Simple Proof for the Riemann Hypothesis

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To my mother

Abstract. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $M_x = \prod_{q \leq x} q$ be the product extending over all prime numbers q that are less than or equal to a natural number $x > 1$. The Riemann hypothesis is the assertion that all non-trivial zeros are complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the Riemann hypothesis. In 2011, Solé and Planat stated that the Riemann hypothesis is true if and only if the inequality $R(M_x) > \frac{e^{\gamma}}{\zeta(2)}$ holds for all $x \geq 5$, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. In this note, using Solé and Planat criterion, we prove that the Riemann hypothesis is true.

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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1.1. *For every* $x > 1$ [6, Theorem 4 (3.15) pp. 70]*:*

$$\theta(x) < \left(1 + \frac{1}{2 \cdot \log x}\right) \cdot x.$$

The following property is based on natural logarithms:

Proposition 1.2. *For* $x > -1$ [3, pp. 1]*:*

$$\log(1 + x) \leq x.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.3. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k *is the* k*th prime number. By definition, we have*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n *denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ *is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.*

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right)$$
$$= \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constants B and H (1874) [4]. The number $B \approx 0.26149$ is the Meissel-Mertens constant where $\gamma = B + H$ [4].

Proposition 1.4. *We have* [2, Lemma 2.1 (1) pp. 359]*:*

$$\sum_{k=1}^{\infty} \left(\log\left(\frac{q_k}{q_k - 1}\right) - \frac{1}{q_k}\right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows [5, pp. 379]:

$$u(x) = \sum_{q>x} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right).$$

On the sum of the reciprocals of all prime numbers not exceeding x, we have:

Proposition 1.5. *For* $x > 1$ [6, Theorem 5 (3.17) pp. 70]*:*

$$-\frac{1}{2 \cdot \log^2 x} < \sum_{q \leq x} \frac{1}{q} - B - \log \log x.$$

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function where $q \mid n$ means the prime q divides n. For $x \geq 2$, a natural number M_x is defined as

$$M_x = \prod_{q \leq x} q.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. We say that $\mathsf{Dedekind}(x)$ holds provided that

$$R(M_x) > \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.6. $\mathsf{Dedekind}(x)$ *holds for all* $x \geq 5$ *if and only if the Riemann hypothesis is true* [7, Theorem 4.2 pp. 5].

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

The following is a key Lemma.

Lemma 2.1. *The inequality* $\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^7$ *holds for all* $x \geq 10^8$.

Proof. By Proposition 1.4, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^7$$

is the same as

$$\sum_{q \leq x} \log \left(\frac{q}{q-1} \right) - B - \log \log \theta(x) \geq H + 6 \cdot \gamma - 7 \cdot \log(\zeta(2))$$

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after of applying the logarithm to the both sides and distributing the terms. In addition,

$$\begin{aligned}\log\log\theta(x) &< \log\log\left(\left(1+\frac{1}{2\cdot\log x}\right)\cdot x\right)\\ &= \log\left(\log\left(1+\frac{1}{2\cdot\log x}\right)+\log x\right)\\ &= \log\left((\log x)\cdot\left(1+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\right)\right)\\ &= \log\log x+\log\left(1+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\right)\\ &\leq \log\log x+\frac{\log\left(1+\frac{1}{2\cdot\log x}\right)}{\log x}\\ &\leq \log\log x+\frac{1}{2\cdot\log^2 x}\end{aligned}$$

by Propositions 1.1 and 1.2. So,

$$\sum_{q\leq x}\log\left(\frac{q}{q-1}\right)-B-\log\log x-\frac{1}{2\cdot\log^2 x}\geq H+6\cdot\gamma-7\cdot\log(\zeta(2)).$$

That is,

$$\sum_{q\leq x}\log\left(\frac{q}{q-1}\right)-B-\log\log x-\frac{1}{2\cdot\log^2 x}-u(x)\geq H-u(x)+6\cdot\gamma-7\cdot\log(\zeta(2)).$$

after subtracting $u(x)$ to the both sides of the inequality. By Proposition 1.4, we can see that

$$\sum_{q\leq x}\left(\frac{1}{q}\right)-B-\log\log x-\frac{1}{2\cdot\log^2 x}-u(x)\geq 6\cdot\gamma-7\cdot\log(\zeta(2)).$$

By Proposition 1.5, we deduce that

$$-\frac{1}{2\cdot\log^2 x}-\frac{1}{2\cdot\log^2 x}-u(x)\geq 6\cdot\gamma-7\cdot\log(\zeta(2)).$$

It is a fact that the inequality

$$-\frac{1}{\log^2 x}-u(x)\geq 6\cdot\gamma-7\cdot\log(\zeta(2))$$

holds for all $x\geq 10^8$ due to

$$-\frac{1}{\log^2 x}-H\geq 6\cdot\gamma-7\cdot\log(\zeta(2))-\sum_{q\leq 10^8}\left(\log\left(\frac{q}{q-1}\right)-\frac{1}{q}\right). \qquad \square$$

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3. Main Insight

This is the main insight.

Lemma 3.1. $\mathsf{Dedekind}(x)$ *always holds for all* $x \geq 10^8$.

Proof. By Lemma 2.1, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^7$$

holds for all $x \geq 10^8$. By Propositions 1.2 and 1.4, the inequality

$$\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^7$$

is equivalent to

$$e^{H-u(x)} \cdot R(M_x) \geq \left(\frac{e^{\gamma}}{\zeta(2)}\right)^7.$$

Certainly, we have

$$\begin{aligned}
\frac{\prod_{q \leq x} \left(\frac{q}{q-1}\right)}{\log \theta(x)} &\geq \left(\prod_{q \leq x} \frac{\left(\frac{q}{q-1}\right)}{e^{\frac{1}{q}}}\right) \cdot \frac{\prod_{q \leq x} \left(1 + \frac{1}{q}\right)}{\log \theta(x)} \\
&= e^{H-u(x)} \cdot \frac{\prod_{q \leq x} \left(1 + \frac{1}{q}\right)}{\log \theta(x)} \\
&= e^{H-u(x)} \cdot \frac{M_x \cdot \prod_{q \mid M_x} \left(1 + \frac{1}{q}\right)}{M_x \cdot \log \log M_x} \\
&= e^{H-u(x)} \cdot \frac{\Psi(M_x)}{M_x \cdot \log \log M_x} \\
&= e^{H-u(x)} \cdot R(M_x)
\end{aligned}$$

using the Propositions 1.2 and 1.4 such that $e^{\frac{1}{q}} \geq \left(1 + \frac{1}{q}\right)$ for every prime q. Consequently, we would have

$$\frac{e^{H-u(x)}}{\left(\frac{e^{\gamma}}{\zeta(2)}\right)^6} \cdot R(M_x) \geq \frac{e^{\gamma}}{\zeta(2)}.$$

We only need to prove that

$$\frac{e^{H-u(x)}}{\left(\frac{e^{\gamma}}{\zeta(2)}\right)^6} < 1$$

holds for all $x \geq 10^8$ to confirm that $\mathsf{Dedekind}(x)$ also holds. Hence, it is enough to show that

$$\frac{e^{H}}{\left(\frac{e^{\gamma}}{\zeta(2)}\right)^6} < 1$$

because of

$$e^{u(x)} > 1$$

for $x \geq 2$ [5, (11) pp. 379]. Next,

$$-B + 6 \cdot \log(\zeta(2)) - 5 \cdot \gamma < 0$$

after applying the logarithm to the both sides of the inequality. Finally, we obtain that

$$\frac{6}{5} < \frac{\gamma + \frac{B}{5}}{\log(\zeta(2))}.$$

Using a simple numerical calculation, we can check that

$$\frac{\gamma + \frac{B}{5}}{\log(\zeta(2))} > 1.26483 > 1.2 = \frac{6}{5}$$

and therefore, the proof is done. □

4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. We already know that $\mathsf{Dedekind}(x)$ holds for all $5 \leq x \leq 10^8$ [5, Theorem 3 (a) pp. 376]. In this way, the Riemann hypothesis must be true as a direct consequence of the Proposition 1.6 and Lemma 3.1. □

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