Exponential stable manifold for some linear system with application to the mean field models

W. Oukil
Faculty of Mathematics.
University of Science and Technology Houari Boumediene.
BP 32 EL ALIA 16111 Bab Ezzouar, Algiers, Algeria.

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Abstract

We study the existence of the *exponential stable manifold* for some linear system and we give some proprieties. We apply the results to the mean field systems similar to the Winfree model in the synchronized state. More precisely, we study it linearized system. This method can be applied to more generalized mean field models.

Keywords: Exponential stability, periodic system, mean field theory, coupled oscillators, synchronization, Winfree model.
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1 Introduction and Main results

In 1967 Winfree [9] proposed a mean field model describing the synchronization of a population of organisms or *oscillators* that interact simultaneously [1, 2, 3, 4, 5, 7].

The main result consists of two parts: *The linear part*, where we will study the stability of a class of perturbed linear systems by decomposing the fundamental matrix. *The Application part* where we study the exponential stability for the Winfree model. The results are within the framework of perturbation theory. In this article, we consider the usual scalar product and the sup-norm as follow: For every vectors $u = (u_i)_{i=1}^m$ and $v = (v_i)_{i=1}^m$ we denote the usual scalar product by

$$\langle u, v\rangle = \sum_{i=1}^{m} u_i v_i, \tag{1}$$

Let $t \in \mathbb{R} \mapsto Q(t) := \{q_{i,j}(t)\}_{\substack{1\le i\le m\\ 1\le j\le m'}}$ be a $m \times m'$ matrix-valued function, we denote the sup-norm of Q as

$$\|Q\| = \sup_{t\in\mathbb{R}} \max_{\substack{1\le i\le m\\ 1\le j\le m'}} |q_{ij}(t)|.$$

In this article the integer $n \in \mathbb{N}^*$ is fixed. We denote by I_n the square identity matrix of order n and $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^n$.

1.1 Main results

We study in this article the following perturbed linear system:

$$\dot{y}(t) = [b(t)I_n + \mathcal{A}(t) + \zeta(t)]y(t), \quad t \ge t' \in \mathbb{R}, \tag{2}$$

where the *perturbation term* $t \mapsto \zeta(t)$ is a square matrix-valued function of order n. The function $t \in \mathbb{R} \mapsto b(t)$ is a continuous real valued function and $t \in \mathbb{R} \mapsto \mathcal{A}(t)$ is a continuous square matrix-valued function of order n satisfying the following *stability assumptions*

- $[H1]$: $b(t)$ and $A(t)$ are 1-periodic,
- $[H2]$: $\mathcal{A}(t) = \{a_{i,j}(t)\}_{1\le i,j\le n}$ is rank one with $a_{i,j}(t) = a_j(t)$, $\forall i, j = 1, \dots, n$,
- $[H3]$: $\alpha := -\int_0^1 b(s)\, ds > 0$,
- $[H4]$: $\int_0^1 b(s) + \langle \mathbb{1}, a(s)\rangle\, ds = 0$, where $a(s) := (a_j(s))_{j=1}^n$, $s \in [0, 1]$.

Define the *fundamental matrix* of a linear system in the following sense

Definition 1. Let $t \mapsto Q(t)$ be a continuous square matrix-valued function of order n. The *fundamental matrix* of the linear system

$$\dot{y} = Q(t)y, \quad t \in \mathbb{R},$$

is the square matrix-valued function $(s, t) \in \mathbb{R} \times \mathbb{R} \mapsto \Psi^{s,t}$ that satisfies

- $\forall t \in \mathbb{R}$, $\Psi^{t,t} = I_n$,
- $\forall (t, s) \in \mathbb{R} \times \mathbb{R}$, $\frac{d}{dt}\Psi^{t,s} = Q(t)\Psi^{t,s}$.

To gain further insights into the behavior of solutions of the linear system (2), we will introduce a class of matrices ζ referred to as *normalizing matrices*, defined as follows

Definition 2. We say that the matrix-valued function ζ is a *normalizing matrix* if the system (2) admits a solution $t \mapsto v(t) := (v_j(t))_{j=1}^n$ such that

$$\inf_{t \geq t'} \|v(t)\| > 0, \quad \text{and} \quad \sup_{t \geq t'} \|v(t)\| < +\infty.$$

We call v a *normalizing* solution of (2) associated to the matrix ζ.

In the following two linear results, we will consider two cases: ζ being a normalizing matrix or arbitrary.

Main Result 3. [General Case] Consider the system (2) with fundamental matrix $R^{s,t}$. Suppose that b and $\mathcal{A}$ satisfy the stability assumptions $[H1 - H4]$. For every $\beta \in (0, \alpha)$ there exist $k > 0$ and $c > 0$ such that for any matrix ζ satisfying $\|\zeta\| < c$ and for any $t \in \mathbb{R}$, there exists a linear form $\psi_t : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall y \in \mathbb{R}^n : \quad \|R^{s,t}[y - \psi_t(y)\mathbb{1}\| < k\|y\| \exp(-\beta(s-t)), \quad s \geq t \geq t'.$$

Moreover, the fundamental matrix $R^{s,t}$ admits the following decomposition

$$\forall y \in \mathbb{R}^n : \quad R^{s,t}y = \psi_t(y)R^{s,t}\mathbb{1} + R^{s,t}[y - \psi_t(y)\mathbb{1}], \quad s \geq t \geq t'.$$

In other words, the manifold $\mathcal{W}_{stab} := \{z \in \mathbb{R}^n, \psi_t(z) = 0\}$ of dimension $n-1$ is exponentially stable. Furthermore, if there exists a solution that does not exponentially decay to zero, then

$$\forall y \in \mathbb{R}^n : \quad \psi_t(y) = \psi_s(R^{s,t}y), \quad s \geq t \geq t'.$$

Main Result 4. Under assumptions of the Main result (I^l), suppose in addition that the matrix-valued function ζ is a normalizing matrix. Then for any $y \in \mathbb{R}^n$ and for any $s \geq t \geq t'$ the linear form $\psi_t : \mathbb{R}^n \to \mathbb{R}$ satisfies the following assertions

- $\psi_s(R^{s,t}v(t)) = 1$,
- $|\psi_t(y)| < k\|y\|$,
- $\psi_t(y) = \psi_s(R^{s,t}y)$,

where $v(t)$ is a normalizing solution of (2) associated to the matrix ζ. Moreover, the fundamental matrix $R^{s,t}$ admits the following decomposition

$$\forall y \in \mathbb{R}^n : \quad R^{s,t}y = \psi_t(y)v(s) + R^{s,t}[y - \psi_t(y)v(t)], \quad s \geq t \geq t'.$$

In other words, the manifold $\mathcal{W}_{stab} := \{z \in \mathbb{R}^n, \psi_t(z) = 0\}$ of dimension $n - 1$ is exponentially stable.

2 Proof of Linear Results (3) and (4): Stability of Perturbed Linear Systems

Consider the system (2). For simplification, we denote

$$c_b = \max_{t\in[0,1]} |b(t)|, \quad c_a = \max_{t\in[0,1]} \sum_{j=1}^{n} |a_j(t)|.$$

Let's denote

$$a(s) := (a_j(s))_{j=1}^n \text{ and } \quad \zeta_i(s) := (\zeta_{i,j}(s))_{j=1}^n, \quad i = 1, \ldots, n, \tag{3}$$

and

$$e^{t,s} := \exp\Big(\int_s^t b(\nu)d\nu\Big), \quad p^{t,s} = e^{t,s}\exp\big(\int_s^t \langle \mathbb{1}, a(\nu)\rangle\, d\nu\Big), \quad \forall t,s \in \mathbb{R}.$$

Recall that we denoted $\mathbb{1} = (1, \ldots, 1) \in \mathbb{R}^n$.

2.1 Proof Tools

To prove the linear results (3) and (4), we consider in this section only the stability assumptions $[H1 - H4]$. On the other hand, this allows us to deduce the Main result (3). Let be $y \in \mathbb{R}^n$ and consider the following non-homogeneous system defined for all $t \geq t'$ as:

$$\begin{cases} \dot{z}^*(t) & = b(t)z^*(t) + \zeta(t)[z_{n+1}(t)\mathbb{1} + z^*(t) + e^{t,t'}y], \\ \dot{z}_{n+1}(t) & = [b(t) + \langle a(t), \mathbb{1}\rangle]z_{n+1}(t) \\ & +\langle a(t), z^*(t) + e^{t,t'}y\rangle, \end{cases} \tag{4}$$

where $z^*(t) := (z_j(t))_{j=1}^n$ and where the function $t \mapsto a(t)$ is defined by equation (3). The goal of introducing the above system is that the part $E^{t',t} = R^{t,t'}[y - \psi_{t'}(y)\mathbb{1}]$ given in the Main results (3) satisfies the following decomposition:

$$E^{t',t}y = z_{n+1}(t)\mathbb{1} + z^*(t) + e^{t,t'}y,$$

where $z(t) := (z^*(t), z_{n+1}(t))$ with $z_{n+1}(t)$ and $z^*(t) = (z_j(t))_{j=1}^n$ is a solution of the coupled non-homogeneous system (4). The idea is to show that $\|z(t)\| < k\exp(-\beta(t-t'))$ with $\beta \in (0, \alpha)$. Here, it is worth noting that the initial condition of $z(t)$ satisfies $z_{n+1}(t')\mathbb{1} + z^*(t') = -\psi_{t'}(y)\mathbb{1}$. The problem to solve is to find this suitable initial condition. To do this, we impose that $z^*(t') = \psi_{t'}(y)(\mathbb{1}, 0)$. First, we will see in this section under what assumptions we will have exponential decay of $z(t)$ to zero. This will allow us to find

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a specific initial condition $\psi_{t'}(y)(\mathbb{1}, 0)$, which in turn will lead to the linear form $\psi_{t'}$ in Section 2.2. We will need three lemmas, with the third Lemma 7 being the main one. The following Lemma enables us to prove Lemma 6, which in turn allows us to prove the main Lemma 7.

Lemma 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function such that*

$$\theta := -\int_0^1 f(s)ds > 0.$$

Let $\beta \in (0, \theta)$, $\omega > 0$, and $\gamma > 0$. Consider the following equation:

$$\frac{d}{dt}\delta(t) = [f(t) + \beta]\delta(t) + \omega\gamma,$$

then there exists $\gamma_* > 0$ such that for all $\gamma < \gamma_*$, the above equation has a solution $\delta(t)$ that is 1-periodic and strictly positive, with $\max_{t\in[0,1]} \delta(t) < 1$. The solution $\delta(t)$ is given by

$$\delta(t) = \frac{\omega\gamma}{1 - \exp(\beta - \theta)} \int_t^{t+1} \exp(\int_s^{t+1} f(\nu) + \beta d\nu)ds.$$

Lemma 6. *Let be $y \in \mathbb{R}^n$ and consider the system (4) with $b(t)$ and $\mathcal{A}$ satisfying the stability assumptions $[H1 - H4]$. Let $\beta \in (0, \alpha)$, then there exist $\omega > 0$ and $c > 0$ such that for any continous matrix-valued function ζ satisfying $\|\zeta\| < c$ and for any solution $z(t) = (z^*(t), z_{n+1}(t))$ of system (4) with an initial condition $z(t') = z \in \mathbb{R}^n$, we have:*

$$\begin{aligned} \forall s > t' \ : \quad & z_{n+1}(s) = 0 \\ & \implies \|z(t)\| < \omega \exp(-\beta(t - t'))[\|z\| + \|y\|], \ \forall t \in [t', s]. \end{aligned}$$

Proof. Let $z(t) = (z^*(t), z_{n+1}(t))$, where $z^*(t) := (z_j(t))_{j=1}^n)$ is a solution to the linear system (4) with the initial condition $z(t') = z \in \mathbb{R}^n$. Suppose that $z_{n+1}(T) = 0$. By integrating (4), we obtain

$$z_{n+1}(t') = -\int_{t'}^T \langle a(s), z^*(s) + e^{s,t'}y\rangle p^{t',s}ds. \tag{5}$$

Let $\beta < \alpha$. Let $c_{z,y} = \|z\| + \|y\|$, so for all $\tilde{c} > 1$, we have $\|z\| < \tilde{c}c_{z,y}$, and there exists $\epsilon > 0$ such that $\|z^*(t)\| < \tilde{c}c_{z,y} \exp(-\beta(t - t'))$. Define

$$T_* = \sup\{t' \leq s \leq T_* : \ \|z^*(s)\| < \tilde{c}c_{z,y} \exp(-\beta(s - t'))\},$$

$$\|z^*(t)\| < \tilde{c}c_{z,y} \exp(-\beta(t - t')) \quad \forall t \in [t', T_*[. \tag{6}$$

The strategy is to find a particular constant M such that $T_* \geq T$. By contradiction, assume that $T_* < T$. By integrating (4) and using (5), we obtain for all $t \in [t', T_*[$

$$\begin{aligned}
|z_{n+1}(t)| &= \left| p^{t,t'} \left[z_{n+1}(t') + \int_{t'}^{t} \langle a(s), z^*(s) + e^{s,t'} y \rangle, p^{t',s} ds \right] \right| \\
&= p^{t,t'} \Big| \int_{t}^{T} \langle a(s), z^*(s) + e^{s,t'} y \rangle p^{t',s} ds \Big| \\
&< c_{z,y} c_a \exp(c_b + c_a) \int_{t}^{T} \exp(-(\beta s - t')) ds \\
&+ c_a c_{z,y} \exp(c_b + c_a) \int_{t}^{T} \exp(-\alpha(s - t') + c_b) ds \\
&< \frac{2 c_a \tilde{c} c_{z,y}}{\beta} \exp(2c_b + c_a) \exp(-\beta(t - t')). \qquad (7)
\end{aligned}$$

Let ζ be a continuous matrix with $\|\zeta\| < c$, then from equation (4), we have the following two inequalities for $z_i(t)$ and $-z_i(t)$

$$\begin{aligned}
|\frac{d}{dt} z_i(t) - b(t) z_i(t)| &< c\tilde{c} c_{z,y} [\frac{2c_a}{\beta} \exp(2c_b + c_a) + 1] \exp(-\beta(t - t')) \\
&\quad + c\tilde{c} c_{z,y} \exp(c_b)) \exp(-\beta(t - t')). \qquad (8)
\end{aligned}$$

Let $z_i(t) = \delta_i(t) \exp(-\beta(t - t'))\tilde{c} c_{z,y}, \quad t \in [t', T_*[$. We have $|\delta_i(t)| \leq 1$ on $[t', T_*[$. Substituting into the last equation, we have

$$|\frac{d}{dt} \delta_i(t) - [b(t) + \beta] \delta_i(t)| < c[\frac{2c_a}{\beta} \exp(2c_b + c_a) + 1 + \exp(c_b))].$$

By definition of T_* there exists $i \in \{1, \ldots, n\}$ such that $|\delta_i(T_*)| = 1$. We will use Lemma 5 to obtain a contradiction.
Lemma 5 implies that there exists $c_* > 0$ such that for every $c \in (0, c_*)$ there exists a strictly positive 1-periodic function solution of the equation

$$\frac{d}{dt} \delta(t) = [b(t) + \beta] \delta(t) + c[\frac{2c_a}{\beta} \exp(2c_b + c_a) + 1 + \exp(c_b))],$$

such that $\max_{t \in [0,1]} \delta(t) < 1$. Let $\tilde{c} > 1$ be such that $\frac{1}{\tilde{c}} < \delta(t')$; thus, $\delta_i(t') \leq \frac{\|z\|}{\tilde{c} c_{z,y}} \leq \frac{1}{\tilde{c}} < \delta(t')$. There exists $\epsilon' > t'$ such that $|\delta_i(t)| < \delta(t)$ on $[t', \epsilon'[$; let $T_0 = \sup\{t' \leq s \leq T_0 : |\delta_i(s)| < \delta(s)\}$. If $T_0 \in (t', T_*)$, then $|\delta_i(T_0)| = \delta(T_0)$. Without loss of generality, assume that $\delta_i(T_0) = \delta(T_0)$, then we obtain

$$\begin{aligned}\frac{d}{dt}|\delta_i(T_0)| &< [b(t)+\beta]|\delta_i(T_0)| + c[\frac{2c_a}{\beta}\exp(2c_b+c_a)+1+\exp(c_b))] \\ &= [b(t)+\beta]\delta(T_0) + c[\frac{2c_a}{\beta}\exp(2c_b+c_a)+1+\exp(c_b))] \\ &= \frac{d}{dt}\delta(T_0).\end{aligned}$$

Contradiction. Thus, $T_0 > T_*$, which implies $\delta_i(T_*) < 1$, contradicting the definition of T_*. Therefore, for all $t \in [t', T]$, we have

$$\|z^*(t)\| < \omega \exp(-\beta(t-t'))[\|z(t')\| + \|y\|]$$

for some constant $\omega = \tilde{c}\max(1, 2\exp(2c_b + c_a)c_a)$. □

The previous Lemma does not provide exponential decay of a solution over $[t', +\infty[$. To achieve this, the strategy in the following Lemma is to consider intervals $[0, T]$ from the previous Lemma, gradually increasing in size. We will approximate a solution $z_y(t)$ with an initial condition $z_y(t')$ using solutions that satisfy the previous lemma. This allows us to demonstrate the existence of a solution $z_y(t)$ that exponentially decays to zero over $[t', +\infty[$. We will also localize the initial condition $z_y(t')$. In order to simplify the notation, denote

$$W := (\mathbb{1}, 0) \in \mathbb{R}^{n+1}.$$

Lemma 7. *Let be $y \in \mathbb{R}^n$ and consider the system (4) with $b(t)$ and $\mathcal{A}$ satisfying the stability assumptions $[H1 - H4]$. Let $\beta \in (0, \alpha)$. Suppose that there exists a sequence solutions $\Big(z_m(t) = (z^*_m(t), z_{n+1,m}(t))\Big)_{m\in\mathbb{N}}$ of (4) with initial condition $z_m(t') = z_{t',m}W$ for some $z_{t',m} \in \mathbb{R}$ such that*

$$z_{n+1,m}(t_m) = 0, \quad \forall m \in \mathbb{N},$$

where $(t_m)_{m\in\mathbb{N}}$ is a real sequence that tends to infinity. Then there exist $k > 0$ and $c > 0$ such that for any continuous matrix-valued function ζ satisfying $\|\zeta\| < c$ there exists a solution $z_y(t) = (z^*_y(t), z_{n+1,y}(t))$ of system (4) with initial condition $z_y(t') = z_y \in \mathbb{R}^n$ such that

$$\|z_y(t)\| < k\exp(-\beta(t-t'))\|y\|, \quad \forall t \geq t'.$$

Moreover, there exists a subsequence $(z_{m_k}(t'))_k$ of $(z_m(t'))_m$ such that

$$z_y(t') = \lim_{k\to+\infty} z_{m_k}(t').$$

Proof. Let $z_m(t)$ and $(t_m)_m$ satisfy the assumptions of this Lemma. The Lemma 6 implies that

$$\exists \omega > 0: \quad \|z_m(t)\| < \omega \exp(-\beta(t-t'))[\|y\| + \|z_m(t')\|], \quad \forall t \in [t', t_m].$$

The idea is to show that there exists $\tilde{c} > 0$ such that $\|z_m(t')\| < \tilde{c}\|y\|$ and then extract a convergent subsequence of $z_m(t)$. Let be $c > 0$ that we determine later. Let ζ be a continuous matrix-valued function such that $\|\zeta\| < c$. Recall that the function $s \mapsto \zeta_i(s)$ is define in the Equation (3). By integrating (4) for all $t \in [t', t_m]$ and all $i = 1, \dots, n$, we have

$$z_{i,m}(t) = e^{t,t'} z_{i,m}(t') + g_{i,m}(t),$$

where the function $g_m := (g_{i,m})_{i=1}^n$ satisfies the flowing equation

$$\begin{aligned} |g_{i,m}(t)| &= |e^{t,t'} \int_{t'}^{t} \langle \zeta_i(s), z_{n+1,m}(s)\mathbb{1} + z_m^*(s) + e^{s,t'} y\rangle e^{t',s} ds| \\ &< c(2\omega + \exp(c_b))[\|y\| + \|z_m(t')\|] e^{t,t'} \int_{t'}^{t} \exp((\alpha - \beta)(s - t')) ds \\ &< c(2\omega + \exp(c_b)) \exp(c_b)[\|y\| + \|z_m(t')\|] \frac{\exp(-\beta(t-t'))}{\alpha - \beta}. \end{aligned}$$

Since $z_{n+1,m}(t_m) = 0$, and by integrating (4), we deduce that $z_{n+1,m}(t_m) = 0$ if and only if

$$\begin{aligned} \int_{t'}^{t_m} \langle a(s), e^{s,t'} y\rangle p^{t',s} ds = &- \int_{t'}^{t_m} p^{t',s} \langle a(s), g_m(s)\rangle ds \\ &- z_{i,m}(t') \int_{t'}^{t_m} \langle a(s), \mathbb{1}\rangle \exp(-\int_{t'}^{s} \langle a(\nu), \mathbb{1}\rangle d\nu) ds. \end{aligned}$$

We deduce that

$$\begin{aligned} &|z_{i,m}(t')| \Big| 1 - \exp(-\int_{t'}^{t_m} \langle a(x), \mathbb{1}\rangle dx) \Big| \\ &< cc_a(2\omega + \exp(c_b)) \exp(c_b) \frac{\|y\| + \|z_{t',m}\|}{\alpha - \beta} \int_{t'}^{t_m} \exp(-\beta(s - t')) p^{t',s} ds \\ &+ | \int_{t'}^{t_m} \langle a(s), e^{s,t'} y\rangle p^{t',s} ds|. \end{aligned}$$

For $c \approx 0$ and $m \to +\infty$, we will have

$$|z_{i,m}(t')| < \frac{\Big[c\frac{c_a(2\omega + \exp(c_b))}{\beta(\alpha-\beta)} + \frac{c_a}{\alpha}\Big] \exp(2c_b + c_a)}{1 - cc_a(2\omega + c_b)\exp(2c_b + c_a)\frac{1}{\beta(\alpha-\beta)}} \|y\|,$$

which implies that $(\|z_m(t)\|)_m$ is uniformly bounded on each interval $[t', t_m]$. Furthermore,

$$\|z_m(t)\| < \omega \exp(-\beta(t-t'))[\|y\| + \|z_m(t')\|] < k \exp(-\beta(t-t'))\|y\|,$$

where

$$k := \omega\Big[1 + \frac{\exp(2c_b + c_a)\Big[c\frac{c_a(2\omega+\exp(c_b))}{\beta(\alpha-\beta)} + \frac{c_a}{\alpha}\Big]}{1 - cc_a(2\omega + c_b)\exp(2c_b + c_a)\frac{1}{\beta(\alpha-\beta)}}\Big].$$

Therefore, we can extract a convergent subsequence that converges to a solution $z_y(t)$ of (4) and satisfies

$$\|z_m(t)\| < k \exp(-\beta(t-t'))\|y\|.$$

□

2.2 Ingredients for the linear form $\psi_{t'}$

In this section, we will show the existence of a family of solutions to the system (4) that satisfies the assumptions of Lemma 7. To do this, we only need to determine the initial conditions $z_m(t') = z_{t',m}W$. Note that it is sufficient to determine the sequence of real numbers $(z_{t',m})_m$. In this section, we consider $b(t)$ and $\mathcal{A}$ satisfying only the stability assumptions $[H1 - H4]$ without distinction in the matrix ζ. This allows us, in particular, to deduce the linear Main result (4). To obtain more information about the solutions $z_m(t)$ of Lemma 7, we will express them in terms of the fundamental matrix. Let $S^{t,t'} = \{s^{t,t'}_{i,j}\}_{\substack{1\le i\le n+1\\ 1\le j\le n}}$ be the fundamental matrix of the homogeneous linear system associated to the system (4) as follows:

$$\begin{cases} \dot{x}^*(t) = b(t)x^*(t) + \zeta(t)[x_{n+1}(t)\mathbb{1} + x^*(t)], \quad x^*(t) = (x_j(t))_{j=1}^n, \\ \dot{x}_{n+1}(t) = [b(t) + \langle a(t), \mathbb{1}\rangle]x_{n+1}(t) + \langle a, x^*(t)\rangle, \end{cases} \tag{9}$$

where $a(s)$ is given by Equation (3). The solution $z(t) = (z^*(t), z_{n+1}(t))$ of (4) with initial condition $z(t') \in \mathbb{R}^n$ can be written in terms of the fundamental matrix as follows:

$$z^*(t) = S^{t,t'}_* z(t') + D^{t,t'}_n y, \tag{10}$$

$$z_{n+1}(t) = p^{t,t'} z_{n+1}(t') + p^{t,t'}\int_{t'}^t \langle a(s), S^{s,t'}_* z(t')\rangle p^{t',s} ds + p^{t,t'}\int_{t'}^t \langle a(s), D^{s,t'}_n y + e^{s,t'} y\rangle p^{t',s} ds, \tag{11}$$

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where $S_*^{t,t'} = \{s_{i,j}^{t,t'}\}_{1\leq i,j\leq n}$ and $D_n^{t,t'}$ is the square diagonal matrix-valued function of diagonal the vector-valued function $(s_{n+1,j}^{t,t'})_{j=1}^n$. As mentioned earlier, we aim to show the existence of solutions to the system (4) that satisfy the assumptions of Lemma 7. In Lemma 7, we have $z_m(t') = z_{t',m}W$, so for a sequence $(T_m)_m$ tending to infinity such that $z_{m,n+1}(T_m) = 0$ and using the notation from the previous equation (11), the sequence of real numbers $(z_{t',m})_m$ must be defined as

$$z_{t',m} = -\frac{\int_{t'}^{T_m} \langle a(s), D_n^{s,t'} y + e^{s,t'} y\rangle p^{t',s} ds}{\int_{t'}^{T_m} \langle a(s), S_*^{s,t'} W\rangle p^{t',s} ds}.$$

In Lemma 8, we will show that this sequence $(z_{t',m})_m$ is well-defined, i.e., the denominator of the quotient on the right-hand side of the equation is nonzero.

Lemma 8. *Let* $S^{t,t'} = \{s_{i,j}^{t,t'}\}_{\substack{1\leq i\leq n+1\\ 1\leq j\leq n}}$ *be the fundamental matrix of the homogeneous linear system* (9) *associated to the system* (4). *Let* $S_*^{t,t'} = \{s_{i,j}^{t,t'}\}_{1\leq i,j\leq n}$. *Suppose that* $b(t)$ *and* $\mathcal{A}$ *satisfy the stability assumptions* $[H1 - H4]$. *Define*

$$H^{t,t'} := \int_{t'}^{t} \langle a(s), S_*^{s,t'} W\rangle p^{t',s} ds, \quad W := (\mathbb{1}, 0),\ t \geq t'. \tag{12}$$

Then, for any $\beta \in (0, \alpha)$, *there exists* $c > 0$ *such that for any continuous matrix-valued function* ζ *with norm* $\|\zeta\| < c$, *there exists* $T_\beta > 0$ *such that for all* $t \geq T_\beta$, *we have* $H^{t,t'} \neq 0$.

Proof. From equation (11), we deduce that $z(t) = (S_*^{t,t'}W - z_{n+1}(t), z_{n+1}(t))$ is a solution of the linear homogeneous system (9) with the initial condition W, where $z_{n+1}(t)$ satisfies

$$z_{n+1}(t) = p^{t,t'} H^{t,t'}, \quad t \geq t'.$$

So, $z(t) = (S_*^{t,t'}W - z_{n+1}(t), z_{n+1}(t))$ is, in particular, a solution of the nonhomogeneous linear equation (4) with $y = 0$ and satisfies the assumptions of Lemma 7. Since $y = 0$, we have $\|z(t)\| \equiv 0$, which contradicts the initial condition $z(t') = W$. □

Therefore, in the following proposition, we show that the system (4) admits a sequence of solutions that satisfy the assumptions of Lemma 7.

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Proposition 9. *Let be $y \in \mathbb{R}^n$ and consider the system (4) with $b(t)$ and $\mathcal{A}$ satisfying the stability assumptions $[H1-H4]$. Let $\beta \in (0, \alpha)$. Then there exist $c > 0$ and $k > 0$, such that for any continuous matrix ζ with $\|\zeta\| < c$ there exists a solution $z_y(t) = (z_y^*(t), z_{n+1,y}(t))$ of (4) such that*

$$\|z_y(t)\| < k \exp(-\beta(t - t'))\|y\|, \quad \forall t \geq t',$$

and such that $z_y(t') = \psi_{t'}(y)W$, where

$$\psi_{t'}(y) = \lim_{t_m \to +\infty} \frac{-1}{H^{t_m,t'}} \int_{t'}^{t_m} \langle a(s), D_n^{s,t'} y + e^{s,t'} y \rangle p^{t',s} ds,$$

and where $(t_m)_m$ is a real sequence that tends to infinity, and $H^{t_m,t'}$ is given by Lemma 8.

Proof. According to Lemma 8, for any sequence $(t_m)_m$ such that $t_m > T_\beta$, we have $H^{t_m,t'} \neq 0$. From (11), the solution $z_m(t) = (z_m^*(t), z_{n+1,m}(t))$ of (4) with the initial condition $z_{m,t'}W$ such that

$$z_{m,t'} = -\frac{1}{H^{t_m,t'}} \int_{t'}^{t_m} \langle a(s), D_n^{s,t'} y + e^{s,t'} y \rangle p^{t',s} ds,$$

satisfies:

$$\begin{aligned} z_{n+1,m}(t_m) = 0 = p^{t',t} z_{m,t'} H^{t_m,t'} \\ + = p^{t',t} \int_{t'}^{t_m} \langle a(s), D_n^{s,t'} y + e^{s,t'} y \rangle p^{t',s} ds. \end{aligned}$$

According to Lemma 7, there exists a solution $z_y(t)$ such that $\|z_y(t)\| < k \exp(-\beta(t - t'))\|y\|$ for all $t \geq t'$, with $z_y(t') = \psi_{t'}(y)W$, where

$$\psi_{t'}(y) = -\lim_{t_{m_k} \to \infty} \frac{1}{H^{t_{m_k},t'}} \int_{t'}^{t_{m_k}} \langle a(s), D_n^{s,t'} y + e^{s,t'} y \rangle p^{t',s} ds.$$

□

2.3 Fundamental matrix decomposition

Finally, we will prove Main results (3) and (4) in the context of linear systems. To ensure consistency in the proof, we will first demonstrate the general case, which is the second linear Main result (3) that does not require the matrix $\zeta(t)$ to be normalizing.

Proof of the second linear Main result (3)*: General Case.*

- Let's show that $\|R^{t,t'}[y+\psi_{t'}(y)\mathbb{1}]\|$ decreases exponentially:

  For any $\beta > 0$ and $\|\zeta\| < c$, where c is defined by Lemma 7, let $\psi_{t'}(y)$ given by the previous Proposition 9. WE have

$$R^{t,t'}y = -\psi_{t'}(y)R^{t,t'}\mathbb{1} + R^{t,t'}[y+\psi_{t'}(y)\mathbb{1}]. \tag{13}$$

  Let $R^{t,t'}[y+\psi_{t'}(y)\mathbb{1}] = z_{n+1}(t)\mathbb{1} + z^*(t) + e^{t,t'}y$ with $z_{n+1}(t)$ being a solution with the initial condition $z_{n+1}(t') = 0$ of the equation

$$\begin{aligned}\dot{z}_{n+1}(t) &= [b(t) + \langle a(t), \mathbb{1}\rangle]z_{n+1}(t)\\ &\quad + \langle a(t), R^{t,t'}[y+\psi_{t'}(y)\mathbb{1}] - z_{n+1}(t)\mathbb{1}\rangle.\end{aligned}$$

  Thus, $z(t) = (z^*(t), z_{n+1}(t))$ is a solution of the nonhomogeneous linear equation (4). From equation (13), we deduce that it has the initial condition $z(t') = W\psi_{t'}(y)$. Proposition 9 implies that

$$\|z(t)\| < k\exp(-\beta(t-t'))\|y\|, \quad \forall t \geq t'.$$

  Hence, $\|R^{t,t'}[y+\psi_{t'}(y)\mathbb{1}]\| < [k+\exp(c_b)]\exp(-\beta(t-t'))\|y\|$ for all $t \geq t'$.

- Let's show that when the system has a solution that does not decay to zero exponentially, then $\psi_t(y) = \psi_s(R^{s,t}y)$:

  Suppose that the system (2) has a solution that does not exponentially decay to zero. Let $v(t)$ be this solution. We have:

$$\begin{aligned}R^{t,s}R^{s,t'}y &= \psi_{t'}(y)v(t) - R^{t,t'}[y - \psi_{t'}(y)v(t')]\\ &= \psi_s(R^{s,t'}y)v(t)\\ &\quad - R^{t,s}[R^{s,t'}y - \psi_s(R^{s,t'}y)v(s)].\end{aligned}$$

  Implies

$$\begin{aligned}&[\psi_{t'}(y) - \psi_s(R^{s,t'}y)]v(t)\\ &\qquad = R^{t,s}[R^{s,t'}\psi_{t'}(y)v(t') + \psi_s(R^{s,t'}y)v(s)].\end{aligned}$$

  Since the right-hand side of the last equation satisfies

$$\begin{aligned}&\|R^{t,s}[R^{s,t'}\psi_{t'}(y)v(t') + \psi_s(R^{s,t'}y)v(s)]\|\\ &\qquad < k[\exp(-\beta(t-t')) + \exp(-\beta(t-s))]\|y\|.\end{aligned}$$

While the left-hand side is a linear form multiplied by the function $v(t)$, which does not decay exponentially to zero, we must have

$$\psi_{t'}(y) - \psi_s(R^{s,t'}y) = 0, \quad \forall t' \in \mathbb{R}, \forall s \geq t'.$$

□

Now, we will prove the first linear result (4). We consider the particular case where $\zeta(t)$ is a normalizing matrix. By Definition 2, the system (2) has a solution $v(t) := (v_j(t))_{j=1}^n$ such that

$$\inf_{t \geq t'} \|v(t)\| > 0, \quad \text{and} \quad \sup_{t \geq t'} \|v(t)\| < +\infty.$$

We denote in the following

$$\alpha_- = \inf_{t \geq t'} \|v(t)\| > 0, \quad \text{and} \quad \alpha_+ = \sup_{t \geq t'} \|v(t)\| < +\infty.$$

We define in the proof below the linear form $\psi_{t'}$ as

$$\tilde{\psi}_{t'}(y) = \frac{\psi_{t'}(y)}{\psi_{t'}(v(t'))},$$

where $\psi_{t'}$ is defined by the previous Proposition 9. We note that by uniqueness, $\tilde{\psi}_{t'}$ is defined as

$$\tilde{\psi}_{t'}(y) = \lim_{t \to \infty} \frac{\int_{t'}^{t} \langle a(s), D_n(s)y + e^{s,t'}y \rangle p^{t',s} ds}{\int_{t'}^{t} \langle a(s), D_n(s)v(t') + e^{s,t'}v(t') \rangle p^{t',s} ds}.$$

Proof of Main result (4).
Without loss of generality, we denote $\psi_{t'} := \tilde{\psi}_{t'}$.

- Let's prove $\psi_t(R^{t,t'}v(t')) = 1$:
  By the definition of $\psi_{t'}$, we have $\psi_t(v(t)) = 1$ for all $t \in \mathbb{R}$. Therefore,

$$\psi_t(R^{t,t'}v(t')) = 1 = \psi_{t'}(v(t')), \quad \forall t' \in \mathbb{R} \quad \forall t \geq t'. \tag{14}$$

- Construction of the linear form ψ_t:
  We have

$$R^{t,t'}v(t') = v(t) = -\psi_{t'}(v(t'))R^{t,t'}\mathbb{1} + R^{t,t'}[v(t') + \psi_{t'}(v(t'))\mathbb{1}].$$

According to Proposition 9, we have $\|\psi_{t'}(v(t'))\| < K\|v(t')\| < k\alpha_+$ for every fixed $t' \in \mathbb{R}$. Furthermore,

$$\begin{aligned}|\psi_{t'}(v(t'))|\|R^{t,t'}\mathbb{1}\| &= \|v(t) - R^{t,t'}[v(t') + \psi_{t'}(v(t'))\mathbb{1}]\| \\ &> \|v(t)\| - \|R^{t,t'}[v(t') + \psi_{t'}(v(t'))\mathbb{1}]\| \\ &> \alpha_- - k\exp(-\beta(t-t'))\|v(t')\| \\ &> \alpha_- - \alpha_+ k\exp(-\beta(t-t')), \quad \forall t' \in \mathbb{R}, .\end{aligned}$$

We integrate over a compact of length ω fixed such that $1 << \omega < +\infty$; let $t = t' + \omega$, we have from (2): $\|R^{t,t'}\mathbb{1}\| < \exp((c_b + c_a + D_*)\omega)$

$$|\psi_{t'}(v(t'))| > \frac{\alpha_- - \alpha_+ k\exp(-\beta\omega)}{\exp((2\alpha + c_b + c_a + D)\omega)} > 0, \ \forall t' \in \mathbb{R}, \ \forall t \geq t'. \quad (15)$$

Let's define

$$\psi_{t'}(y) = \frac{\psi_{t'}(y)}{\psi_{t'}(v(t'))}.$$

- From (15) and according to Proposition 9, there exists $k_1 > 0$ such that $|\psi_{t'}(y)| < k_1\|y\|$.
- Exponential decay:
  According to equation (14) for all $y \in \mathbb{R}$

$$\begin{aligned}\phi^t(y) &= \psi_{t'}(y)v(t) - R^{t,t'}[y - \psi_{t'}(y)v(t')] \\ &= \psi_{t'}(y)v(t) \\ &+ R^{t,t'}\Big[y - \psi_{t'}(y)\mathbb{1} - (\psi_{t'}(y)v(t') - \frac{\psi_{t'}(y)}{\psi_{t'}(v(t'))}\psi_{t'}(v(t'))\mathbb{1})\Big] \\ &= \psi_{t'}(y)v(t) \\ &+ R^{t,t'}\Big[y - \psi_{t'}(y)\mathbb{1} - (\psi_{t'}(y)v(t') - \psi_{t'}(\psi_{t'}(y)v(t'))\mathbb{1})\Big].\end{aligned}$$

  According to the previously demonstrated linear Main result (3), there exists $k_* > 0$ such that we have

$$\begin{aligned}&\|R^{t,t'}[y - \psi_{t'}(y)\mathbb{1}]\| < k_*\exp(-\beta(t-t'))\|y\|, \quad \forall t \geq t' \\ &\|R^{t,t'}[\psi_{t'}(y)v(t') - \psi_{t'}(\psi_{t'}(y)v(t'))\mathbb{1}] < k_*\exp(-\beta(t-t'))\|y\|,\end{aligned}$$

- Finally, let's show that $\psi_{t'}(y) = \psi_s(R^{s,t'}y)$: We have

$$\begin{aligned}R^{t,s}R^{s,t'}y &= \psi_{t'}(y)v(t) - R^{t,t'}[y - \psi_{t'}(y)v(t')] \\ &= \psi_s(R^{s,t'}y)v(t) \\ &- R^{t,s}[R^{s,t'}y - \psi_s(R^{s,t'}y)v(s)].\end{aligned}$$

Therefore,

$$[\psi_{t'}(y) - \psi_s(R^{s,t'}y)]v(t)$$
$$= R^{t,s}[R^{s,t'}\psi_{t'}(y)v(t') + \psi_s(R^{s,t'}y)v(s)].$$

Since $\min_{t \geq t'} \|v(t)\| = \alpha_- > 0$ and as the right-hand side of the last equation verifies

$$\|R^{t,s}[R^{s,t'}\psi_{t'}(y)v(t') + \psi_s(R^{s,t'}y)v(s)]\|$$
$$< k[\exp(-\beta(t-t')) + \exp(-\beta(t-s))]\|y\|.$$

Thus,

$$\psi_{t'}(y) - \psi_s(R^{s,t'}y) = 0, \quad \forall t' \in \mathbb{R}, \forall s \geq t'.$$

□

We have studied a class of linear systems perturbed by a matrix-valued function ζ. We have shown that the space $\mathbb{R}^n$ decomposes into a direct sum of subspaces $\mathbb{R}\mathbb{1} \oplus \mathcal{W}$ where $\mathcal{W}$ is a manifold of dimension $n-1$ and exponentially stable. In the next section, we will use these results to demonstrate the stability for the Winfree model.

3 Non-linear result: Exponential stability for the Winfree model

The Winfree model is given by the following differential equation

$$\begin{cases} \dot{x}_i = \omega_i - \kappa\sigma(x)R(x_i), \ t \geq 0, \ x = (x_i)_{i=1}^n, \\ \sigma(x) := \frac{1}{n}\sum_{j=1}^n P(x_j), \ \forall x = (x_i)_{i=1}^n \in \mathbb{R}^n, \\ \sup_{x \in \mathbb{R}} P(x)R(x) > 0, \quad P, R \in C^2(\mathbb{R}) \text{ are } 2\pi\text{-periodic}, \end{cases} \tag{16}$$

where $n \geq 1$ is the number of oscillators, $\sigma(x)$ is the mean field interaction, $x_i(t)$ is the phase of the i-th oscillator, and $x(t) = (x_i(t))_{i=1}^n$ is the global state of the system. We assume that the *natural frequencies* are chosen indifferently in some interval about $\omega = 1$,

$$\omega_i \in (1-\gamma, 1+\gamma), \quad \text{where} \quad \gamma \in (0,1). \tag{17}$$

The *coupling strength* κ is taken in the interval $(0,1)$. We first define the notions of invariance and stability. Let $n \in \mathbb{N}^*$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Denote DF its Jacobian and assume

$$\max\{\sup_{z\in\mathbb{R}^n} \|F(z)\|, \sup_{z\in\mathbb{R}^n} \|DF(z)\|\} < \infty.$$

where $\|.\|$ is the usual matrix norm. Let $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$ be the flow of the autonomous system

$$\dot{x} = F(x),\ t \geq 0. \tag{18}$$

Definition 10 (Invariance)**.** Let $C \subset \mathbb{R}^n$ be an open set. We say that C is ϕ^t*-positively invariant* for the system (18), if $\phi^t(C) \subset C$ for all $t \geq 0$.

Definition 11 (Stability)**.** Let $C \subset \mathbb{R}^n$ be an open set. We say that the system(18) is ϕ^t*-positively stable on* C, if C is ϕ^t-positively invariant and

$$\begin{aligned}&\exists\lambda > 0,\ \forall x \in C,\ \exists\delta > 0,\ \forall y \in C :\\ &\qquad \|x-y\| < \delta \implies \|\phi^t(x) - \phi^t(y)\| \leq \lambda\|x-y\|, \quad \forall t \geq 0.\end{aligned}$$

Definition 12 (Exponential Stability)**.** Let $C \subset \mathbb{R}^n$ be an open set. We say that the system (18) is ϕ^t*-positively exponential stable on* C, if it is ϕ^t-positively stable on C and there exist $\beta > 0$ such that for every $x \in C$ there exists a subset $C_x \subset C$ such that

$$\begin{gathered}\forall y \in C_x,\ \exists k_{x,y} > 0 :\\ \|\phi^t(x) - \phi^t(y)\| < k_{x,y}\exp(-\beta(t-t_0)), \quad \forall t \geq 0.\end{gathered}$$

Let Φ^t be the flow of the Winfree model (16). The existence of a synchronization state in the Winfree model is proved in [8] for every number n of oscillators and every choice of natural frequencies. Using the positive invariant cone, the stability, as defined in the Definition 11, is proved in [8] independently of the number of oscillators and the distribution of the natural frequencies. We recall the main synchronization hypothesis used in [8] ,

$$\int_0^{2\pi} \frac{P(s)R'(s)}{1-\kappa P(s)R(s)}\, ds > 0, \quad \forall\kappa \in (0,\kappa_*), \tag{H}$$

where κ_* is the *locking bifurcation critical* parameter defined by

$$\kappa_* := \Big(\sup_{x\in\mathbb{R}} P(x)R(x)\Big)^{-1}. \tag{19}$$

We proved in [7] there exists $D_* > 0$ such that for every $D \in (0, D_*)$ there exists an open set

$$U_D \subset U := \Big\{(\gamma, \kappa) \in (0,1) \times (0, \kappa_*) : 1 - \gamma - \frac{\kappa}{\kappa_*} > 0\Big\}$$

containing in its closure $\{0\} \times [0, \kappa_*]$, such that for every $n \geq 1$ and every parameter $(\gamma, \kappa) \in U_D$ there exists a C^2 2π-periodic function $\Delta_{\gamma,\kappa} : \mathbb{R} \to (0, D)$ and a Φ^t-positively invariant open set $C^n_{\gamma,\kappa,D}$ independent of choice of the natural frequencies $(\omega_i)_{i=1}^n$,

$$C^n_{\gamma,\kappa,D} := \Big\{x = (x_i)_{i=1}^n \in \mathbb{R}^n \,:\, \max_{i,j} |x_j - x_i| < \Delta_{\gamma,\kappa}\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big)\Big\}. \qquad (20)$$

We proved in [8] for every $D \in (0, D_*)$, for every parameter $(\gamma, \kappa) \in U_D$, for every $n \geq 1$ and every choice of natural frequencies $(\omega_i)_{i=1}^n$ as in (17), the Winfree model (16) is Φ^t-positively stable on $C^n_{\gamma,\kappa,D}$. We prove in the following Theorem

Theorem 13. *Consider the Winfree model* (16) *and assume that hypothesis* (H) *is satisfied. Then there exists $D > 0$ such that for every parameter $(\gamma, \kappa) \in U_D$, for every $n \geq 1$ and every choice of natural frequencies $(\omega_i)_{i=1}^n$ as in* (17)*, the Winfree model* (16) *is Φ^t-positively exponential stable on $C^n_{\gamma,\kappa,D}$.*

We consider the Winfree model (16) and its associated flow Φ^t. We recall that the Winfree model satisfies the hypothesis (H). The *linearized Winfree model* is given by

$$\begin{cases} \dfrac{dy}{dt} = D\mathcal{W}(\Phi^t(x))y, \quad t \geq 0, \quad y = (y_i)_{i=1}^n, \\ \mathcal{W}_i(x) := \omega_i - \kappa\sigma(x)R(x_i), \quad x = (x_i)_{i=1}^n \in \mathbb{R}^n, \\ \dfrac{\partial \mathcal{W}_i}{\partial x_j} = -\kappa\Big[\sigma(x)R'(x_i)\delta_{i,j} + \dfrac{R(x_i)P'(x_j)}{n}\Big]. \end{cases} \qquad (21)$$

We fix $(\gamma, \kappa) \in U$ and an initial condition $x_* \in C^n_{\gamma,\kappa,D}$ defined in (20). We denote by $R^{t,0}_{x_*}$ the fundamental matrix of (21). Let $x(t) = \Phi^t(x_*)$ be the solution of (16) starting at x_*, and

$$\mu(t) := \frac{1}{n}\sum_{i=1}^n x_i(t), \quad \forall t \geq 0.$$

The main idea of the proof is to rewrite the linearized Winfree model by making a change of time $t \leftrightarrow s$ and a linear change of the tangent vectors $y \leftrightarrow z$. In Lemme 3.1 of [7] we proved that the velocity of μ is strictly positive,

$$\frac{d\mu}{dt} \geq 1 - \kappa/\kappa_* - \gamma - \kappa MD > 0, \tag{22}$$

and for every $x \in\in C^n_{\gamma,\kappa,D}$ we have

$$\inf_{t\geq 0} \| \frac{d}{dt}\Phi^t(x)\| > 0. \tag{23}$$

Let be $s_* := \mu(0)$. The map

$$t \in [0, +\infty) \mapsto \mu(t) \in [s_*, +\infty)$$

is a smooth diffeomorphism admitting as inverse map

$$s \in [s_*, +\infty) \mapsto \tau(s) \in [0, +\infty).$$

Define for $t = \tau(s) \Leftrightarrow s = \mu(t)$,

$$\begin{aligned} v(s) &:= \frac{d\mu}{dt}(t), \\ h_{x,i}(t) &:= \frac{\kappa\sigma(x(t))R'(x_i(t))}{v(s)} - \frac{\kappa P(s)R'(s)}{1-\kappa P(s)R(s)}, \\ h_{x,i,j}(s) &:= -\frac{\kappa R(x_i(t))P'(x_j(t))}{v(s)} + \frac{\kappa P'(s)R(s)}{1-\kappa P(s)R(s)}. \end{aligned}$$

Using the change of variable $z_i(s) = y_i \circ \tau(s)$, equation (21) becomes,

$$\begin{aligned} \frac{dz_i}{ds}(s) &= \frac{\kappa P(s)R'(s)}{1-\kappa P(s)R(s)} z_i + \tilde{h}_{x,i}(t) z_i \\ &\quad - \frac{1}{n}\sum_{j=1}^{n} \big(\frac{\kappa P'(s)R(s)}{1-\kappa P(s)R(s)} - h_{x,i,j}(s) \big) z_j, \ \forall s \geq s_*, \ \forall i := 1, \ldots, n, \end{aligned}$$

In other words,

$$\begin{gathered} \frac{d}{ds} z(s) = [b(s)I_N + \mathcal{A}(s) + \zeta_x(s)] z(s), \\ z(t) := (z_i(t)_{i=1}^n, \quad z(0) = z, \quad t \geq 0, \end{gathered} \tag{24}$$

where $\mathcal{A}(s) = \{a_{i,j}(s) = a_j(s)\}_{1\leq i,j\leq N}$ is a rank-1 matrix, and $b(s)$ is a function defined as follows:

$$a_j(s) = -\frac{1}{n}\frac{\kappa P'(s)R(s)}{1-\kappa P(s)R(s)},\ j = 1,\dots,n, \quad \text{and} \quad b(s) = \frac{\kappa P(s)R'(s)}{1-\kappa P(s)R(s)},$$

satisfying

$$\int_0^1 b(s) + \sum_{j=1}^{n} a_j(s)ds = \ln\Big(\frac{1-\kappa P(1)R(1)}{1-\kappa P(0)R(0)}\Big) = 0,$$

and by the hypothesis (H) we have

$$\int_0^1 b(s)ds < 0,$$

meaning the stability assumptions $[H1 - H4]$ are satisfied. The system (21) can be written in the form of the linear systems (2). The fact, $C^n_{\gamma,\kappa,D}$ is Φ^t-positively invariant, then

$$\max_{1\leq i\leq n}\sup_{t\geq 0}|x_i(t)-\mu(t)| < \sup_{t\geq 0}\Delta_{\gamma,\kappa}(\mu(t)) < D,$$

The there exists $c_* > 0$ such that

$$\forall x \in C^n_{\gamma,\kappa,D}: \quad \|\zeta_x\| < c_* D. \tag{25}$$

In order to simplify the notation, denote

$$\forall x \in\in C^n_{\gamma,\kappa,D}: \quad w_x(t) := \frac{d}{dt}\Phi^t(x), \quad \forall t \geq 0.$$

Remark that for every $x \in\in C^n_{\gamma,\kappa,D}$ the function $t \mapsto w_x(t)$ is a solution of the system (21). Respectively to a change of variables mentioned above, the vector-valued function $v_x(s) = w_x \circ \tau(s)$ is solution of the system (24). Since the filed of the Winfree model is uniformly bounded and by the Equation (23) we deduce that for every $x \in\in C^n_{\gamma,\kappa,D}$ the solution $s \mapsto v_x(s)$ is a normalizing solution of (24) associated to the matrix ζ_x as defined in the Definition 2. In other words, the matrix-valued function ζ_x is a normalizing matrix.

Proof of Theorem (13). According to the Equation (25) an the Main result (4), we deduce for every $\beta \in (0,\alpha)$ there exist $k > 0$ and and $D > 0$ such that for every $n \geq 1$ and every parameter $(\gamma, \kappa) \in U_D$ and every $x \in C^n_{\gamma,\kappa,D}$ there exists a linear form $\psi_{x,s} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall y \in \mathbb{R}^n : \quad \|R^{s,s_*}_x[y - \psi_{x,s_*}(y)v_x(s_*)]\| < k\|y\| \exp(-\beta(s - s_*)), \quad \forall s \geq s_*, \tag{26}$$

where $(s, s') \in \mathbb{R} \times \mathbb{R} \mapsto R^{s,s'}_x$ is the fundamental matrix of the system (24). Furthermore, the solution $z(s)$ with initial condition $z(s_*) = z \in \mathbb{R}^n$ can be written as follows:

$$\begin{aligned} z(s) = R^{s,s_*}_x z = \psi_{x,s_*}(z)v_x(s) \\ + R^{s,s_*}[z - \psi_{x,s_*}(z)v_x(s_*)], \quad \forall s \geq s_*. \end{aligned}$$

Without loss of generality, denote $\psi_{x,\mu(0)} := \psi_x$ and $R^{\mu(t)}_x := R^{\mu(t),\mu(0)}_x$. Respectively to the considered change of variables, we get

$$\begin{aligned} y(t) = d\Phi^t(x)y = R^{\mu(t)}_x y = \psi_x(y)w_x(t) \\ + R^{\mu(t)}_x[y - \psi_x(y)w_x(0)], \quad y(0) = y, \quad \forall t \geq 0. \end{aligned} \tag{27}$$

where we recall that $w_x(t) = \frac{d}{dt}\Phi^t(x)$. Without loss of generality, let $x \in C^n_{\gamma,\kappa,D}$ fixed. The idea is to find trajectories of the form $s \in [0,1) \mapsto z(s)$ connecting $z(0) := x$ and $z(1) := y \in C^n_{\gamma,\kappa,D}$ such that for all $s \in [0,1]$, the quantity $\frac{dz(s)}{ds}$ lies in the kernel of the linear form $\psi_{z(s)}$. This allows us to use equation (27) and cancel the term that does not exponentially decay to zero as follows:

$$\begin{aligned} \Phi^t(y) - \Phi^t(x) &= \int_0^1 d\Phi^t(z(s))\frac{dz(s)}{ds}ds \\ &= \int_0^1 \psi_{z(s)}\Big(\frac{dz(s)}{ds}\Big)\frac{d}{dt}\Phi^t(z(s))ds \\ &+ \int_0^1 R^{\mu(t)}_{z(s)}\Big[\frac{dz(s)}{ds} - \psi_{z(s)}\Big(\frac{dz(s)}{ds}\Big)\frac{d}{dt}\Phi^0(z(s))\Big]ds \end{aligned}$$

By the Main result (4), we have

$$\psi_{z(\xi,s)}\Big(\frac{d}{dt}\Phi^0(z(\xi,s)))\Big) = 1,$$

For that the trajectory $\frac{dz(s)}{ds}$ lies in the kernel of the linear form $\psi_{z(s)}$, it is sufficient that for a given $\xi \in \mathbb{R}^n$ the trajectory $z(s) := z(\xi, s)$ satisfies the following differential equation:

$$\frac{d}{ds}z(\xi,s) = \xi - \psi_{z(\xi,s)}(\xi)\frac{d}{dt}\Phi^0(z(\xi,s))), \ z(\xi,0) = x, \ s \in [0,1]. \tag{28}$$

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By the Main result (4), we have

$$\forall y \in C^n_{\gamma,\kappa,D},\ \forall \xi \in \mathbb{R}^n: \quad |\psi_y(\xi)| < k\|\xi\|, \tag{29}$$

and by definition of the field of the Winfree model, we have

$$\tilde{k} := \sup_{y\in\mathbb{R}^n} |\frac{d}{dt}\Phi^0(y)| < +\infty,$$

then

$$\|\frac{d}{ds}z(\xi,s)\| < (1+k\tilde{k})\|\xi\|, \quad \forall s \in (0,1),\ \forall \xi \mathbb{R}^n.$$

Since $z(\xi,0) = x \in C^n_{\gamma,\kappa,D}$. Then there exists $\epsilon_x > 0$

$$\forall \xi \in \mathbb{R}^n, \quad \|\xi\| < \epsilon_x : \quad z(\xi,1) \in C^n_{\gamma,\kappa,D}.$$

Define

$$B_{\epsilon_x} := \{\xi \in \mathbb{R}^n : \quad \|\xi\| < \epsilon_x\}.$$

By definition of the field of the Winfree model, the term $\frac{d}{dt}\Phi^0(z(\xi,s)))$ in the Equation (28), implies that there exists an open non-empty subset $\tilde{B}_{\epsilon_x} \subset B_{\epsilon_x}$ such that

$$\forall \xi \in \tilde{B}_{\epsilon_x} : \quad z(\xi,1) \neq x.$$

Define the subset $C_x \subset C^n_{\gamma,\kappa,D}$ as

$$C_x := \Big\{z(\xi,1) : \quad \xi \in \tilde{B}_{\epsilon_x}\Big\}.$$

The equations (29) and (27) implies that the Winfree model (16) is Φ^t-positively strong stable on $C^n_{\gamma,\kappa,D}$. For all $y \in C_x$:

$$\|\Phi^t(y) - \Phi^t(x)\| < kk_{x,y}\exp(-\beta\mu(t)), \quad \forall t \geq 0,$$

where $k_{x,y} := \max_{s\in[0,1]} \sup_{\xi\in B_{\epsilon_x}} \|\frac{dz(s,\xi)}{ds}\|$. We recall that by Equation (22) we have

$$\tilde{\beta} := \inf_{t\geq 0} \frac{d\mu}{dt} > 0,$$

which implies $\mu(t) > \tilde{\beta}t$ for all $t \geq 0$. □

In conclusion, this section has highlighted the relationship between the synchronization hypothesis and the stability hypothesis. The Winfree Model satisfying the synchronization hypothesis can be linearized around synchronized orbits and satisfy the stability hypothesis, which allows us to deduce the exponential stability.

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