An Equivalent to the Riemann Hypothesis

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To my mother

Abstract. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. We prove that the Riemann hypothesis is true whenever there exists a large enough positive number x_0 such that for all $x > x_0$ we obtain the approximate value of

$$\sum_{n \leq \alpha_x} \frac{1}{n} - \sum_{n \leq \frac{x}{\log x}} \frac{e^{-\gamma}}{n \cdot \left(\log(n \cdot \log n) - 1 + \frac{(\log \log n - 2)}{\log n} + O\left(\frac{(\log \log n)^2}{(\log n)^2}\right)\right)}$$

is lesser than or equal to $e^{-\gamma} \cdot \left(\gamma - B - \frac{1}{2 \cdot (x-1)}\right)$ where $B \approx 0.26149$ is the Meissel-Mertens constant and $\alpha_x = \left(\log x + \frac{0.0222 \cdot \log x}{\log \log x}\right)$. Since the approximate expression goes to 0 as x tends to infinity, then we deduce that the Riemann hypothesis must be true.

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1. Introduction

In mathematics, the Prime-counting function $\pi(x)$ is given by

$$\pi(x) = \sum_{q \leq x} 1$$

with the sum extending over all prime numbers q that are less than or equal to x.

Proposition 1.1. *For* $x \geq 17$ [11, Corollary 1 (3.5) pp. 69]*:*

$$\pi(x) > \frac{x}{\log x},$$

where log *is the natural logarithm.*

Proposition 1.2. *An approximation for the nth prime number is*

$$q_n = n \cdot \left(\log(n \cdot \log n) - 1 + \frac{(\log\log n - 2)}{\log n} + O\left(\frac{(\log\log n)^2}{(\log n)^2}\right)\right).$$

Specifically, for $n \geq 6$ [11, Corollary (3.13) pp. 69]*:*

$$q_n < n \cdot (\log(n \cdot \log n)).$$

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right) = \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constants B and H (1874) [5]. We define $H = \gamma - B$ such that $B \approx 0.26149$ is the Meissel-Mertens constant [5].

Proposition 1.3. *We have* [2, Lemma 2.1 (1) pp. 359]*:*

$$\sum_{k=1}^{\infty} \left(\log\left(\frac{q_k}{q_k - 1}\right) - \frac{1}{q_k}\right) = \gamma - B = H,$$

where q_k *is the kth prime number.*

For $x \geq 2$, the function $u(x)$ is defined as follows [7, pp. 379]:

$$u(x) = \sum_{q > x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q}\right).$$

Proposition 1.4. *We have* [7, (11) pp. 379]*:*

$$0 < u(x) \leq \frac{1}{2 \cdot (x - 1)}.$$

On the sum of the reciprocals of all prime numbers not exceeding x, we have:

Proposition 1.5. *For* $x \geq 2278383$ [3, Theorem 5.6 (1) pp. 243]*:*

$$-\frac{0.2}{\log^3 x} \leq \sum_{q \leq x} \frac{1}{q} - B - \log\log x \leq \frac{0.2}{\log^3 x}$$

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

Proposition 1.6. *Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \ldots < q_r$ with natural numbers $a_1, \ldots, a_r$ as exponents. Then* [4, Lemma 1 pp. 2],

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n.$$

The Ramanujan's Theorem stated that if the Riemann hypothesis is true, then the previous inequality holds for large enough n. Next, we have the Robin's Theorem:

Proposition 1.7. $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true* [10, Theorem 1 pp. 188].

Unconditionally on Riemann hypothesis, we have:

Proposition 1.8. $\mathsf{Robin}(n)$ *holds for all natural numbers $10^{10^{13.11485}} \geq n > 5040$* [9, Theorem 5 pp. 6].

Proposition 1.9. *For $x \geq 1$* [11, Corollary 2 (3.31) pp. 71]*:*

$$\prod_{q \leq x} \frac{q}{q - 1} < e^{\gamma} \cdot \sum_{m \leq x} \frac{1}{m},$$

where m denotes a natural number.

In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [8]. Superabundant numbers were also studied by Leonidas Alaoglu and Paul Erdős (1944) [1]. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [2, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following property for the superabundant numbers:

Proposition 1.10. *If n is superabundant, then n is a Hardy-Ramanujan integer* [1, Theorem 1 pp. 450].

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Proposition 1.11. *Let n be a large enough superabundant number such that q is the largest prime factor of n. Then* [6, Corollary 4.16 pp. 16]*:*

$$q < (\log n) \cdot \left(1 + \frac{0.0222}{\log \log n}\right).$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \ \textit{for} \ \ (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 1.12. *Every colossally abundant number is superabundant* [1, pp. 455]*.*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis could be false.

Proposition 1.13. *If the Riemann hypothesis is false, then there exist infinitely many colossally abundant numbers $n > 5040$ such that* $\mathsf{Robin}(n)$ *fails (i.e.* $\mathsf{Robin}(n)$ *does not hold)* [10, Proposition pp. 204]*.*

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many superabundant numbers n such that* $\mathsf{Robin}(n)$ *fails.*

Proof. This is a direct consequence of Propositions 1.7, 1.12 and 1.13. □

3. Main Insight

This is the main insight.

Theorem 3.1. *The Riemann hypothesis is true whenever there exists a large enough positive number n_0 such that for all $n > n_0$ we obtain the approximate value of*

$$\sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{m \leq \frac{n}{\log n}} \frac{e^{-\gamma}}{m \cdot \left(\log(m \cdot \log m) - 1 + \frac{(\log \log m - 2)}{\log m} + O\left(\frac{(\log \log m)^2}{(\log m)^2}\right)\right)}$$

is lesser than or equal to $e^{-\gamma} \cdot \left(H - \frac{1}{2 \cdot (n-1)}\right)$ *where* $\alpha_n = \left(\log n + \frac{0.0222 \cdot \log n}{\log \log n}\right)$.

Proof. Let $n > 5040$ be a counterexample such that $\mathsf{Robin}(n)$ does not hold. We know this number could be a large enough superabundant number by Lemma 2.1. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of this superabundant number n as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents according to Proposition 1.10. We know that $n > 10^{10^{13.11485}}$ by Proposition 1.8. Under our supposition, we have

$$\sigma(n) \geq e^{\gamma} \cdot n \cdot \log \log n.$$

By Proposition 1.5, we notice that

$$\sum_{q \leq n} \frac{1}{q} \leq \log \log n + B + \frac{0.2}{\log^3 n}.$$

We can see that

$$\sum_{q \leq n} \log \left(\frac{q}{q-1} \right) - \sum_{q \leq n} \left(\log \left(\frac{q}{q-1} \right) - \frac{1}{q} \right) \leq \log \log n + B + \frac{0.2}{\log^3 n}$$

which is

$$\sum_{q \leq n} \log \left(\frac{q}{q-1} \right) - H < \log \log n + B + \frac{0.2}{\log^3 n}$$

and

$$\sum_{q \leq n} \log \left(\frac{q}{q-1} \right) < \log \log n + \gamma + \frac{0.2}{\log^3 n}$$

by Proposition 1.3. That is the same as

$$n \cdot \sum_{q \leq n} \log \left(\frac{q}{q-1} \right) < n \cdot \log \log n + \left(\gamma + \frac{0.2}{\log^3 n} \right) \cdot n$$

after multiplying both sides by the superabundant number n. We know that

$$\left(\gamma + \frac{0.2}{\log^3 n} \right) \cdot n \leq (e^{\gamma} - 1) \cdot n \cdot \log \log n$$

for $n > 10^{10^{13.11485}}$. Consequently, we obtain that

$$\sigma(n) > n \cdot \sum_{q \leq n} \log \left(\frac{q}{q-1} \right)$$

by transitivity since

$$e^{\gamma} \cdot n \cdot \log \log n \geq n \cdot \log \log n + \left(\gamma + \frac{0.2}{\log^3 n} \right) \cdot n.$$

In this way, we have

$$f(n) > \sum_{q \leq n} \log \left(\frac{q}{q-1} \right)$$

which is

$$\prod_{q \leq \alpha_n} \left(\frac{q}{q-1} \right) > \sum_{q \leq n} \log \left(\frac{q}{q-1} \right)$$

by Proposition 1.6 and 1.11 since

$$\prod_{q \leq q_k} \left(\frac{q}{q-1} \right) > f(n)$$

and

$$q_k < (\log n) \cdot \left(1 + \frac{0.0222}{\log \log n} \right) = \left(\log n + \frac{0.0222 \cdot \log n}{\log \log n} \right)$$

for large enough superabundant number n. That would be

$$e^{\gamma} \cdot \sum_{m \leq \alpha_n} \frac{1}{m} > \sum_{q \leq n} \log \left(\frac{q}{q-1} \right)$$

since

$$\prod_{q \leq \alpha_n} \left(\frac{q}{q-1} \right) < e^{\gamma} \cdot \sum_{m \leq \alpha_n} \frac{1}{m}$$

by Proposition 1.9. So, we would have

$$e^{\gamma} \cdot \sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{q \leq n} \frac{1}{q} > H - u(n)$$

by Proposition 1.3. By Proposition 1.4, we have

$$e^{\gamma} \cdot \sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{q \leq n} \frac{1}{q} > H - \frac{1}{2 \cdot (n-1)}.$$

That is equivalent to

$$e^{\gamma} \cdot \sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{m \leq \frac{n}{\log n}} \frac{1}{q_m} > H - \frac{1}{2 \cdot (n-1)}$$

since

$$\pi(n) > \frac{n}{\log n}$$

by Proposition 1.1. Hence, it is enough to show that

$$\sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{m \leq \frac{n}{\log n}} \frac{e^{-\gamma}}{q_m} > e^{-\gamma} \cdot \left(H - \frac{1}{2 \cdot (n-1)} \right)$$

which means that the approximate expression

$$\sum_{m \leq \alpha_n} \frac{1}{m} - \sum_{m \leq \frac{n}{\log n}} \frac{e^{-\gamma}}{m \cdot \left(\log(m \cdot \log m) - 1 + \frac{(\log \log m - 2)}{\log m} + O\left(\frac{(\log \log m)^2}{(\log m)^2} \right) \right)}$$

would be greater than $e^{-\gamma} \cdot \left(H - \frac{1}{2 \cdot (n-1)} \right)$ by Proposition 1.2. However, that contradicts the fact that n could be a superabundant number as large as we want and thus, it could happen that $n > n_0$ from our pre-conditions. This contradiction implies that it cannot exist infinitely many superabundant numbers n such that $\mathsf{Robin}(n)$ fails and therefore, the Riemann hypothesis should be true using a proof by contraposition from Lemma 2.1. $\square$

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4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. Since the approximate expression of Theorem 3.1 goes to 0 as x tends to infinity, then we deduce that the Riemann hypothesis must be true. This is because of the upper bound $\frac{n}{\log n}$ is exponentially larger than the number $\alpha_n = \left(\log n + \frac{0.0222 \cdot \log n}{\log \log n}\right)$ and

$$\frac{e^{-\gamma}}{m \cdot \left(\log(m \cdot \log m) - 1 + \frac{(\log \log m - 2)}{\log m} + O\left(\frac{(\log \log m)^2}{(\log m)^2}\right)\right)} \approx \frac{1}{m}$$

for each value of $m \geq 6$ by Proposition 1.2 since $e^{\gamma} \approx 0$. □

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