Short Proof for the Riemann Hypothesis

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To my mother

Abstract. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order n. There are several statements equivalent to the Riemann hypothesis. We state that if for every large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that $R(N_{n'}) \leq R(N_n)$, then the Riemann hypothesis is true. In this note, using our criterion, we prove that the Riemann hypothesis is true.

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1. Introduction

The Riemann hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. The following property is based on natural logarithms:

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Proposition 1.1. *For* $x \geq 1$ [3, Theorem 1.1 (13) pp. 3]*:*

$$\frac{1}{x+0.5} < \log\left(1+\frac{1}{x}\right) < \frac{1}{x+0.4}.$$

We define $S(x) = \theta(x) - x$.

Proposition 1.2. $S(x)$ *changes sign infinitely often* [8, pp. 1]*. Besides, we have* [8, pp. 1]*:*

$$S(x) \sim 0 \quad as \quad (x \to \infty).$$

Proposition 1.3. *For* $x \geq 121$ [7, (5) (6) pp. 378]*:*

$$\frac{S(x)}{x \cdot \log x} - \frac{S^2(x)}{x^2 \cdot \log x} \leq \log\log\theta(x) - \log\log x \leq \frac{S(x)}{x \cdot \log x}$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.4. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k *is the* k*th prime number (We also use the notation* q_n *to denote the* n*th prime number). By definition, we have*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n *denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ *is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.*

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right)\\ &= \int_1^{\infty}\left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx.\end{aligned}$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. In mathematics, $\Psi(n) = n \cdot \prod_{q \mid n}\left(1+\frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n}\left(1+\frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log\theta(q_n).$$

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Next, we have Solé and Planat Theorem:

Proposition 1.5. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann hypothesis is true* [9, Theorem 4.2 pp. 5].

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ is satisfied.

Proposition 1.6. *Unconditionally on Riemann hypothesis, we know that* [9, Proposition 3. pp. 3]*:*

$$\lim_{n \to \infty} R(N_n) = \frac{e^\gamma}{\zeta(2)}.$$

On the sum of the reciprocals of all prime numbers not exceeding x:

Proposition 1.7. *For* $x \geq 2278383$ [2, Theorem 5.6 (1) pp. 243]*:*

$$-\frac{0.2}{\log^3 x} \leq \sum_{q \leq x} \frac{1}{q} - B - \log \log x \leq \frac{0.2}{\log^3 x}$$

where $B \approx 0.26149$ *is the Meissel-Mertens constant* [6, (17.) pp. 54].

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [4]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [5]:

$$f(x) = \Omega_R(g(x)) \text{ as } x \to \infty \text{ if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$

$$f(x) = \Omega_L(g(x)) \text{ as } x \to \infty \text{ if } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_\pm(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as [10]:

$$f(x) = \Omega_+(g(x)) \text{ if } \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [7, Theorem 3 pp. 376]:

$$f(x) = e^\gamma \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Finally, we have the Nicolas Theorem:

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Proposition 1.8. *If the Riemann hypothesis is false then there exists a real b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$* [7, Theorem 3 (c) pp. 376]*:*

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there exist some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The function g is defined as [9, Theorem 4.2 pp. 5]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

By Proposition 1.5, the Riemann hypothesis is false whenever there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$. It was proven the following bound [9, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

By Proposition 1.8, if the Riemann hypothesis is false, then there exists a real number $0 < b < \frac{1}{2}$ such that there exist infinitely many natural numbers $x \geq 2$ for which $\log f(x) = \Omega_{+}(x^{-b})$. Actually Nicolas proved that $\log f(x) = \Omega_{\pm}(x^{-b})$, but we only need to use the notation Ω_{+} under the domain of the natural numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the variable k disappears in our previous expression since we do not need it anymore. In this way, if the Riemann hypothesis is false, then there exist infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, then it would be infinitely many natural numbers x_0

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such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. Actually, we know that

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

3. Main Insight

This is the main insight.

Lemma 3.1. *The Riemann hypothesis is true whenever for every large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. By Lemma 2.1, if the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for every large enough prime number q_n, then there exists an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails. By Proposition 1.6, this is a contradiction with the fact that

$$\liminf_{n \to \infty} R(N_n) = \lim_{n \to \infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

By definition of the limit inferior for any positive real number ε, only a finite number of elements of the sequence $R(N_n)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This contradicts the existence of such previous infinite subsequence and thus, the Riemann hypothesis must be true. □

4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. By Lemma 3.1, the Riemann hypothesis is true whenever

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for large enough prime numbers $q_{n'} > q_n$. That is the same as

$$\log\log\theta(q_{n'}) - \log\log\theta(q_n) \geq \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right).$$

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By Propositions 1.1, 1.2, 1.3 and 1.7, for every large enough prime number q_n, there exists always another prime $q_{n'} > q_n$ such that

$$\begin{aligned}
&\log\log\theta(q_{n'}) - \log\log\theta(q_n)\\
&= \log\log(q_{n'}) - \log\log(q_n) + (\log\log\theta(q_{n'}) - \log\log(q_{n'}))\\
&- (\log\log\theta(q_n) - \log\log(q_n))\\
&\geq \log\log(q_{n'}) - \log\log(q_n) + \varepsilon_{(q_{n'},q_n)}\\
&= \frac{0.2}{\log^3 q_n} + B + \log\log(q_{n'}) - B - \frac{0.2}{\log^3 q_n} - \log\log(q_n) + \varepsilon_{(q_{n'},q_n)}\\
&\geq \sum_{q_n < q \leq q_{n'}} \frac{1}{q + 0.4}\\
&> \sum_{q_n < q \leq q_{n'}} \log\left(1 + \frac{1}{q}\right),
\end{aligned}$$

when

$$\varepsilon_{(q_{n'},q_n)} = \frac{S(q_{n'})}{q_{n'} \cdot \log q_{n'}} - \frac{S^2(q_{n'})}{q_{n'}^2 \cdot \log q_{n'}} - \frac{S(q_n)}{q_n \cdot \log q_n}$$

and

$$S(q_{n'}) > 0,$$

where

$$S(q_n) \sim 0 \quad as \quad (n \to \infty),$$

and

$$\frac{1}{q + 0.4} > \log\left(1 + \frac{1}{q}\right).$$

Consequently, the inequality

$$R(N_{n'}) \leq R(N_n)$$

always holds for a pair of sufficiently large prime numbers $q_{n'} > q_n$ and therefore, the Riemann hypothesis is true. □

5. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are considered to be true under the assumption of the Riemann hypothesis and some of them that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. A proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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References

[1] Ayoub, R.: Euler and the Zeta Function. The American Mathematical Monthly **81**(10), 1067–1086 (1974). https://doi.org/10.2307/2319041

[2] Dusart, P.: Explicit estimates of some functions over primes. The Ramanujan Journal **45**, 227–251 (2018). https://doi.org/10.1007/s11139-016-9839-4

[3] Ghosh, A.: An asymptotic formula for the Chebyshev theta function. Notes on Number Theory and Discrete Mathematics **25**(4), 1–7 (2019). https://doi.org/10.7546/nntdm.2019.25.4.1-7

[4] Hardy, G.H., Littlewood, J.E.: Some problems of diophantine approximation: Part II. The trigonometrical series associated with the elliptic ϑ-functions. Acta mathematica **37**(1), 193–239 (1914)

[5] Hardy, G.H., Littlewood, J.E.: Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Mathematica **41**, 119–196 (1916)

[6] Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. reine angew. Math. **1874**(78), 46–62 (1874). https://doi.org/10.1515/crll.1874.78.46

[7] Nicolas, J.L.: Petites valeurs de la fonction d'Euler. Journal of Number Theory **17**(3), 375–388 (1983). https://doi.org/10.1016/0022-314X(83)90055-0

[8] Platt, D.J., Trudgian, T.S.: On the first sign change of $\theta(x) - x$. Mathematics of Computation **85**(299), 1539–1547 (2016). https://doi.org/10.1090/mcom/3021

[9] Solé, P., Planat, M.: Extreme values of the Dedekind ψ function. Journal of Combinatorics and Number Theory **3**(1), 33–38 (2011)

[10] Tenenbaum, G.: Introduction to Analytic and Probabilistic Number Theory, vol. 163. American Mathematical Soc. (2015)

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