An Exposition of Polygonal Approximation of Circle

PALASH KHANRA

Abstract – This article attempts to discuss a journey of creating an infinite number of circles from a single circle, using its tangents with a pattern (P_X; X≥3) and a few other consequences. Different techniques are available to create new circles from a given circle. But while using those techniques, people are unknowingly using the technique described in this article.

Keywords and phrases . n-Regular polygons; Circle generation; Approximations

1 Introduction.

Polygons, along with lines and circles, constitute the earliest collection of Geometric figures studied by humans. By polygon, we mean a closed figure with a finite number n of sides (n vertices)(n $\geq$ 3) each of which is a line segment. Increasing the value of the number of sub-intervals into which domain is divided, increases the accuracy of the approximation.

The first work of circle approximation dates back to Archimedes of Syracuse (287-212BC). Archimedes approximated the area of a circle by using the Pythagorean theorem to find the areas of two regular polygons: the polygon inscribed within the circle and the polygon within which the circle was circumscribed [1]. Some mathematicians use parametric polynomial curves and Quadratic B$\acute{e}$zier curves to approximate circular arcs. In 1993, L Yong-Kui worked on circle approximation and its generation. He introduces a new algorithm for the generation of the circle, using the intersecting polygon instead of the inscribed polygon, which greatly reduces the error [2]. In recent years 2011, J$\acute{o}$zef Borkowski had done work on 'Minimization of Maximum Errors In Universal Approximation of The Unit Circle By a Polygon' [3].

Researchers gave many algorithms and interesting ways to approximate circles. Very few worked on the generation of circles. Now some queries! is it possible to create infinitely many successor and predecessor circles from a single circle, using its tangents? If yes! then what is the most detailed process? Is there any pattern behind the radius of the successor circles and gaps between any two nearby successor circles? If there is any pattern then can we use them to understand something else? This article will try to illustrate and resolve all those queries.

2 Procedure.

Let us consider that, we have a single circle (Figure 1) and any tangents of the circle look like Figure 2. Here, we will deal with the regular polygonal pattern of tangents.

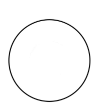

Figure 1: circle

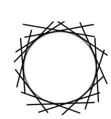

Figure 2: Circle with it's tangents

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2.1 Visualization to the Process.

The triangular pattern of tangents is the simplest regular polygonal pattern. Consider three tangents with triangular patterns on the given single circle. For $i \in N$, define, b_i : end-intersecting points of tangents

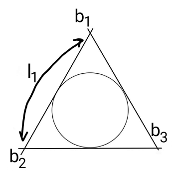

Figure 3: n=1

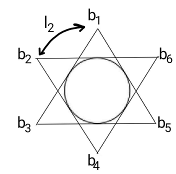

Figure 4: n=2

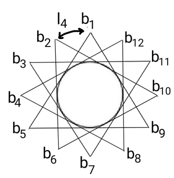

Figure 5: n=4

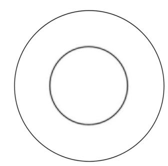

Figure 6: 1st successor circle, $n \longrightarrow \infty$

on the circle. It is assumed that all end-intersecting points are distributed in a precise way. Meaning, for each $n = k$, $k \in N$; $d_u(b_1, b_2) = d_u(b_2, b_3) = ... = d_u(b_i, b_{i+1})$; for all $i \in N$. The d_u symbolizes the Euclidean distance [1]. Define $d_u(b_i, b_{i+1}) = l_k$ for $n = k$. The above geometric treatments give us a decreasing sequence here, $(L_n)_{n=1}^{\infty} = \{l_1, l_2, l_3, ..., l_n, ...\}$ where, $l_1 > l_2 > l_3 > ... > l_n > ....$ It can be easily checked that this sequence converges to 0 (using the Least Upper Bound property) and then we will have our 1_{st} successor circle and the set of all $b_i{}'s$ becoming dense in R. The 2_{nd} successor circle can be obtained by applying the same idea of tangents pattern (here, triangular) on the 1_{st} successor circle.
Let, c_i: for all $i \in N$, are the end intersecting points of tangents on the 1_{st} successor circle. It is assumed that for each n=k: $k \in N$; $d_u(c_1, c_2) = d_u(c_2, c_3) = ... = d_u(c_i, c_{i+1})$; for all $i \in N$. Define for $n = k$, $d_u(c_i, c_{i+1}) = m_k$. Then a decreasing sequence can be obtained and it is defined

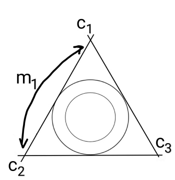

Figure 7: n=1

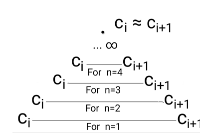

Figure 8: Varying lengths of $m_i's$

by, $(M_n)_{n=1}^{\infty} = \{m_1, m_2, m_3, ..., m_n, ...\}$. Now, $d_u(c_i, c_{i+1}) \longrightarrow 0$ when $n \longrightarrow \infty$; for all $i \in N$ and it can be easily shown that, this sequence converges to 0. So, now we have our 2_{nd} successor circle.
Applying the same procedure on $(n-1)_{th}$ successor circle, we can obtain n_{th} successor circle. Infinitely many successor circles can be obtained, repeating this sequential process infinitely many times.

[1] In the whole article, we will use d_u as a notation of Euclidean distance.

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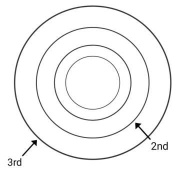

Figure 9: 2nd and 3rd successor circles

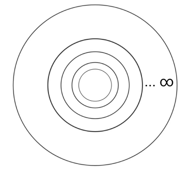

Figure 10: Infinitely many successor circles

Let's deal with the 2_{nd} simplest regular polygon i.e. the square pattern of tangents on circles. Assumed that, the end-intersecting points (say $b_i{}'s$) of tangents are distributed in a precise way. Define, $d_u(b_i, b_{i+1}) =$

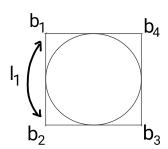

Figure 11: n=1

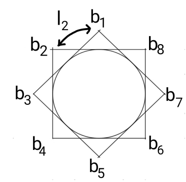

Figure 12: n=2

l_k , for all $n = k : k \in N$. It can be easily checked that a decreasing sequence of $l_k{}'s$, $(L_n)_{n=1}^{\infty} =$

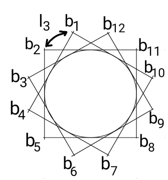

Figure 13: n=3

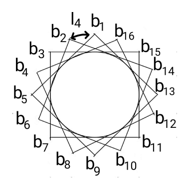

Figure 14: n=4

$\{l_1, l_2, l_3, ..., l_n, ...\}$ can be obtained and converged to 0 and then we will have our 1_{st} successor circle and the set of all $b_i{}'s$ becoming dense in R. Let, c_i: for all $i \in N$, are the end intersecting points of tangents on the 1_{st} successor circle. Assumed that, for each n=k : k belongs to N ; $d_u(c_1, c_2) = d_u(c_2, c_3)$ = ... $= d_u(c_i, c_{i+1})$; for all i belongs to N. Define, $d_u(c_i, c_{i+1}) = m_k$, for $n = k; k \in N$. Here also the below-

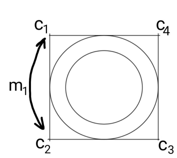

Figure 15: n=1

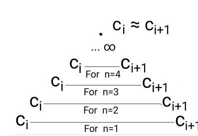

Figure 16: Varying length's of $m_i{}'s$

defined sequence $(M_n)_{n=1}^{\infty} = \{m_1, m_2, m_3, ..., m_n, ...\}$ converges to 0 and the set of all $c_i{}'s$ becoming dense in R. Then our 2_{nd} successor circle is obtained. Infinitely many successor circles can be obtained, repeating this sequential process infinitely many times.

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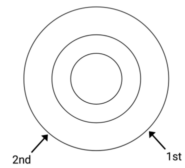

Figure 17: 1_{st} and 2_{nd} successor circle

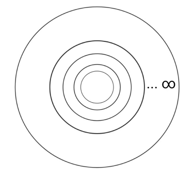

Figure 18: Infinitely many successor circles

In previous portions, the triangular pattern and square pattern are discussed. But the interesting fact is, an infinite number of circles can be obtained with any polygonal pattern (applying sequential idea). In polygonal approximation, increasing the value of the number of sub-intervals into which the domain is divided, increases the accuracy of approximation.

2.2 Consequences.

Let P_X (X $\geq$ 3): regular polygon with X number of vertices.
Let T_1: time to get the 1_{st} successor circle from the given single circle using the P_3 pattern of tangents. Whenever we apply the P_3 pattern for the 1_{st} time, then 3 end-intersecting points arise. Applying the P_3 pattern for the 2_{nd} time, we have 6 end-intersecting points. Continuing this process, we will have 9, 12, 15, ... so on. Observe that, a sequence arises here and it is defined by, $(P_{3n})_{n=1}^{\infty} = \{3, 6, 9, 12, ..., 3n, ...\}\ i.e. (3n)_{n=1}^{\infty}$
Now, T_1 depends on the rate of convergence of sequence $(\frac{1}{P_{3n}})_{n=1}^{\infty}$ [2].
Let, T_2 be the time to get the 1_{st} successor circle using P_4 pattern of tangents. It can be easily checked that a sequence of end-intersecting points of tangents can be obtained and it is defined by, $(P_{4n})_{n=1}^{\infty} = \{4, 8, 12, 16, ..., 4n, ...\}\ i.e. (4n)_{n=1}^{\infty}$ Here, T_2 depends on the rate of convergence of sequence $(\frac{1}{P_{4n}})_{n=1}^{\infty}$. Proceeding in a similar way, it can be observed that, if $T_K : K \in N$, be the time to get the 1_{st} successor circle from the single circle using P_{K+2} pattern then T_K depends on the rate of convergence of sequence, $(\frac{1}{P_{(K+2)n}})_{n=1}^{\infty}$. We need to compare all those sequences, on which $T_K{}'s$ depends. It can be easily checked that, for all $n \in N$, $\frac{1}{3n} > \frac{1}{4n} > \frac{1}{5n} > ... > \frac{1}{Kn} > ...$ here, any $\frac{1}{Kn}$ defines the terms of sequence $(\frac{1}{Kn})_{n=1}^{\infty}$. Arranging those above sequences according to their rate of convergence then we have,

$$(\frac{1}{3n})_{n=1}^{\infty}, (\frac{1}{4n})_{n=1}^{\infty}, (\frac{1}{5n})_{n=1}^{\infty}, ..., (\frac{1}{(Kn)})_{n=1}^{\infty}, ...$$

rate of converges decreases when K increases. Now, we construct a sequence of all $T_K{}'S$ defined by, $(T_n)_{n=1}^{\infty} = \{T_1, T_2, T_3, ..., T_n, ...\} : T_1 > T_2 > T_3 > ... > T_K > ...$ So, it is decreasing in order and converging to 0. Thus by increasing n in P_n, the time to get our 1_{st} successor circle from the given single circle, can be reduced and Thus increasing n in P_n, reduces the time to get infinitely many successor circles from the single circle. Let, T_{P_K}: Time to get an infinite number of circles from a single circle using P_K pattern (K $\geq$ 3). So, $T_{P_3} > T_{P_4} > T_{P_5} > ... > T_{P_K} > ....$ Define, P_∞ : the polygon with infinitely many vertices (here, P_∞ is just a notation). The idea of polygonal approximation of a circle tells us that the regular polygon with infinite vertices is an approximation to a circle **[2]**. An interesting fact is that if we apply the P_∞ pattern of the tangents on the circle, almost without taking any time, our 1_{st} successor circle can be obtained. More precisely, for P_∞ on the given single circle, almost without taking any time, we will get infinitely many successor circles and they almost merge into a single one. Basically, the decreasing

[2] Rate of convergence is how quickly the terms of the sequence converges to 0

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sequence, $(T_{P_n})_{n=1}^{\infty} = \{T_{P_1}, T_{P_2}, ..., T_{P_n}, ...\}$ converges to 0.

Let's deal with the thought that, how the distance between any two nearby successive circles depends on the radius of the given single circle and how the radius of the successive circles vary Let, r be the radius of the given single circle. Let, b_1 , b_2 , b_3 , ... b_n , ... are eip [3] of tangents (for, P_X pattern) and u_1 , u_2 , u_3 , ... u_n, ... are the ctip [4]. Therefore, we will join those points with lines (we take only those lines which pass through the origin o). Now joining points are, (i) eip and eip (ii) eip and ctip (iii) ctip and ctip. Therefore, if we have n joining lines (passing through the origin), then the whole circle is divided into 2n equal parts. Interestingly, for the P_X pattern; X=n, we will have 2n equal parts of the circle. At first, we will deal with the P_3 pattern (Triangular pattern) of tangents. Here, b_i'S lies on the circumference of the 1_{st} successor circle

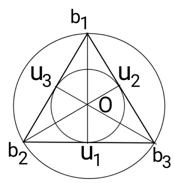

Figure 19: Points position

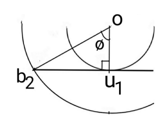

Figure 20

and ou_1 = ou_2 = ou_3 = r , i.e. the radius of the given single circle and ob_1 = ob_2 = ob_3 = r_1 ; radius of the 1_{st} successor circle. For P_3 pattern we have 3 vertices. So the whole circle would divide into 6 equal parts. One of those equal parts is, $\triangle u_1 o b_2$. It can be observed that, ou_1 is perpendicular to b_2u_1. Now, $\phi = \frac{360}{6} = 60$ degree (Since, 6 equal parts), therefore $\cos\frac{\pi}{3} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{3}}$ Let, r_2 be the radius of the 2nd successor circle. Here, ob_2 = r_2 , ou_1 = r_1. It can be easily checked that, $r_2 = \frac{r_1}{\cos\frac{\pi}{3}} = \frac{r}{\left(\cos\frac{\pi}{3}\right)^2}$. Basically, we have a sequence of radius of successor circles, $\left\{ r, \frac{r}{\cos\frac{\pi}{3}}, \frac{r}{\left(\cos\frac{\pi}{3}\right)^2}, ... \right\} = \left(\frac{r}{\left(\cos\frac{\pi}{3}\right)^{j-1}} \right)_{j=1}^{\infty}$, Where $\frac{r}{\left(\cos\frac{\pi}{3}\right)^j}$

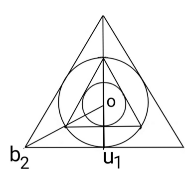

Figure 21

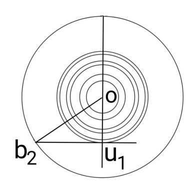

Figure 22: nth successor circle

, is the radius of the j_{th} successor circle obtained using P_3 pattern. The distance between any two successive circles (including a single circle also) is defined by, $\left[\frac{1}{\left(\cos\frac{\pi}{3}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{3}\right)^{j-1}} \right] r$. For the P_4 pattern, the whole circle can be divided into 8 equal parts. One of those equal parts is $\triangle u_1 o b_2$. It can be easily seen that,

[3]eip: end-intersecting points

[4]ctip: circle-tangent intersecting points

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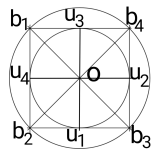

Figure 23: Points position

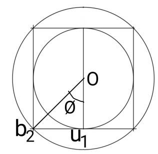

Figure 24

ou_1 is perpendicular to b_2u_1 (here, ou_1 = r, $ob_2 = r_1$). Now, $\phi = \frac{360}{8}$ = 45 degrees (Since, 8 equal parts) therefore, $\cos\frac{\pi}{4} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{4}}$. Let, r_2 be the radius of the 2_{nd} successor circle. Here, $ob_2 = r_2$, $ou_1 = r_1$. It can be easily checked that, $r_2 = \frac{r_1}{\cos\frac{\pi}{4}} = \frac{r}{\left(\cos\frac{\pi}{4}\right)^2}$. Here, a sequence of radius of successor

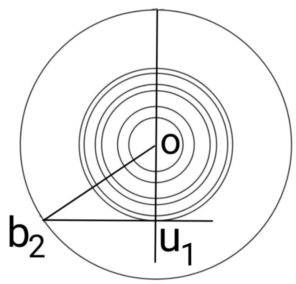

Figure 25: nth successor circle

circles can be obtained and each radius is depending on the radius of the given single circle. That sequence is defined below, $\left\{ r, \frac{r}{\cos\frac{\pi}{4}}, \frac{r}{\left(\cos\frac{\pi}{4}\right)^2}, ... \right\} = \left(\frac{r}{\left(\cos\frac{\pi}{4}\right)^{j-1}}\right)_{j=1}^{\infty}$ Where, $\frac{r}{\left(\cos\frac{\pi}{4}\right)^{j}}$ is the radius of the j_{th} successor circle, obtained using P_4 pattern. The distance between any two successive circles (including a single circle also) is, $\left[\frac{1}{\left(\cos\frac{\pi}{4}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{4}\right)^{j-1}}\right]r$. Now, we want to propose a new theorem here.

Theorem 1. *If* $(P_X; X \geq 3)$ *be any X-regular polygonal pattern of tangents on a single circle, then the* $(j+1)_{th}$ *term of the sequence of radii of generations of circle i.e. the radius of the* j_{th} *successor circle, is defined by,* $\frac{r}{\left(\cos\frac{\pi}{X}\right)^{j}}$ *and the distance between any two nearby successor circles (including single circle also) is defined by,* $\left[\frac{1}{\left(\cos\frac{\pi}{X}\right)^{j}} - \frac{1}{\left(\cos\frac{\pi}{X}\right)^{j-1}}\right]r$.

Let, the statement of the above theorem be our P(m), for some $m \in N$. To prove this statement we will use the 'Induction Hypothesis'.

Proof. Our previous discussions show that $P(1)$ and $P(2)$ are true for $X = 3$ and $X = 4$, respectively. Let us assume that, P(K) is true. This means we are applying here, P_{K+2} pattern of tangents. A sequence

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of radius of the successor circles depending on the radius of the given circle, can be obtained. Then the radius of the j_{th} successor circle is, $\frac{r}{\left(\cos\frac{\pi}{K+2}\right)^j}$ and the distance between any two nearby successor circles (including a single circle also) is,

$$\left[\frac{1}{\left(\cos\frac{\pi}{K+2}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{K+2}\right)^{j-1}}\right]r$$

Now, we need to show that, P(K+1) is also true.
Let, $b_1, b_2, b_3, ..., b_{K+1}$ are eip of the tangents of the given single circle. So, all those points lie on the circumference of the 1_{st} successor circle. Let, u_1 be any of the ctip and lies on the circumference of the single circle. Therefore taking lines passing through the origin, we join those points. Joining points are, (i). eip and eip (ii) eip and ctip (iii) ctip and ctip.

Here we are using, P_{K+3} pattern of tangents. Thus, the circle can be broken into $2(K+3)$ equal parts

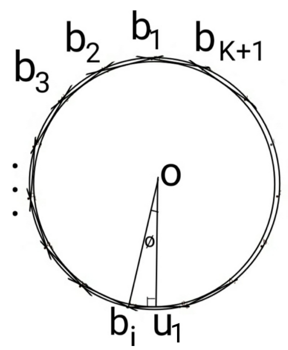

Figure 26

and each of those equal parts is a 'right triangle'. One of those equal parts is, $\triangle ou_1b_i$, where. ou_1 = r , ob_i = r_1. It can be easily checked that, $\phi = \frac{360}{2(K+3)}$ degree. Therefore, $\cos\frac{\pi}{(K+3)} = \frac{r}{r_1} \Rightarrow r_1 = \frac{r}{\cos\frac{\pi}{K+3}}$.
r_2 i.e. the radius of the 2_{nd} successor circle, can be obtained by repeating the same process on the 1_{st} successor circle. So now, $r_2 = \frac{r_1}{\cos\frac{\pi}{K+3}} = \frac{r}{\left(\cos\frac{\pi}{K+3}\right)^2}$. Basically, we have a sequence of radius of successor circles, $\left\{r, \frac{r}{\cos\frac{\pi}{K+3}}, \frac{r}{\left(\cos\frac{\pi}{K+3}\right)^2}, ...\right\} = \left(\frac{r}{\left(\cos\frac{\pi}{K+3}\right)^{j-1}}\right)_{j=1}^{\infty}$. Here, the radius of the j_{th} successor circle is, $\frac{r}{\left(\cos\frac{\pi}{K+3}\right)^j}$, and the distance between any two nearby successor circles (including the given single circle also) is, $\left[\frac{1}{\left(\cos\frac{\pi}{K+3}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{K+3}\right)^{j-1}}\right]r$. So, our proposed statement holds for this pattern also.
Hence, P(K+1) is true. Thus, our statement P(m) is true for all m belongs to N. □

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For the P_∞ pattern of tangents on the single circle then we will just take the limit on the radius and distance formulae. Therefore the radius of the j_{th} successor circle (for P_∞ pattern) is, $\lim_{X\to\infty} \frac{r}{\left(\cos\frac{\pi}{X}\right)^j} = \frac{r}{\left(\lim_{X\to\infty}\cos\frac{\pi}{X}\right)^j} = \frac{r}{1} = r$ and the distance between any two successive circles (including the single circle also) is, $\lim_{X\to\infty}\left[\frac{1}{\left(\cos\frac{\pi}{X}\right)^j} - \frac{1}{\left(\cos\frac{\pi}{X}\right)^{j-1}}\right] r = [1-1]r = 0$. So, all the infinitely many successor circles have almost the same radius, as what the single circle has, i.e. all the infinitely many successors circles almost [5]merge into a single one, almost without taking any time. Interestingly, if our given circle has zero radius then whatever P_X pattern we apply to it, it will always remain a single point.

3 Discussion.

The article describes how one can obtain infinitely many circles from a single circle, using its tangents with a pattern. It also introduces one new theorem regarding the gaps between nearby successor circles and the patterns of radius. This idea can be applied to computer-aided designs. There may be other many applications that need to think about. Without the approach discussed in this article, we can create new circles from a given single circle. We can just draw a triangle (or, any other polygon with sides touching a given circle), and therefore using a compass we can draw another circle. That idea simply can be realized by the idea of rotating that polygon and then one can visualize the formation of a new circular path. The rotating idea simply lies behind the idea discussed in this article. That's why our idea is more general.

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[5]We mentioned 'almost', because we use polygonal approximation to get such observation and also in Limit, we are not interested in the arrival, rather we are interested in approaching .

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