n^2 \sum_{k=1}^{n} \frac{1}{k}$$

$$\sum_{k=1}^{n} \varphi(k)\tau(k) \leq \frac{1}{12}n(n+1)(2n+7)$$

Proof. Since $\varphi(k) \leq k$, $\sum_{k=1}^{n} \varphi(k)\tau(k) \leq \sum_{k=1}^{n} k\tau(k) \leq n\sum_{k=1}^{n} \tau(k)$. Since Corollary2.2 and $[\frac{n}{k}] \leq \frac{n}{k}$, $n\sum_{k=1}^{n} \tau(k) = n\sum_{k=1}^{n}[\frac{n}{k}] \leq n\sum_{k=1}^{n} \frac{n}{k} = n^2 \sum_{k=1}^{n} \frac{1}{k}$. Therefore, $\sum_{k=1}^{n} \varphi(k)\tau(k) \leq n^2 \sum_{k=1}^{n} \frac{1}{k}$ is proved.

Next, because of $\tau(k) \leq \frac{k}{2}+1$, $\sum_{k=1}^{n} \varphi(k)\tau(k) \leq \sum_{k=1}^{n} k\tau(k) \leq \sum_{k=1}^{n} k(\frac{k}{2}+1)) = \frac{1}{12}n(n+1)(2n+7)$. Therefore, $\sum_{k=1}^{n} \varphi(k)\tau(k) \leq \frac{1}{12}n(n+1)(2n+7)$ is proved. $\square$

From the corollary above, at least

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \varphi(k)\tau(k)}{n^3} \leq \frac{1}{6}$$

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is satisfied. Therefore, we have a next question that $\sum_{k=1}^{n}\varphi(k)\tau(k)$ is $\mathbb{O}(n^2)$ or not. There is a similar formula $n^2 \geq \sum_{k=1}^{n}\sum_{a+b=k}\frac{\frac{k}{gcd(a,b)}}{\varphi(\frac{k}{gcd(a,b)})} = \sum_{k=1}^{n}\sigma(k) = \sum_{k=1}^{n}\sum_{d|k}\varphi(d)\tau(\frac{k}{d}) = \sum_{k=1}^{n}\varphi(k)\sum_{i\leq[\frac{n}{k}]}\tau(i) = \sum_{k=1}^{n}\sum_{a+b=k}\tau(gcd(a,b)) \geq \frac{n(n+1)}{2}$ from [2, p.17], Theorem2.1, and Corollary 2.10, but it seems not to give us the answer.

Also, with Menon's identity and Corollary 2.3, we have the next corollary, and it seems to have some relations to this question [7].

Corollary 2.12.

$$\varphi(n)\tau(n) = \sum_{\substack{a+b=n\\ gcd(a,b)=1}} gcd(a-1,b+1) = \sum_{a+b=n}\frac{\varphi(n)}{\varphi(\frac{n}{gcd(a,b)})}$$

Proof. From Menon's identity, $\varphi(n)\tau(n) = \sum_{\substack{1\leq i\leq n\\ gcd(i,n)=1}} gcd(i-1,n) = \sum_{\substack{a+b=n\\ gcd(a,b)=1}} gcd(a-1,b+1)$ holds. From Corollary 2.3, $\varphi(n)\tau(n) = \sum_{a+b=n}\frac{\varphi(n)}{\varphi(\frac{n}{gcd(a,b)})}$ holds. From the both formulas, the proposition is proved. □

Comparing to $\frac{n(n+1)}{2} = \sum_{k=1}^{n}\varphi(k)[\frac{n}{k}]$ and considering the changing pace of functions $\varphi(k)$, $[\frac{n}{k}]$, and $\tau(k)$, it seems $\sum_{k=1}^{n}\varphi(k)\tau(k)$ is more than or equals to $\frac{n(n+1)}{2}$. Basically, $\varphi(k)$ increases, $[\frac{n}{k}]$ reduces, and $\tau(k)$ increases.

We list all the sequences compared to $\sum_{k=1}^{n}\varphi(k)\tau(k)$ below. The order is from up to bottom: $\sum_{k=1}^{n}\varphi(k)\tau(k)$, $\frac{n(n+1)}{2}$, $\sum_{k=1}^{n}\varphi(k)\tau(k) - \frac{n(n+1)}{2}$, n^2-n+1, and n^2. The list sequences are counted until $n \leq 20$.

$$1,3,7,13,21,29,41,57,75,91,111,135,159,183,215,255,287,323,359,407$$
$$1,3,6,10,15,21,28,36,45,55,66,78,91,105,120,136,153,171,190,210$$
$$0,0,1,3,6,8,13,21,30,36,45,57,68,78,95,119,134,152,169,197$$
$$1,3,7,13,21,31,43,57,73,91,111,133,157,183,211,241,273,307,343,381$$
$$1,4,9,16,25,36,49,64,81,100,121,144,169,196,225,256,289,324,361,400$$

Although only just the lead 20 counts, it also seems $\sum_{k=1}^{n}\varphi(k)\tau(k)$ is more than or equals to $\frac{n(n+1)}{2}$.

Interesting thing is that $\sum_{k=1}^{n}\varphi(k)\tau(k) - \frac{n(n+1)}{2}$ is similar to $\frac{n(n+1)}{2}$ with delaying two counts. Therefore, from the assumption $\frac{(n-1)(n-2)}{2} \fallingdotseq \sum_{k=1}^{n}\varphi(k)\tau(k) - \frac{n(n+1)}{2}$, we list a sequence n^2-n+1 above. The head parts of the sequences are similar, but gradually $\sum_{k=1}^{n}\varphi(k)\tau(k)$ becomes increasing from n^2-n+1.

n^2 is also similar on the head part, but gradually $\sum_{k=1}^{n}\varphi(k)\tau(k)$ becomes increasing from n^2 too. The following list show the counts from 40 to 50 of $\sum_{k=1}^{n}\varphi(k)\tau(k)$ and n^2.

$$1799,1879,1975,2059,2179,2323,2411,2503,2663,2789,2909$$
$$1600,1681,1764,1849,1936,2025,2116,2209,2304,2401,2500$$

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