

# Where the truth lies: a paraconsistent approach to Bayesian epistemology

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## Abstract

Bayesian epistemology has close connections to inductive reasoning, accepting the view that inductive inferences should be analyzed in terms of epistemic probabilities. An important precept of Bayesian epistemology is the dynamics of belief change, with change in belief resulting from updating procedures based on new evidence.

The inductive relations between evidence  $E$  and hypotheses or theories  $H$  are essential, particularly the notions of plausibility, confirmation, and acceptability, which are critical but subject to several difficulties. As a non-deductive process, Bayesian reasoning cannot itself be subjected to strict deductive logic, but it can take advantage of an enabling logical environment. The present paper proposes that paraconsistent and paracomplete logics can be helpful for some questions of Bayesian epistemology, even to the point of being relevant in developing a legitimate *paraconsistent Bayesian epistemology*. By developing a novel probability theory based on the Logic of Evidence and Truth ( $LET_F$ ), a logic that deals with evidence for or against a judgment, including contradictory or missing evidence, we allow for the possibility of quantifying the degree of evidence attributed to a proposition through novel probability measures.

We illustrate, through examples, some ways to address challenging problems in the area by using paraconsistent and paracomplete paradigms. This topic is of significant interest not only in the philosophy of science but also in Artificial Intelligence and other emerging trends such as probabilistic networks and related models (see [8])<sup>1</sup>.

Keywords: Probability theories; Bayesian epistemology; Inductive reasoning; Conditionalization; Evidence; Paraconsistency; Paracompleteness.

## 1 Introduction and motivations

Paraconsistency is a philosophical position regarding logical consequence: its main thesis says that not every contradiction entails arbitrary consequences, thereby rejecting the law of *Ex Contradictione Quodlibet* (ECQ) or *Law of Explosion* (LoE). There are several concepts of paraconsistency, and these basically can be reduced to weak and strong paraconsistent logics where the validity of adjunctive and non adjunctive rules is essential. In contrast with the usual adjunctive rule of standard logic, a non-adjunctive system is a system that does not validate adjunction (i.e.,  $A, B \not\vdash A \wedge B$ ).

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Paraconsistent logics are useful to prevent and control human fallibility. Human or artificial agents make mistakes, maintain false beliefs or information, and review theories incoherently. Paraconsistent logic prevents absurd conclusions from being derived from an irrelevant error.

One of the ways to define weak paraconsistent logics is to endorse a non-adjunctive rule. When  $B$  is  $\neg A$ , non-adjunctivism means that an agent can have contradictory beliefs, believing  $A$  and  $\neg A$ , but not believing a contradiction such as  $A \wedge \neg A$ ; in other words, a non-adjunctivist agent can maintain contradictory beliefs, while avoiding belief in a contradiction.

Adjunctivists, at the other extreme (and there are degrees of adjunctivism, see, e.g., [19] and [12]) hold that if someone believes that  $A$  is the case and also believes that  $B$ , then he or she is committed to the conjunction of  $A$  and  $B$ . Adjunctivism is not free of criticism, and is sometimes blamed for probabilistic paradoxes such as the Lottery Paradox (see [12]). In fact, belief measured by standard probabilistic is non-adjunctive: if  $A$  and  $B$  both have high probability, does not mean that  $A \wedge B$  also has high probability. The axioms of probability say exactly the contrary: the probability of  $A \wedge B$  is smaller than the minimum of the probabilities of  $A$  and  $B$ .

In spite of this, as defended in [13], there is virtue in adopting an adjunctive logic because in this way one can assign a non-zero probability to contradictions such as  $A \wedge \neg A$ , thus allowing us to *measure* an agent's belief in a contradiction. This is especially relevant if the agent is a body of, say, 1,000 randomly chosen people in an election poll. For example: if 60% of respondents prefer candidate  $A$  to  $B$ , while 70% prefer  $B$  to  $A$ , there is a 30% percentage of contradiction. Measuring the gradation of contradiction is something quite important; in this case, a 30% contradiction is far more worrying than one of 5%.

The Logic of Evidence and Truth  $LET_F$ , an extension of the Belnap-Dunn's logic of First-Degree Entailment ( $FDE$ ), is extensively treated in [18].  $LET_F$  was developed to offer an intuitive reading of reasoning in terms of preservation of evidence and truth, in a language enriched with the connectives  $\circ$  for consistency and  $\bullet$  for inconsistency, interpreted respectively as *classicality* and *non-classicality* of a proposition. A probabilistic semantics developed on top of  $LET_F$  permits us to quantify and measure the degree of evidence attributed to a proposition. In this way, a probability measure  $P$  on  $LET_F$  quantifies the amount  $P(\alpha)$  of evidence attributed to proposition  $\alpha$ . In addition the connective  $\circ$  of classicality (or coherence, or consistency) that is part of the language of  $LET_F$  permits us to qualify the degree of confidence on the evidence for a proposition  $\alpha$ .

To make reading easier, we will indulgently use Greek letters for logic, and uppercase Latin letters where probability is involved,

The title of this article is obviously a *double entendre*, but sometimes the truth really does lie a bit, and sometimes the truth lies elsewhere, where we are not looking for it. Probability based on paraconsistency and paracompleteness can be, we believe, good keys for reasoning about this.

## 2 Extending Bayesian epistemology with the logic of evidence and truth $LET_F$

One of the main principles of Bayesian epistemology is that rational agents hold beliefs of different strength, which must satisfy the axioms of probability. Therefore, belief and change in the belief due to new evidence can be represented by probabilistic attributions.

The principle that governs how the credence of an agent in a given hypothesis should change upon receiving new evidence for or against that hypothesis is the *Principle of Conditionalization* (PrincCond), based on the notion of conditional probability.

The probability assigned to the hypothesis before receiving new evidence is called *prior probability*; the probability afterwards is called *posterior probability*. This change of opinion for a hypothesis  $H$  in the face of new evidence  $E$  is expressed by (PrincCond) in the following way:

$$P_{\text{posterior}}(H) = P_{\text{prior}}(H|E) = P_{\text{prior}}(H \wedge E)/P_{\text{prior}}(E)$$

Conditionalization is not an axiom, nor a definition, but a *modus operandi* that incorporates the dynamics of the revision of an agent's credence: the posterior belief in a hypothesis  $H$  at a time  $t' > t$  after learning  $E$  should equal her prior credence in  $H$  at a time  $t$ , if she had supposed  $E$ , effected through the notion of conditional probability.

Other procedures that can be regarded as similar *modus operandi* are the optimization process of minimizing cross-entropy, as the measure of the dissimilarity between two probability distributions, and the Kullback-Leibler (KL) divergence as way of comparing two probability distributions.

What we propose, however, is a more radical move; it's a way to treat directly not only mere dissimilarities, but deeper contradictions.

In less informal terms, (PrincCond) is expressed as

$$P_B(A) = P(A|B) =_{def} P(A \wedge B)/P(B)$$

under the proviso that  $P(B) > 0$ .

Although the main concept of Bayesianism is already contained in the famous posthumous work of Thomas Bayes, in mathematical terms Bayes' Theorem, at least in the standard case, is in fact a trivial consequence of the definition of conditional probability, respecting the laws of probability:

**Bayes' Theorem:**  $P_E(H) = (P(H)/P(E)) \cdot P_H(E)$

What is stunning is that a mathematical triviality could have such profound philosophical importance. The present paper intends to extend the orthodox view on Bayesianism by combining the traditional view with the paraconsistent paradigm. There are already several proposals suggesting the combining of probabilistic reasoning with the paraconsistent paradigm. G. Priest in [17] outlines the interest and feasibility of a probability theory from the point of view of paraconsistency, and E. Mares in [13] focuses on paraconsistent Bayesianism. A paraconsistent theory of probability based on the Logics of Formal Inconsistency (LFIs) is investigated in [2], while [11] describes a metric for uncertain probabilities, and [6] defines a concept of probability over Łukasiewicz logic with rational truth-constants and explores the possibility of non-trivial reasoning through this schema.

This paper contributes to this trend, but using a richer underlying logic: Priest does not formally lean on any basic logic, while Mares suggests a more elaborate rule of conditionalization, relying on a version (called D4) of the four-valued First-Degree Entailment. Our proposal subsumes those attempts and offers new alternatives to the conceptual problems of Bayesian epistemology.

The backbone of our proposal is the logic  $LET_F$ , an adjunctive paraconsistent and paracomplete logic that is a member of the family of the Logics of Formal Inconsistency and Undeterminedness. The language  $L$  of  $LET_F$  is defined in the usual way, closed under the connectives  $\circ, \bullet, \neg, \wedge$ , and  $\vee$ .

## 2.1 A brief account of $LET_F$ and its virtues

The syntactic machinery of  $LET_F$  is defined over the language  $L$  by the following natural deduction rules:

**Definition 1.** *The Logic of Evidence and Truth  $LET_F$*

$$\begin{array}{c} \frac{\alpha \quad \beta}{\alpha \wedge \beta} \wedge I \quad \frac{\alpha \wedge \beta}{\alpha} \wedge E \quad \frac{\alpha \wedge \beta}{\beta} \\[10pt] \frac{\alpha}{\alpha \vee \beta} \vee I \quad \frac{\beta}{\alpha \vee \beta} \quad \frac{\begin{array}{c} [\alpha] \\ \vdots \\ \gamma \end{array} \quad \begin{array}{c} [\beta] \\ \vdots \\ \gamma \end{array}}{\alpha \vee \beta \quad \gamma} \vee E \\[10pt] \frac{\neg \alpha}{\neg(\alpha \wedge \beta)} \neg \wedge I \quad \frac{\neg \beta}{\neg(\alpha \wedge \beta)} \quad \frac{\begin{array}{c} [\neg \alpha] \\ \vdots \\ \gamma \end{array} \quad \begin{array}{c} [\neg \beta] \\ \vdots \\ \gamma \end{array}}{\neg(\alpha \wedge \beta) \quad \gamma} \neg \wedge E \end{array}$$

$$\begin{array}{c}
\frac{\neg\alpha \quad \neg\beta}{\neg(\alpha \vee \beta)} \neg \vee I \quad \frac{\neg(\alpha \vee \beta)}{\neg\alpha} \neg \vee E \quad \frac{\neg(\alpha \vee \beta)}{\neg\beta} \\
\\
\frac{\alpha}{\neg\neg\alpha} DN \quad \frac{\neg\neg\alpha}{\alpha} \\
\\
\frac{\circ\alpha \quad \bullet\alpha}{\beta} Cons \quad \frac{}{\circ\alpha \vee \bullet\alpha} Comp \\
\\
\frac{\circ\alpha \quad \alpha \quad \neg\alpha}{\beta} EXP^\circ \quad \frac{\circ\alpha}{\alpha \vee \neg\alpha} PEM^\circ
\end{array}$$

The definition of a deduction of  $\alpha$  from a set of premises  $\Gamma$  in  $\Gamma \vdash_{LET_F} \alpha$ , is the usual one for natural deduction systems.

A valuation semantics for  $LET_F$  is obtained by the following clauses (the first five define a valuation semantics of  $FDE$  and are equivalent to its traditional four-valued semantics, see [18] for details):

**Definition 2.** A function  $v : L_1 \rightarrow \{0, 1\}$  is a  $LET_F$ -valuation if it satisfies the following clauses:

- v1.  $v(\alpha \wedge \beta) = 1$  iff  $v(\alpha) = 1$  and  $v(\beta) = 1$ ,
- v2.  $v(\alpha \vee \beta) = 1$  iff  $v(\alpha) = 1$  or  $v(\beta) = 1$ ,
- v3.  $v(\neg(\alpha \wedge \beta)) = 1$  iff  $v(\neg\alpha) = 1$  or  $v(\neg\beta) = 1$ ,
- v4.  $v(\neg(\alpha \vee \beta)) = 1$  iff  $v(\neg\alpha) = 1$  and  $v(\neg\beta) = 1$ ,
- v5.  $v(\alpha) = 1$  iff  $v(\neg\neg\alpha) = 1$ ,
- v6.  $v(\bullet\alpha) = 1$  iff  $v(\circ\alpha) = 0$ ,
- v7. If  $v(\circ\alpha) = 1$ , then  $v(\alpha) = 1$  if and only if  $v(\neg\alpha) = 0$ .

The valuation semantics above is sound, complete, and provides a decision procedure for  $LET_F$  (cf. [18]). From now on, when there is no risk of ambiguity,  $\vdash$  and  $\models$  will be used in place of  $\vdash_{LET_F}$  and  $\models_{LET_F}$ .

The choice of  $LET_F$  brings several advantages: the first is complete epistemic freedom – since  $LET_F$  is a paraconsistent and paracomplete logic, agents under this logic can believe in contradictions and at the same time are not obliged to believe in all classical tautologies. Second,  $LET_F$  is able to express the distinction between an inconsistent context where there is some positive evidence for  $\alpha$  (i.e.,  $\alpha$  holds,  $\neg\alpha$  does not hold, but  $\alpha$  is not true) from a consistent context where only  $\alpha$  holds because it is true. Third, the connective  $\circ\alpha$  acts as a classicality operator. When  $\circ\alpha$  holds, excluded middle and explosion are valid, that is:  $\alpha, \neg\alpha, \circ\alpha \vdash \beta$  although  $\alpha, \neg\alpha \not\vdash \beta$ , and  $\circ\alpha \vdash \alpha \vee \neg\alpha$ , while  $\not\vdash \vee\neg\alpha$ . The connective  $\bullet$  is a non-classicality operator, dual to the classicality operator  $\circ$ . These features are expressed by the above rules  $EXP^\circ$  and  $PEM^\circ$ .

The following rules related to  $\bullet$  can be proved in  $LET_F$ :

$$\frac{\alpha \quad \neg\alpha}{\bullet\alpha} \bullet R1 \quad \frac{}{\alpha \vee \neg\alpha \vee \bullet\alpha} \bullet R2$$

$\bullet R1$  means that conflicting evidence about  $\alpha$  implies the non-classicality of  $\alpha$ , but  $\bullet\alpha$  does not imply that  $\alpha$  and  $\neg\alpha$  hold.  $\bullet R2$  expresses the trichotomic character of lack of evidence: either there is evidence for  $\alpha$ , or for  $\neg\alpha$ , or  $\bullet\alpha$  holds.

In [18] a more complete account of  $LET_F$  is developed; here only some basic properties are listed.

**Fact 3.**  $\bullet$  An implication  $\alpha \rightarrow \beta$  can be defined in  $LET_F$  as  $\neg\alpha \vee \beta$ , but Modus Ponens and the Deduction Theorem do not necessarily hold for  $\rightarrow$ .

Nevertheless, both properties are recovered for the defined implication  $\rightarrow$  for classical propositions.

- *Recovering Modus Ponens*  
In  $LET_F$ , for classical propositions, Modus Ponens holds:  $\circ\alpha, \alpha, \neg\alpha \vee \beta \models \beta$ .
- *Recovering the Deduction Theorem*  
In  $LET_F$ , the following form of the Deduction Theorem holds:  $\circ\alpha, \alpha \models \beta$  implies  $\circ\alpha \models \neg\alpha \vee \beta$ .
- *In virtue of the rules  $EXP^\circ$  and Cons of  $LET_F$  (see Def. 1), propositions  $\circ\alpha \wedge \alpha \wedge \neg\alpha$  and  $\circ\alpha \wedge \bullet\alpha$  act as a bottom particles*

$LET_F$  is an extension of the four-valued logic  $FDE$  introduced and extensively investigated by N. Belnap and M. Dunn, and in this sense is also an extension of the logic  $D4$  of E. Mares in [13].

Since  $LET_F$  distinguishes conclusive from non-conclusive evidence, it is able to express the following six scenarios:

- When  $\bullet\alpha$  holds, four scenarios of non-conclusive evidence can be expressed:
  1. Only evidence that  $\alpha$  is true:  $\alpha$  holds,  $\neg\alpha$  does not hold
  2. Only evidence that  $\alpha$  is false:  $\neg\alpha$  holds,  $\alpha$  does not hold
  3. No evidence at all: both  $\alpha$  and  $\neg\alpha$  do not hold
  4. Conflicting evidence: both  $\alpha$  and  $\neg\alpha$  hold
- When  $\circ\alpha$  holds, two scenarios are added, classical truth and falsity:
  5.  $\alpha$  holds and  $\neg\alpha$  does not hold, which means that  $\alpha$  is true
  6.  $\neg\alpha$  holds and  $\alpha$  does not hold, which means that  $\alpha$  is false

It should be noted that  $\circ\alpha$  does not imply  $\alpha$ ; rather, it implies that exactly one alternative between  $\alpha$  and  $\neg\alpha$  holds. Also,  $\bullet\alpha$  does not imply that  $\alpha$  and  $\neg\alpha$  hold simultaneously; indeed, according to  $\bullet R2$ , if  $\alpha$  and  $\neg\alpha$  do not hold then  $\bullet\alpha$  holds. Moreover,  $\circ\alpha$  is a sufficient condition for the classicality of  $\alpha$ , since  $\circ\alpha \vdash \alpha \vee \neg\alpha$  and  $\circ\alpha \vdash \neg(\alpha \wedge \neg\alpha)$ , although  $\bullet\alpha$  is *not* a sufficient condition for the non-classicality of  $\alpha$  since  $\bullet\alpha \not\vdash \alpha \wedge \neg\alpha$ . Nonetheless,  $\bullet\alpha$  is a necessary condition for the non-classicality of  $\alpha$ , because  $\alpha \wedge \neg\alpha \vdash \bullet\alpha$  (proofs in [18]).

A point to be clarified is that, although  $LET_F$  is an adjunctive paraconsistent logic, it does not endorse any dialetheist thesis, as there is no underlying claim that some contradictions are really true. Contradictions in  $LET_F$  are understood epistemically, in the sense that our language, or our communication, or our thoughts, or our beliefs, may be contradictory, independently of whether or not there are ‘real’ contradictions in the universe.

A new theory of probability is formally definable over  $LET_F$ , which leads naturally to a variation of Bayesianism not yet developed in any previous work. We argue here that this model is of general interest and that it is relevant to the opening up of new routes of research.

### 3 Probabilistic semantics for $LET_F$

The main idea behind the probabilistic semantics for  $LET_F$  is that the value of a probabilistic statement  $P(\alpha)$  measures the amount of evidence available for  $\alpha$ .

A paraconsistent scenario holds when there is conflicting evidence for  $\alpha$ , and so  $P(\alpha) + P(\neg\alpha) > 1$ . A paracomplete scenario occurs when there is little or no evidence at all for or against  $\alpha$ , and so  $P(\alpha) + P(\neg\alpha) < 1$ .

A classical scenario is re-established when  $P(\circ\alpha) = 1$ , or equivalently  $P(\bullet\alpha) = 0$ . In this case,  $P(\alpha) + P(\neg\alpha) = 1$  and also  $P(\alpha \wedge \neg\alpha) = 0$  and  $\alpha$  is subject to classical probability laws.

### 3.1 $LET_F$ probability distributions

**Definition 4.** Given a logic  $\mathcal{L}$ , with a derivability relation  $\vdash$  and a language  $L$ , a probability distribution for  $\mathcal{L}$  is a real-valued function  $P : L \mapsto \mathbb{R}$  satisfying the following conditions:

1. *Non-negativity:*  $0 \leq P(\alpha) \leq 1$  for all  $\alpha \in L$
2. *Tautologicity:* If  $\vdash \alpha$ , then  $P(\alpha) = 1$ ;
3. *Anti-Tautologicity:* If  $\alpha \vdash$ , then  $P(\alpha) = 0$
4. *Comparison:* If  $\alpha \vdash \beta$ , then  $P(\alpha) \leq P(\beta)$
5. *Finite additivity*  $P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta)$ .

An obvious consequence of Comparison is:

**Fact 5.** If  $\alpha \vdash \beta$  and If  $\beta \vdash \alpha$ , then  $P(\alpha) = P(\beta)$ .

**Definition 6.**  $LET_F$ -probability distribution

Let  $\Sigma = \{\alpha_1, \dots, \alpha_n, \dots\}$  be a (finite or infinite) collection of propositions in the language  $L_2$  of  $LET_F$ . A  $LET_F$ -probability distribution over  $\Sigma$  is an assignment of probability values  $P$  to the elements of  $\Sigma$  that can be extended to a full probability function  $P : L_2 \mapsto \mathbb{R}$  in the sense of Definition 4.

In terms of evidence, a conditional probability statement  $P(\alpha/\beta)$  is to be read as a measure of how much the evidence available for  $\beta$  affects the evidence for  $\alpha$ .

Some specific theorems on  $LET_F$ -probability are the following, with the caveat that  $P(\beta) \neq 0$  in all cases where  $P(\alpha/\beta)$  is mentioned (proofs in [18]):

**Theorem 7.**

1.  $P(\alpha/\beta) + P(\neg\alpha/\beta) - P(\bullet\alpha/\beta) = 1$
2.  $P(\alpha/\beta) + P(\neg\alpha/\beta) = 1$ , if  $P(\circ\alpha) = 1$
3.  $P(\beta/\circ\beta) + P(\neg\beta/\circ\beta) = 1$
4.  $P(\alpha \wedge \neg\alpha) \leq P(\bullet\alpha)$
5.  $P(\circ\alpha) \leq P(\alpha \vee \neg\alpha)$
6.  $P(\circ\alpha) = 1 - P(\bullet\alpha)$
7.  $P(\circ\alpha \vee (\alpha \wedge \neg\alpha)) \leq P(\alpha \vee \neg\alpha)$
8. If  $P(\circ\alpha) = 1$  or  $P(\bullet\alpha) = 0$  then  $P(\neg\alpha) = 1 - P(\alpha)$
9. If  $P(\circ\alpha) = 1$  or  $P(\bullet\alpha) = 0$  then  $P(\alpha \vee \neg\alpha) = 1$  and  $P(\alpha \wedge \neg\alpha) = 0$ .

There have been other attempts in the direction of developing a more directly paraconsistent statistics, as in [21], but a much more complicated mathematical machinery is required.

## 4 Paraconsistent Conditionalization and paraconsistent versions of Bayes' Theorem

The Dutch Book Argument (the view that interprets probability theory as a rational system of belief, by which an agent's degrees of belief should conform to the probability calculus on the pain of being vulnerable to a sure betting loss, originated with Frank Ramsey and Bruno de Finetti), the Principal Principle (the view that rational agents should conform their credence to the chances, originated with David Lewis), and Bayesian Conditionalization (originated with Thomas Bayes and discussed above), are the most standard precepts of Bayesian epistemology, a rational attitude that does not replace traditional epistemology, but rather complements it. Alternative Bayesian statistical significance measures, however, were introduced in [14].

The chief objective of this proposal is not to evaluate or criticise the framing of Bayesian epistemology, but to expand the dynamics of Bayesian Conditionalization, along the similar lines of E. Mares' proposal in [13]. Taking into consideration that  $\alpha \wedge \neg\alpha \vdash \bullet\alpha$  and that  $\circ\alpha \vdash \alpha \vee \neg\alpha$ , two new and quite general Paraconsistent Rules of Conditionalization are developed, and their interest and utility are illustrated:

**Definition 8.** *Paraconsistent Rules of Conditionalization (PRC)*

( $\neg$ -PrincCond): For  $P(A) > P(A \wedge \neg A)$ ,

$$P_A^-(B) = \frac{P(B \wedge A) - P(B \wedge A \wedge \neg A) - P(A \wedge B \wedge \neg B) + P(A \wedge \neg A \wedge B \wedge \neg B)}{P(A) - P(A \wedge \neg A)}$$

( $\bullet$ -PrincCond): For  $P(A) > P(\bullet A)$ ,

$$P_A^\bullet(B) = \frac{P(B \wedge A) - P(B \wedge \bullet A) - P(\bullet B \wedge A) + P(\bullet B \wedge \bullet A)}{P(A) - P(\bullet A)}$$

The idea behind these definitions is straightforward. In  $\neg$ -PrincCond, it is to discard the contradictions affecting  $A$  and  $B$ , while respecting a form of the counting principle of Inclusion-Exclusion. In the case of  $\bullet$ -PrincCond, inconsistencies are being discarded, again respecting a form of the counting principle of Inclusion-Exclusion. In the environment of the logic  $LET_F$ , contradictions and inconsistencies are not necessarily the same (see, e.g., [18] for a discussion concerning this difference).

The main contrast between ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) is that the first is concerned with uncertain evidence caused by a contradiction (informational gluts), while the second refers to uncertainty caused by either contradictory or missing evidence (informational gaps or gluts).

In other words, the intuition behind ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) is that in ( $\neg$ -PrincCond) the conditionalization takes into account the measure of contradictory information, while ( $\bullet$ -PrincCond) takes into account the measure of inconsistent information, or in terms of evidence, takes into account the measure of non-classicality of  $A$ .

Differently from the standard case where the familiar Bayes' Theorem follows directly from the definition of conditional probability, in the paraconsistent case two versions of Bayes' Theorem can be proved in virtue of ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond).

**Theorem 9.** *LET<sub>F</sub>-Versions of Bayes' Theorem (BT)*

$\neg BT$  : For  $P(A) > P(A \wedge \neg A)$  and  $P(B) > P(B \wedge \neg B)$ ,

$$P_A^-(B) = P_B^-(A) \cdot \frac{(P(B) - P(B \wedge \neg B))}{(P(A) - P(A \wedge \neg A))}$$

$\bullet BT$  : For  $P(A) > P(\bullet A)$  and  $P(B) > P(\bullet B)$ ,

$$P_A^\bullet(B) = P_B^\bullet(A) \cdot \frac{(P(B) - P(\bullet B))}{(P(A) - P(\bullet A))}$$



*Proof.* For  $\neg$ BT: since  $P(A) > P(A \wedge \neg A)$  and  $P(B) > P(B \wedge \neg B)$ ,

$$P_A^-(B) = P_A^-(B) \cdot \frac{(P(B) - P(B \wedge \neg B))}{(P(B) - P(B \wedge \neg B))} = P_A^-(B) \cdot \frac{(P(B) - P(B \wedge \neg B))}{(P(A) - P(A \wedge \neg A))}$$

For  $\bullet$ BT the argument is similar.  $\square$

The above versions of Bayes' theorem accompany the Paraconsistent Rules of Conditionalization ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond), in the sense that  $\neg$ BT is designed to be used in cases of uncertain evidence caused by a contradiction, while  $\bullet$ BT is thought to be applied in the general case of uncertainty caused by either contradictory or missing evidence.

The notion of probability employed in Bayesian epistemology models personal belief, contrary to the idea of objective chance: it is subjective or epistemic, not necessarily anchored in data. Prior probability, rather than data, is attached to an agent's belief.

Assuming that there is a distinction between asserting the negation of a proposition and rejecting it, it seems reasonable, following the intuitions of [13], to use ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) to update propositions that are accepted and whose negations are rejected.

There are of course other paraconsistent logics that equate contradiction and inconsistency, and in those cases ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) would coincide, but at the cost of admitting additional principles (see [2] and [3]). We prefer to deal with a basic case.

In a rough estimate one could set up  $P(\alpha \wedge \neg \alpha) = P(\bullet \alpha)$ , but the rules give different update values that usually provide useful information.

The following results show the dynamic character of the paraconsistent Bayesian update.

**Theorem 10.** ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) get closer to the standard Rule of Conditionalization as much as the evidence in favor of the classicality of  $A$  and  $B$  increases.

*Proof.* For ( $\neg$ -PrincCond): reasoning for the component  $A$ , if  $P(\circ A) \geq \epsilon$ , then  $P(\bullet A) \leq 1 - \epsilon$ , in view of Theorem 7. Since  $A \wedge \neg A \vdash \bullet A$ , then  $P(A \wedge \neg A) \leq 1 - \epsilon$ . Analogous for the component  $B$ .

A similar reasoning applies for ( $\bullet$ -PrincCond), for the component  $A$ , since  $B \wedge A \wedge \neg A \vdash B \wedge \bullet A \vdash \bullet A$ , then  $P(B \wedge A \wedge \neg A) \leq P(B \wedge \bullet A) \leq 1 - \epsilon$ . Analogous for the component  $B$ .  $\square$

**Theorem 11.** ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) get closer in value as much as the evidence in favor of the classicality of  $A$  increases,

*Proof.* As in the previous result, if  $P(\circ A) \geq \epsilon$ , then  $P(\bullet A) \leq 1 - \epsilon$ . Since  $A \wedge \neg A \vdash \bullet A$ ,  $0 \leq P(\bullet A) - P(A \wedge \neg A) \ll 1 - \epsilon$ . Hence  $(P(A) - P(\bullet A)) - (P(A) - P(A \wedge \neg A)) \ll 1 - \epsilon$ .

Similarly,  $(P(B \wedge A) - P(B \wedge \bullet A) - (P(B \wedge A) - P(B \wedge A \wedge \neg A))) \ll 1 - \epsilon$ .  $\square$

Inspired by the proposal in [13], some immediate results can be easily obtained.

A proposition  $A$  is said to be  $\neg$ -normal (respectively,  $\bullet$ -normal w.r.t. the probability function  $P$  if  $P(A) - P(A \wedge \neg A) > 0$  (respectively,  $P(A) - P(\bullet A) > 0$ ).

Clearly, if  $A$  is  $\bullet$ -normal it is also  $\neg$ -normal, since  $P(A \wedge \neg A) \leq P(\bullet A)$ .

**Theorem 12.** If  $A$  is  $\neg$ -normal and  $P$  is a probability function, then:

1.  $P_{A \wedge \neg A}^-(A) = 0$
2.  $P_A^-(\neg A) = P_{\neg A}^-(A) = 0$
3.  $P_A^-(A) = P_A^\bullet(A) = 1$
4.  $P_{\neg A}^\bullet(A) = P_A^\bullet(\neg A)$



*Proof.*

- 1) Since  $P(A) > P(A \wedge \neg A)$ , an easy calculation gives  $P_{A \wedge \neg A}^{\neg}(A) = 0$ , using Fact 5.
- 2) Suppose  $A$  is  $\neg$ -normal. For the first case, by definition of  $(\neg\text{-PrincCond})$ ,

$$P_A^{\neg}(\neg A) = \frac{P(\neg A \wedge A) - P(\neg A \wedge A \wedge \neg A) - P(A \wedge \neg A \wedge \neg \neg A) + P(A \wedge \neg A \wedge \neg A \wedge \neg \neg A)}{P(A) - P(A \wedge \neg A)}$$

Clearly,  $P(\neg A \wedge A) = P(\neg A \wedge A \wedge \neg A)$  and  $P(A \wedge \neg A \wedge \neg \neg A) = P(A \wedge \neg A \wedge \neg \neg A)$  by Fact 5, and since  $A$  is  $\neg$ -normal,

$$P_A^{\neg}(\neg A) = \frac{P(\neg A \wedge A) - P(\neg A \wedge A) - P(A \wedge \neg A \wedge \neg \neg A) + P(A \wedge \neg A \wedge \neg \neg A)}{P(A) - P(A \wedge \neg A)} = 0$$

For the second case the proof is analogous. For items (3) and (4) the proof follows from a routine calculation.  $\square$

It is worth noting that, while  $P_{A \wedge \neg A}(A)$  is undefined in the standard case (due to the restriction on the ratio formula of conditional probability), this value is defined as  $P_{A \wedge \neg A}^{\neg}(A) = 0$  via  $(\neg\text{-PrincCond})$ , as shown above.

This indicates a substantial difference from the classical case, and also from E. Mares rule of conditionalization (RC\*) in [13], which obtains  $Pr_{A \wedge \neg A}^*(A) = 1$ .

Theorem 12 says that for  $\neg$ -normal propositions, there is no gain in conditionalizing via rule  $(\neg\text{-PrincCond})$  on new evidence after  $\neg A$  (the same for conditionalizing via rule  $(\neg\text{-PrincCond})$  on new evidence after  $A$ ). An analogous result holds for  $(\bullet\text{-PrincCond})$ . In other words, in such cases  $A$  and  $\neg A$  are independent.

However, conditionalizing  $A$  on new evidence after  $\neg A$  (or  $\neg A$  on new evidence after  $A$ ) taking into account the degree of non-classicality of  $A$  (that is,  $\circ A$ ) via  $(\bullet\text{-PrincCond})$  may give some relevant values:  $P_A^{\bullet}(\neg A)$  and  $P_{\neg A}^{\bullet}(A)$  are not necessarily zero.

It is relevant to know under which conditions the values obtained by updating via the Paraconsistent Rules of Conditionalization define a legitimate probability distribution. If  $P$  is a probability function and  $A$  is  $\neg$ -normal. then  $P_A^{\neg}$  is a  $LET_F$ -probability function.

**Theorem 13.** *If  $P$  is a probability function and  $A$  is  $\neg$ -normal, then  $P_A^{\neg}$  is a  $LET_F$ -probability function.*

*Proof.* Essentially the same as in [13], Theorem 8.  $\square$

## 5 Some illustrative examples

Perhaps the distinction between  $\bullet A$  and  $A \wedge \neg A$  for a proposition (event)  $A$  and the respective probabilities could be elusive if not looked at more closely. The examples below can be helpful.

### Example 14. *Rolling dice with Pinocchia*

Suppose your friend Pinocchia rolls a fair 6-sided die, without giving you any information. If she asks you about your unconditional credence that the die came up 4, your most rational answer would be  $\frac{1}{6}$ ; this is your prior hypothesis  $H$ .

If she asks you to suppose that the die came up even, your most rational attitude would be to update your prior hypothesis to a posterior  $H$ , now the degree of belief in '4' conditioned on the new evidence  $E$  that it is even, thus updating your credence to  $\frac{1}{3}$ .

This is explained by a simple application of the standard (PrincCond), expressed by:

$$P_{\text{post}} = P_E(H) = P_{\text{prior}}(H \wedge E) / P_{\text{prior}}(E)$$

Now she says: “The die came up even, this is now a fact”. It happens, however, that Pinocchio lies sometimes, and you know she has lied in 8% of the previous die games you played together: her statements involve a contradiction (in 8% of the cases she says the die came up even, when it actually came up odd, or vice-versa).

Standard conditionalization cannot withstand contradictions in a natural way, while paraconsistent conditionalization through  $(\neg\text{-PrincCond})$  (see Section 4) can be applied artlessly.

In this example:  $P(E) = \frac{1}{2}$ ,  $P(H \wedge E) = \frac{1}{6}$  as before, while the hypothesis  $H$  behaves classically, so  $P(H \wedge \neg E) = 0$ . The reasoning, however, depends crucially on  $x = P(E \wedge \neg E)$ .

In the present case  $P(E \wedge \neg E) = 0.08$ , and reasoning by  $(\neg\text{-PrincCond})$  gives:

$$P_E^-(H) = \frac{P(H \wedge E) - P(H \wedge E \wedge \neg E)}{P(E) - P(E \wedge \neg E)}$$

It remains to evaluate  $P(H \wedge E \wedge \neg E)$ . A reasonable evaluation is to regard  $H$  and  $E \wedge \neg E$  as independent, and in this case  $P(H \wedge E \wedge \neg E) = \frac{1}{6} \cdot x$ .

By  $(\neg\text{-PrincCond})$ :

$$P_E^-(H) = \frac{P(H \wedge E) - P(H) \cdot P(E \wedge \neg E)}{P(E) - P(E \wedge \neg E)} =$$

$$P_E^-(H) = \left(\frac{1}{6} - \frac{1}{2}x\right) / \left(\frac{1}{2} - x\right) = \frac{1}{3} \cdot \frac{(1-3x)}{(1-2x)}$$

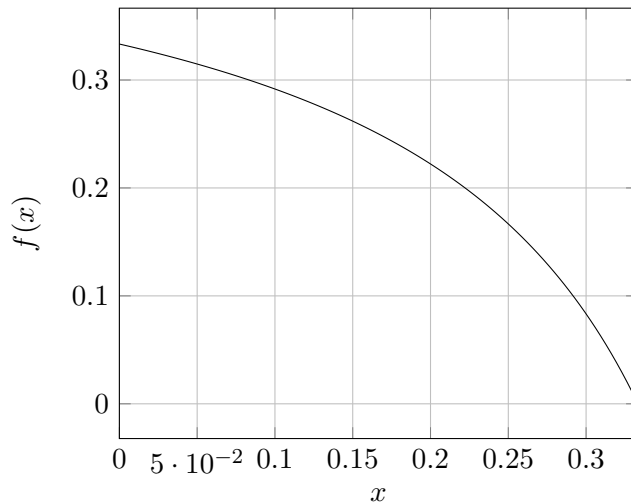
In order to maintain the coherence of probability theory ( $P\text{-SAT}$ ), in the present case the existence conditions that guarantee  $0 \leq P_E^-(H) \leq 1$  are:

$$\frac{1}{3} \cdot \frac{(1-3x)}{(1-2x)} \geq 0 \text{ and thus } 0 \leq x \leq \frac{1}{3}$$

In our example  $x = 0.08$  and thus  $P_E^-(H) = 0.3$ , so the degree of credence has decreased in view of the risk posed by Pinocchio’s lie.

The updated credence decreases non-linearly as the probability of lies increases, as shown by the graph of  $f(x) = \frac{1}{3} \cdot \frac{(1-3x)}{(1-2x)}$ :

Plot of  $f(x)$



**Remark 15.** As the function  $f(x) = \frac{1}{3} \cdot \frac{(1-3x)}{(1-2x)}$  shows, when  $x = P(E \wedge \neg E) = 0$  this means that Pinocchio is telling no lies, and so  $f(0) = \frac{1}{3}$ , i.e., the update proceeds as in the standard case. Analogously, when  $x = P(E \wedge \neg E) = \frac{1}{3}$ , then Pinocchio is lying to the maximum acceptable probability, and  $f(\frac{1}{3}) = 0$ . Again the update proceeds as in the standard case, under the knowledge of a complete lie. For the in-between measures, the Bayesian learning (or the process by which an agent forms and updates beliefs), even under contradictions and inconsistencies, is governed by the function  $f(x)$ .

It is not impossible that, by using standard conditionalization, learning under contradictions and inconsistencies can also be feasible. In any case, our model is quite natural and direct, and offers a sensible way to cope with contradictions and inconsistencies in a reasoned way.

Another aspect that raises discussion is whether or not, by telling a lie, Pinocchio is uttering a contradiction. She is certainly contradicting reality when she says that the die landed even, while it came out odd. For those who prefer to look at this as an inconsistency, the application of ( $\bullet$ -PrincCond) (instead of ( $\neg$ -PrincCond)) offers an immediate solution.

Even if the philosophical distinction between a contradiction and an inconsistency is not always absolutely clear, our theoretical framework recognizes that there is a distinction, and this is clearly reflected in the logic  $LET_F$ .

## 6 Extending Bayesian Confirmation Theory

Paraconsistent Bayesian Confirmation Theory is an account of the support that an evidence confers to a theory (or hypothesis), taking into consideration the probability of classicality of this evidence, by means of the probability of the consistency ( $\circ$ ) and the inconsistency ( $\bullet$ ) operators.

Confirmation hides a gigantic difficulty, because it works on the non-deductive part of scientific theories, usually the more intricate part.

A simple version of standard Bayesian Confirmation Theory defines an evidence  $E$  to confirm a hypothesis  $H$  for an agent under a certain circumstance, if and only if the agent's credence on  $H$  would be raised if the agent conditionalizes on  $E$ ; or, in direct terms: in standard terms, based on (PrincCond), evidence  $E$  confirms a hypothesis  $H$  iff the following holds:

**Definition 16.** (*PrincCond*)-Confirmation Theory

An evidence  $E$  confirms classically a hypothesis  $H$  iff

$P_{\text{posterior}}(H) = P_{\text{prior}}(H/E) > P_{\text{prior}}(H)$ , or equivalently, iff  $P(H \wedge E)/P(E) > P(H)$ .

In terms of ( $\bullet$ -PrincCond) and ( $\neg$ -PrincCond), two versions of Paraconsistent Bayesian Confirmation Theory can be defined as follows, restricting attention to the more plausible cases where  $B$  plays the role of consistent (a.k.a. 'classical') theory, with  $P(\circ B) = 1$ , being confirmed by evidence  $A$  possibly under the condition that  $P(\circ A) < 1$ .

It is relevant to make it clear that under the above assumption that  $P(\circ B) = 1$ , ( $\neg$ -PrincCond) and ( $\bullet$ -PrincCond) (recall Definition 8) simplify to the formulas below, since  $P(\circ B) = 1$  implies  $P(B \wedge \neg B) = 0$ :

( $\neg$ -PrincCond) for  $P(\circ B) = 1$ :

$$P_A^-(B) = \frac{P(B \wedge A) - P(B \wedge A \wedge \neg A)}{P(A) - P(A \wedge \neg A)}$$

where  $P(A) > P(A \wedge \neg A)$

( $\bullet$ -PrincCond) for  $P(\circ B) = 1$ :

$$P_A^\bullet(B) = \frac{P(B \wedge A) - P(B \wedge \bullet A)}{P(A) - P(\bullet A)}$$

where  $P(A) > P(\bullet A)$

**Definition 17.** ( *$\neg$ -PrincCond*)-Confirmation Theory for  $P(\circ H) = 1$

An evidence  $E$  confirms a hypothesis  $H$  relative to ( $\neg$ -PrincCond) iff

$$P_E^-(H) > P(H) \text{ iff } \frac{P(H \wedge E) - P(H \wedge E \wedge \neg E)}{P(E) - P(E \wedge \neg E)} > P(H)$$

for  $P(E) > P(E \wedge \neg E)$

**Definition 18.** ( $\bullet$ -PrincCond)-Confirmation Theory for  $P(\circ H) = 1$

An evidence  $E$  confirms a hypothesis  $H$  relative to ( $\bullet$ -PrincCond) iff

$$P_E^\bullet(H) > P(H) \text{ iff } \frac{P(H \wedge E) - P(H \wedge \bullet E)}{P(E) - P(\bullet E)} > P(H)$$

for  $P(E) > P(\bullet E)$

Of course a full version of paraconsistent confirmation theory can be used if  $P(\circ H) \neq 1$  and  $P(\circ E) \neq 1$ , and clearly the standard confirmation measure is attained when  $P(\circ H) = 1$  and  $P(\circ E) = 1$ .

The last section of this paper will be devoted to showing some applications of the paraconsistent approach to Bayesian epistemology with regard to Confirmation Theory, and to some criticisms of the standard (PrincCond).

## 7 Comparing the Principles of Conditionalization

The standard Principle of Conditionalization (PrincCond) has been criticized for being unrealistic. An alternative form of conditionalization, proposed by R. Jeffrey ([10]), adjusts the formula to take the probability of the evidence into account, but has also been a target of criticisms ([20]).

According to [20], a troublous and pertinacious challenge for Bayesian Confirmation Theory is the Problem of Old Evidence (POE). Suppose that a certain phenomenon  $E$  occurs unexplained by the available scientific theories, until a theory  $H$  is discovered and accounts for  $E$ . The observable  $E$ , previous to the theory, is called “old evidence”. It seems clear that  $E$  confirms  $H$ , if they are independent. A famous and well-studied case of old evidence in science is the Mercury perihelion anomaly. The dynamics of Mercury could not be explained by Newtonian mechanics, until Einstein’s General Theory of Relativity was proposed, and among other things was able to explain the perihelion shift.

Standard Bayesian calculus cannot account for the Mercury perihelion anomaly, nor for several other problems of old evidence (cf. [20]). Indeed, when  $E$  is old evidence, the prior degree of belief in  $E$  is maximal, that is,  $P(E) = 1$ . But under such assumption, it follows that the posterior probability of  $H$  is never confirmed by  $E$  using standard Confirmation Theory. A simple calculation shows that, since  $P(E) = 1$  :

$$P_E(H) = P(H \wedge E) = P(E/H) \cdot P(H) \leq P(H)$$

Hence  $E$  does not confirm  $H$ , even if  $E$  is a well-known observational anomaly that is explained by  $H$ , which shows, if not the failure of standard Bayesian belief theory, at least the failure of standard Confirmation Theory.

Paraconsistent Bayesian conditionalization and the ( $\bullet$ -PrincCond)-confirmation measure offer a way out of this problem, by taking into account the degree of non-classicality (inconsistency) of  $E$ ,  $P(\bullet E)$ .

According to Definition 18, the evidence  $E$  confirms the hypothesis  $H$  relative to ( $\bullet$ -PrincCond) iff

$$P_E^\bullet(H) = \frac{P(H \wedge E) - P(H \wedge \bullet E)}{P(E) - P(\bullet E)} > P(H)$$

under the proviso that  $P(E) > P(\bullet E)$ .

Since  $P(E) = 1$ , the proviso is attended. If  $P(\bullet E) > 0$  then  $P_E^\bullet(H)$  is not necessarily less than  $P(H)$ , and the Problem of Old Evidence does not necessarily arise.

But even the standard Confirmation Theory stands to gain from this new approach. As an illustration of the reach of Paraconsistency Conditionalization, let us revisit a case suggested in [17] involving the conditions under which a hypothesis  $H$  is disconfirmed, or confirmed, when the negation  $\neg E$  of one of its consequences is observed. Suppose that  $H$  implies some observational consequence  $E$  so that  $P(E/H) = 1$ . Suppose that  $\neg E$  is observed.

Evidence  $\neg E$  confirms a hypothesis  $H$  in this case (see section 6) iff

(1)

$$c = P(H/\neg E)/P(H) > 1$$

By the definition of conditional probability,

$$P(H/\neg E).P(\neg E) = P(H \wedge \neg E) = P(\neg E/H).P(H)$$

hence

(2)

$$c = P(\neg E/H)/P(\neg E)$$

From

$$P(E \vee \neg E) = P(E) + P(\neg E) - P(E \wedge \neg E)$$

it follows

(3)

$$P(\neg E) = P(E \vee \neg E) - P(E) + P(E \wedge \neg E)$$

Conditionalizing this expression over  $H$  obtains:

(4)

$$P(\neg E/H) = P(E \vee \neg E/H) - P(E/H) + P(E \wedge \neg E/H)$$

But  $P(E/H) = P(E \wedge H)/P(H) = 1$  by hypothesis, and  $E \wedge H \vdash (E \vee \neg E) \wedge H$ , hence  $P(E \wedge H)/P(H) \leq P(E \vee \neg E \wedge H)/P(H) = 1$  and consequently  $P(E \vee \neg E/H) = 1$ . Thus:

(5)

$$P(\neg E/H) = P(E \wedge \neg E/H)$$

Plugging (4) and (5) into (2) results in:

$$C = P(E \wedge \neg E/H)/P(E \vee \neg E) - P(E) + P(E \wedge \neg E) = \\ P(E \wedge \neg E/H)/P(\neg E)$$

In the standard theory of probability,  $C$  is evaluated as zero, since  $P(E \wedge \neg E)$  is zero, so  $H$  is peremptorily disconfirmed when the negation of one of its consequences is observed, without even considering the coherence or consistency of that negated consequence.

Paraconsistent Bayesian Confirmation proposes a more elastic approach to confirmation. Taking into account, in this case, the degree of classicality of  $E$ , namely,  $P(\circ E)$ .

The key points are the properties  $P(E \wedge \neg E) \leq P(\bullet E)$ ,  $P(\circ E) \leq P(E \vee \neg E)$ , and  $P(\circ E) = 1 - p(\bullet E)$ .

It is to be noted that  $\circ E$  does not imply  $E$ , it just implies that  $E$  and  $\neg E$  cannot hold simultaneously; also,  $\bullet E$  does not imply that  $E$  and  $\neg E$  hold simultaneously.

If  $P(\circ E) = 1$ , it is enjoying maximum classicality and then  $P(E \wedge \neg E) = 0$ , so the paraconsistent confirmation agrees with the standard confirmation measure, and  $H$  is disconfirmed.

If  $P(\circ E) < 1$ , then  $C > 0$ , even if not necessarily greater than 1, but varies according to  $P(\circ E)$ .

However, if  $H$  entails  $\neg E$ , as suggested by [17],  $H$  entails  $E \wedge \neg E$  because  $H$  implies  $E$  by hypothesis; so that  $P(E \wedge \neg E/H) = 1$  and  $C = 1/P(\neg E)$ , unless the anomalous  $\neg E$  is such that  $P(\neg E) = 1$ ,  $C > 1$  and  $H$  is confirmed, as expected.

The problem of Old Evidence has received several attempted solutions and alternative proposals, some authors even suggesting the abandoning of the Bayesian way. Clark Glymour ([5]), for instance, offers several reasons why not to follow Bayesian kinematics when changing rational belief.

On the other hand, when trying to provide a solution to it, Daniel Garber ([7]) argues that the Problem of Old Evidence is generated by the requirement that the Bayesian agent be logically omniscient, a requirement usually thought to follow from coherence. He shows how the requirement of logical omniscience can be relaxed in a way consistent with coherence, and shows how this can lead us to a solution of the Problem of Old Evidence. His solution is quite involved –it occupies more than 30 pages of subtle reasoning, in a tangled web of arguments. Stephan Hartmann and Branden Fitelson ([9]) show that some of the assumptions made in Garber’s solution to the Problem of Old Evidence can be relaxed, resulting in a more general approach.

Our proposed solution offers a more straightforward approach, that only takes into account the possibility that the old evidence might be dubious, despite being accepted with higher probability.

## 8 Conclusions

One of the tenets of  $LET_F$  is that there are no true contradictions, but that there may be contexts, aside from the normal and well-behaved ones, subject to conflicting evidence or to absence of any evidence. The probabilistic semantics attached to  $LET_F$  works as a natural quantification of degrees of evidence.

The fact that  $LET_F$  has no theorems of the form  $\circ A$  (cf [18]) means that the degree of classicality of  $A$ ,  $P(\circ A)$ , and consequently  $P(\bullet A)$ , comes from outside the formal system. Pieces of evidence can be contradictory, or incomplete, or neither contradictory nor incomplete but still non-conclusive.

Nevertheless,  $LET_F$  is not the unique alternative to building a paraconsistent and paracomplete approach to probability and its philosophy: other approaches as, e.g., in [3] supply such logics with a more standard algebraic counterpart, which makes possible a probabilistic treatment in terms of lattices or sigma-algebra on the lines of [16]. This is, however, left for further development.

There are some important issues that have not been addressed in this paper. One of them is the ‘zero-denominator problem’, which concerns the requirement  $P(B) > 0$  in the ratio formula  $P(A/B)$ . This restriction puts up a serious obstacle for some developments of probability. K. Popper in [15] proposes an alternative axiomatic theory of probability where conditional probability is taken as primitive, avoiding such restriction. It seems quite natural to explore the possibilities of combining Popper’s approach with the proposal expounded here and examining its consequences, although some aspects of conditional probability solving some problems concerning the ratio formula in the spirit of Popper are already met by our theory (see note in Section 4). Another question is to evaluate more deeply the expanded notion of conditionalization advanced in [10] in the light of our proposal. Still another hard question is that of investigating how the paraconsistent Bayes rules could be extended to continuous random variables, possibly integrating countable additivity.

Another promising line of research is to investigate the role of the paraconsistent probabilities as developed in this paper to quantum theory, as hinted by several authors (see, e.g., [1], [4], and [22]).

The proposal developed in this paper is just the outline of a theory yet to be explored. There are other versions of Bayesianism and other problems that deserve to be investigated, but we hope the analysis started here can point to fruitful directions.

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