New Criterion for the Riemann Hypothesis

Frank Vega

To my mother

Abstract. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order n. There are several statements equivalent to the Riemann hypothesis. We state that if for each large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that $R(N_{n'}) \leq R(N_n)$, then the Riemann hypothesis is true. In this note, using our criterion, we prove that the Riemann hypothesis is true.

Mathematics Subject Classification (2010). Primary 11M26; Secondary 11A41, 11A25.

Keywords. Riemann hypothesis, prime numbers, Riemann zeta function, Chebyshev function.

1. Introduction

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1.1. *We have* [9, pp. 1]*:*

$$\theta(x) \sim x \quad as \quad (x \to \infty).$$

We know the following inequalities:

https://doi.org/10.33774/coe-2023-2f203-v7 **ORCID:** https://orcid.org/0000-0001-8210-4126 Content not peer-reviewed by Cambridge University Press. Lic

Proposition 1.2. *For $x < 1.79$ [7, pp. 1]:*

$$1 + x \leq e^x \leq 1 + x + x^2.$$

Proposition 1.3. *The inequality $(1+x)^r \leq 1 + x^r$ holds for $-1 < x < 0$ and $0 < r < 1$ under the domain of real numbers* [6, pp. 49].

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.4. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number). By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) = \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n.

Definition 1.5. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) \geq \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

https://doi.org/10.33774/coe-2023-2f203-v7 **ORCID:** https://orcid.org/0000-0001-8210-4126 Content not peer-reviewed by Cambridge University Press. Lic

We define $R(n) = \frac{\Psi(n)}{n \cdot \log\log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) \geq \frac{e^{\gamma}}{\zeta(2)}$ is satisfied.

Proposition 1.6. *Unconditionally on Riemann hypothesis, we know that* [10, Proposition 3. pp. 3]*:*

$$\lim_{n \to \infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.7. *For all prime numbers* $q_n > 5$ [3, Theorem 1.1. pp. 358]*:*

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) < e^{\gamma} \cdot \log \theta(q_n).$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [4]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [5]:

$$f(x) = \Omega_R(g(x)) \ \textit{as} \ x \to \infty \ \textit{if} \ \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$

$$f(x) = \Omega_L(g(x)) \ \textit{as} \ x \to \infty \ \textit{if} \ \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_{\pm}(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as:

$$f(x) = \Omega_+(g(x)) \ \textit{if} \ \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [8, Theorem 3 pp. 376], [2, (5.5) pp. 111]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Finally, we have the Nicolas Theorem:

Proposition 1.8. *If the Riemann hypothesis is false then there exists a real* b *with* $0 < b < \frac{1}{2}$ *such that, as* $x \to \infty$ [8, Theorem 3 (c) pp. 376], [2, Theorem 5.29 pp. 131]*:*

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there exist some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The function g is defined as [10, Theorem 4.2 pp. 5]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We claim that $\mathsf{Dedekind}(q_n)$ fails whenever there exists some real number $x_0 \geq 5$ for which $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ and q_n is the greatest prime number such that $q_n \leq x_0$. It was proven the following bound [10, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

By Proposition 1.8, if the Riemann hypothesis is false, then there is a real number $0 < b < \frac{1}{2}$ such that there exist infinitely many numbers x for which $\log f(x) = \Omega_{+}(x^{-b})$. Actually Nicolas proved that $\log f(x) = \Omega_{\pm}(x^{-b})$, but we only need to use the notation Ω_{+} under the domain of the real numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0) \colon \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0) \colon \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the value of k is not necessary in the statement above. In this way, if the Riemann hypothesis is false, then there exist infinitely many wide apart numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, then it would be infinitely many wide apart real numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some real number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. The reason is because of the equality of the following terms:

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

https://doi.org/10.33774/coe-2023-2f203-v7 **ORCID:** https://orcid.org/0000-0001-8210-4126 Content not peer-reviewed by Cambridge University Press. Lic

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

3. Main Insight

This is the main insight.

Lemma 3.1. *The Riemann hypothesis is true whenever for each large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. By Lemma 2.1, if the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for each large enough prime number q_n, then there exists an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails. By Proposition 1.6, this is a contradiction with the fact that

$$\liminf_{n\to\infty} R(N_n) = \lim_{n\to\infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

By definition of the limit inferior for any positive real number ε, only a finite number of elements of $R(N_n)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This contradicts the existence of such previous infinite subsequence and thus, the Riemann hypothesis must be true. □

4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. By Lemma 3.1, the Riemann hypothesis is true if for all primes q_n (greater than some threshold), the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for some prime $q_{n'} > q_n$. That is the same as

$$\frac{\prod_{q\leq q_{n'}}\left(1+\frac{1}{q}\right)}{\log\theta(q_{n'})} \leq \frac{\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)}{\log\theta(q_n)}$$

and

$$\frac{\prod_{q\leq q_{n'}}\left(1+\frac{1}{q}\right)}{\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)} \leq \frac{\log\theta(q_{n'})}{\log\theta(q_n)}$$

https://doi.org/10.33774/coe-2023-2f203-v7 ORCID: https://orcid.org/0000-0001-8210-4126

which is

$$\log\log\theta(q_{n'}) - \log\log\theta(q_n) \geq \sum_{q_n < q \leq q_{n'}} \log\left(1+\frac{1}{q}\right)$$

after of applying the logarithm to the both sides and distribute the terms. That is equivalent to

$$1 - \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} \geq \frac{\sum_{q_n < q \leq q_{n'}} \log\left(1+\frac{1}{q}\right)}{\log\log\theta(q_{n'})}$$

after dividing both sides by $\log\log\theta(q_{n'})$. This is possible because of the prime number $q_{n'}$ is large enough and thus, the real number $\log\log\theta(q_{n'})$ would be greater than 0. We can apply the exponentiation to the both sides in order to obtain that

$$\exp\left(1 - \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right) \geq \left(\prod_{q_n < q \leq q_{n'}} \left(1+\frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}$$

and

$$e \geq \exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right) \cdot \left(\prod_{q_n < q \leq q_{n'}} \left(1+\frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}.$$

We can take an arbitrary large enough prime $q_{n'}$ such that

$$\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} \approx 0.$$

Certainly, we could have

$$\exp\left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right) \approx 1$$

for an arbitrary prime number $q_{n'}$ much greater than q_n. For large enough prime $q_{n'}$, we have

$$e = (\log\theta(q_{n'}))^{\frac{1}{\log\log\theta(q_{n'})}}$$

since $e = x^{\frac{1}{\log x}}$ for $x > 0$. Hence, it is enough to show that

$$\log\theta(q_{n'}) \gg \prod_{q_n < q \leq q_{n'}} \left(1+\frac{1}{q}\right),$$

where $\gg$ means "much greater than". That is equal to

$$e^{\gamma} \cdot \log\theta(q_{n'}) \gg e^{\gamma} \cdot \prod_{q_n < q \leq q_{n'}} \left(1+\frac{1}{q}\right).$$

By Proposition 1.7, we know that

$$e^{\gamma} \cdot \log\theta(q_{n'}) > \prod_{q \leq q_{n'}} \left(1+\frac{1}{q}\right).$$

So, we deduce that

$$1 \gg e^{\gamma} \cdot \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

which is trivially true since

$$\lim_{n \to \infty} \left(e^{\gamma} \cdot \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}\right) = 0.$$

This is because of

$$(\log \theta(q_n))^{-1} > \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know that

$$\lim_{n \to \infty} \left(e^{\gamma} \cdot (\log q_n)^{-1}\right) = 0$$

is true since

$$\theta(q_n) \sim q_n \quad as \quad (n \to \infty)$$

by Proposition 1.1. Actually, the point here is the statement

$$(\log \theta(q_n))^{-1} > \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

must be true for large enough n which is equal to say that $R(N_n) > 1$ holds indeed. By Proposition 1.6, there exists a value of m_0 so that for all natural numbers $m \geq m_0$

$$\liminf_{m \to \infty} R(N_m) - \epsilon = \frac{e^{\gamma}}{\zeta(2)} - \epsilon < R(N_m) < \frac{e^{\gamma}}{\zeta(2)} + \epsilon = \limsup_{m \to \infty} R(N_m) + \epsilon$$

for every arbitrary and absolute value $\epsilon > 0$ by definition of limit superior and inferior due to

$$\liminf_{m \to \infty} R(N_m) = \limsup_{m \to \infty} R(N_m) = \lim_{m \to \infty} R(N_m).$$

In this way, it should exist some value of n_0 so that for all natural numbers $n \geq n_0$ we obtain that $R(N_n) > 1$ since $\frac{e^{\gamma}}{\zeta(2)} > 1$. We would have

$$1 + \epsilon_1 = \exp\left(\frac{\log \log \theta(q_n)}{\log \log \theta(q_{n'})}\right)$$

and

$$e \cdot (1 - \epsilon_2) = \left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)\right)^{\frac{1}{\log \log \theta(q_{n'})}}.$$

We only need to prove that

$$e \geq (1 + \epsilon_1) \cdot e \cdot (1 - \epsilon_2)$$

which is the same as

$$\epsilon_2 \geq \frac{\epsilon_1}{\epsilon_1 + 1}.$$

https://doi.org/10.33774/coe-2023-2f203-v7 **ORCID:** https://orcid.org/0000-0001-8210-4126

We could check that

$$0 < \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} \leq \epsilon_1 \leq \frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} + \left(\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})}\right)^2 < 1$$

by Proposition 1.2 since $\frac{\log\log\theta(q_n)}{\log\log\theta(q_{n'})} < 1.79$. In addition, we can see that

$$1 - e^{-1} \cdot \left(\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}} = \epsilon_2$$

which is

$$1 - \left(\frac{\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)}{\log\theta(q_{n'})}\right)^{\frac{1}{\log\log\theta(q_{n'})}} = \epsilon_2$$

as we explained before. By Proposition 1.7, we know that

$$\frac{\prod_{q_n < q \leq q_{n'}} \left(1 + \frac{1}{q}\right)}{\log\theta(q_{n'})} < \frac{e^\gamma}{2 \cdot \zeta(2)}$$

for a sufficiently large prime q_n. As a consequence, we obtain that

$$1 - \left(\frac{e^\gamma}{2 \cdot \zeta(2)}\right)^{\frac{1}{\log\log\theta(q_{n'})}} < \epsilon_2.$$

Putting all together, we show that

$$1 - \left(\frac{e^\gamma}{2 \cdot \zeta(2)}\right)^{\frac{1}{\log\log\theta(q_{n'})}} \geq \frac{\epsilon_1}{\epsilon_1 + 1}$$

which is

$$1 \geq \frac{\epsilon_1}{\epsilon_1 + 1} + \left(\frac{e^\gamma}{2 \cdot \zeta(2)}\right)^{\frac{1}{\log\log\theta(q_{n'})}}.$$

That is equivalent to

$$1 \geq \frac{\epsilon_1}{\epsilon_1 + 1} + \left(1 + \left(\frac{e^\gamma}{2 \cdot \zeta(2)} - 1\right)\right)^{\frac{1}{\log\log\theta(q_{n'})}}.$$

and

$$0 \geq \frac{\epsilon_1}{\epsilon_1 + 1} + \left(\frac{e^\gamma}{2 \cdot \zeta(2)} - 1\right)^{\frac{1}{\log\log\theta(q_{n'})}}$$

by Proposition 1.3. Finally, we know that the right hand side of

$$0.545 > \frac{e^\gamma}{2 \cdot \zeta(2)}$$

is smaller than

$$1 + \left(-\frac{\epsilon_1}{\epsilon_1 + 1}\right)^{\log\log\theta(q_{n'})} \approx 1$$

as long as $q_{n'}$ gets larger and larger in relation to q_n under the consideration that $-1 < \left(-\frac{\epsilon_1}{\epsilon_1+1}\right) < 0$. Now, the proof is done. □

https://doi.org/10.33774/coe-2023-2f203-v7 **ORCID:** https://orcid.org/0000-0001-8210-4126 Content not peer-reviewed by Cambridge University Press. Lic

5. Conclusion

Practical uses of the Riemann hypothesis include many propositions that are considered to be true under the assumption of the Riemann hypothesis and some of them that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas.

References

[1] Ayoub, R.: Euler and the Zeta Function. The American Mathematical Monthly **81**(10), 1067–1086 (1974). https://doi.org/10.2307/2319041

[2] Broughan, K.: Euler's Totient Function, Encyclopedia of Mathematics and its Applications, vol. 1, pp. 94–143. Cambridge University Press (2017). https://doi.org/10.1017/9781108178228.007

[3] Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. Journal de Théorie des Nombres de Bordeaux **19**(2), 357–372 (2007). https://doi.org/10.5802/jtnb.591

[4] Hardy, G.H., Littlewood, J.E.: Some problems of diophantine approximation: Part II. The trigonometrical series associated with the elliptic ϑ-functions. Acta mathematica **37**(1), 193–239 (1914)

[5] Hardy, G.H., Littlewood, J.E.: Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Mathematica **41**, 119–196 (1916)

[6] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press (1988)

[7] Kozma, L.: Useful Inequalities. Kozma's Homepage, Useful inequalities cheat sheet. `http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf` (2011), Accessed 21 December 2023

[8] Nicolas, J.L.: Petites valeurs de la fonction d'Euler. Journal of Number Theory **17**(3), 375–388 (1983). https://doi.org/10.1016/0022-314X(83)90055-0

[9] Platt, D.J., Trudgian, T.S.: On the first sign change of $\theta(x)-x$. Mathematics of Computation **85**(299), 1539–1547 (2016). https://doi.org/10.1090/mcom/3021

[10] Solé, P., Planat, M.: Extreme values of the Dedekind ψ function. Journal of Combinatorics and Number Theory **3**(1), 33–38 (2011)

Frank Vega
NataSquad
10 rue de la Paix
FR 75002 Paris
France
e-mail: `vega.frank@gmail.com`