Non trivial zeros of the Zeta function using the differential equations

W. Oukil
Faculty of Mathematics.
University of Science and Technology Houari Boumediene.
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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Abstract

In this paper, we investigate a relation between the differential equations and the non trivial zeros of the Zeta function.

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1 Main result

Consider the representation of the Riemann Zeta function ζ defined by the Abel summation formula [[1], page 14 Equation 2.1.5] as

$$\zeta(s) := -\frac{s}{1-s} - s\int_1^{+\infty} \frac{\{t\}}{t^{1+s}}dt, \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*, \tag{1}$$

where $\{t\}$ is the fractional part of the real t. We prove the following Theorem.

Theorem 1. *Consider the function ζ defined by the Equation* (1)*. For every $\tau \in \mathbb{R}^*$ and $r \in (\frac{1}{2}, 1)$ we have*

$$|\zeta(r+i\tau)| \neq 0.$$

Thanks to the Riemann functional equation we deduce that any *non trivial zero* of the Zeta function has a real part equal to $\frac{1}{2}$, where the *non trivial zeros* are defined in the following sense

Definition 2. Consider the function ζ defined by the Equation (1). Let be $s \in \mathbb{C}$. We say that s is a *non trivial zero of the function ζ* if

$$|\zeta(s)| = 0 \quad \text{and} \quad \Re(s) \in (0,1), \quad \Im(s) \in \mathbb{R}^*.$$

2 Main Proposition

For every $r \in (0,1)$ and $\tau \in \mathbb{R}^*$ the Equation (1) implies,

$$\frac{\zeta(r+i\tau)}{r+i\tau} = -\frac{1}{1-r-i\tau} - \int_1^{+\infty} u^{-i\tau-1-r}\{u\}du,$$

equivalent to

$$\frac{\zeta(r+i\tau)}{r+i\tau} = -\frac{1}{1-r-i\tau} - \int_0^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}du. \qquad (2)$$

The aim is to studies the differential equation of solutions the functions

$$t \mapsto \psi_{\tau,r}(z,t) := \exp((r+i\tau)t)\Big[z + \int_0^t \exp(-(r+i\tau)u)\{\exp(u)\}du\Big],$$
$$z \in \mathbb{C},\ t \geq 0.$$

We focus only on the bounded solutions (there is a unique bounded solution. All other solutions are oscillating and diverge to infinity in norm). More precisely, the strategy to prove the Theorem 1, is to use the Laplace transformation and prove that $\sup_{t\geq 0} |\psi_{\tau,r}(\frac{1}{1-r-i\tau}, t)| < +\infty$ implies $2r \leq 1$. In other words $|\frac{\zeta(r+i\tau)}{r+i\tau}| = 0$ implies $2r \leq 1$.
For every $\tau \in \mathbb{R}^*$ and $r \in (0,1)$ we consider the following differential equation

$$\frac{d}{dt}x = (r+i\tau)x + \{\exp(t)\}, \qquad (3)$$
$$t \in \mathbb{R}_+/\mathbb{N}, \quad x(0) = \frac{1}{1-r-i\tau}, \quad x : \mathbb{R}_+ \to \mathbb{C}.$$

Lemma 3. *For every $\tau \in \mathbb{R}^*$ and $r \in (0,1)$ there exists a unique continuous solution $\psi_{\tau,r}(t) : \mathbb{R}_+ \to \mathbb{C}$ of the differential equation (3). Further,*

$$\psi_{\tau,r}(t) = \exp((r+i\tau)t)\Big[\frac{1}{1-r-i\tau} + \int_0^t \exp(-(r+i\tau)u)\{\exp(u)\}du\Big].$$

Proof. Let be $r \in (0,1)$ and $\tau \in \mathbb{R}^*$ fixed. The Equation (3) is a non-homogeneous linear differential equation. The unique continuous solution $\psi_{\tau,r}(t) : \mathbb{R}_+ \to \mathbb{C}$ such that $\psi_{\tau,r}(0) = \frac{1}{1-r-i\tau}$ is given by

$$t \mapsto \exp((r+i\tau)t)\Big[\frac{1}{1-r-i\tau} + \int_0^t \exp(-(r+i\tau)u)\{\exp(u)\}du\Big],$$

□

Proposition 4. *Let be $\tau \in \mathbb{R}^*$ and $r \in (0,1)$. Consider the continuous solution $\psi_{\tau,r}(t) : \mathbb{R}_+ \to \mathbb{C}$ of the differential equation* (3)*. Suppose that $2r > 1$, then $\sup_{t\geq 0} |\psi_{\tau,r}(t)| = +\infty$.*

Proof. Let be $r \in (\frac{1}{2}, 1)$ and $\tau \in \mathbb{R}^*$. In order to simplify the notation, denote the function $\Phi_{\tau,r} : \mathbb{R}_+ \to \mathbb{C}$ as

$$\Phi_{\tau,r}(t) := \int_0^t \exp((r+i\tau)u) \int_0^u \exp(-(r+i\tau)v)\{\exp(v)\}dvdu, \quad \forall t \geq 0.$$

By the Lemma 3

$$\begin{aligned}\int_0^t \psi_{\tau,r}(u)du &= \Phi_{\tau,r}(t) + \frac{1}{1-r-i\tau}\int_0^t \exp((r+i\tau)u)du \\ &= \Phi_{\tau,r}(t) + \frac{1}{1-r-i\tau}\frac{1}{r+i\tau}\Big[\exp((r+i\tau)t) - 1\Big], \quad \forall t \geq 0.\end{aligned} \tag{4}$$

Using the fact $r \in (\frac{1}{2}, 1)$, to prove the present Proposition, it is sufficient to prove that there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$|\Phi_{\tau,r}(t_n)| \leq \frac{\exp(rt_n)}{r^2+\tau^2}, \quad \forall n \in \mathbb{N}.$$

The function $\Phi_{\tau,r} : \mathbb{R}_+ \to \mathbb{C}$ satisfies

$$\frac{d^2}{dt^2}\Phi_{\tau,r} = (r+i\tau)\frac{d}{dt}\Phi_{\tau,r} + \{\exp(t)\},$$
$$t \in \mathbb{R}_+/\mathbb{N}, \quad |\frac{d}{dt}\Phi_{\tau,r}(0)| = 0.$$

We obtain the new following linear differential equation

$$\frac{d}{dt}\Phi_{\tau,r} = (r+i\tau)\Phi_{\tau,r} + \int_0^t \{\exp(u)\}du,, \tag{5}$$
$$t \in \mathbb{R}_+/\mathbb{N}, \quad |\Phi_{\tau,r}(0)| = 0.$$

Consider the Laplace transformation, defined as

$$\mathcal{L}[g(t)](x) := \int_0^{+\infty} \exp(-xt)g(t)dt, \quad \forall x > r.$$

By the Equation (5), we get

$$\mathcal{L}[\Phi_{\tau,r}(t)](x) = \frac{\int_0^{+\infty}\exp(-xt)\Big[\int_0^t\{\exp(u)\}du\Big]dt}{x-r-i\tau}, \quad \forall x > r. \tag{6}$$

Which can be written as

$$\mathcal{L}[\Phi_{\tau,r}(t)](x) = \mathcal{L}[\phi_{1,\tau,r}(t)](x)\mathcal{L}[\phi_{2,\tau,r}(t)](x), \tag{7}$$

where

$$\begin{aligned}
\mathcal{L}[\phi_{1,\tau,r}(t)](x) &= \frac{1}{x}\frac{1}{x-r-i\tau}, \\
\mathcal{L}[\phi_{2,\tau,r}(t)](x) &= x\int_0^{+\infty} \exp(-xt)\Big[\int_0^t \{\exp(u)\}du\Big]dt
\end{aligned}$$

Then

$$\begin{aligned}
\phi_{1,\tau,r}(t) &= \frac{1}{r+i\tau}\Big[\exp((r+i\tau)t)-1\Big], \\
\phi_{2,\tau,r}(t) &= \frac{d}{dt}\int_0^t \{\exp(u)\}du = \{\exp(u)\}.
\end{aligned}$$

Using the convolution formula for the Equation (7), we obtain

$$\begin{aligned}
\Phi_{\tau,r}(t) &= \int_0^t \phi_{1,\tau,r}(u)\phi_{2,\tau,r}(t-u)du \\
&= \frac{1}{r+i\tau}\int_0^t \Big[\exp((r+i\tau)u)-1\Big]\Big\{\exp(t-u)\Big\}du.
\end{aligned}$$

Then there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$|\Phi_{\tau,r}(t_n)| \leq \frac{\exp(rt_n)}{r^2+\tau^2}, \quad \forall n \in \mathbb{N}.$$

Since $r \in (\frac{1}{2}, 1)$, by the Equation (4) we get $\sup_{t>0} |t^{-1}\int_0^t \psi_{\tau,r}(u)du| = +\infty$. □

3 Proof of the Theorem 1

Proof of the Theorem 1. Let be $r \in (0,1)$ and $\tau \in \mathbb{R}^*$. Suppose that $|\zeta(r+i\tau)| = 0$. By the Equation (2) we have,

$$\frac{\zeta(r+i\tau)}{r+i\tau} = -\frac{1}{1-r-i\tau} - \int_0^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}du.$$

Then

$$\Big|\frac{1}{1-r-i\tau} + \int_0^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}du\Big| = 0.$$

Implies

$$\frac{1}{1-r-i\tau} + \int_0^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}$$
$$= -\int_t^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}du, \quad \forall t \geq 0.$$

$$\sup_{t\geq 0}\Big| \exp((r+i\tau)t)\Big[\frac{1}{1-r-i\tau} + \int_0^t \exp(-(r+i\tau)u)\{\exp(u)\}du\Big]\Big|$$
$$= \sup_{t\geq 0}\Big| \exp((r+i\tau)t)\int_t^{+\infty} \exp(-(r+i\tau)u)\{\exp(u)\}du\Big|$$
$$\leq \sup_{t\geq 0}\Big[\exp(rt)\int_t^{+\infty}\exp(-ru)du\Big] \leq \frac{1}{r}. \tag{8}$$

Let $\psi_{\tau,r}(t) : \mathbb{R}_+ \to \mathbb{C}$ be the unique continuous solution of the differential equation (3). By Lemma 3 and the Equation (8) we have,

$$\sup_{t\geq 0} |\psi_{\tau,r}(t)| < +\infty,$$

Thanks to the Main Proposition 4 we get $2r \leq 1$. □

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).