Incomplete Minkowski space as a source of gravitational interaction

Vladimir V.Komen
Independent researcher
vkomen@wit.ru

Introduction

The main idea of this paper is the violation of the completeness of the space-time continuum under the influence of gravitating bodies in it. The idea is borrowed from the notion of the optical density of the medium (eikonal), from where, in fact, the variational principles were introduced into physics. The formalism developed below allows us to explain from other positions the observed physical effects: gravitational time dilation, deflection of light rays in the gravitational field, precession of Newtonian orbits.

1. Mathematical prerequisites

Consider a dense subset $\overline{\mathbf{R}}^3$ of the Cartesian space $\mathbf{R}^3$, such that its complement to the complete space $\mathbf{R}^3$ has nonzero measure. Since the set of non-zero measure is "discarded" from the complete $\mathbf{R}^3$, when determining the lengths of curves via the curvilinear integral [Lebesgue] in such an incomplete space, we will obtain smaller values than in the complete $\mathbf{R}^3$.

In the following we will consider such dense subsets of Cartesian space $\mathbf{R}^3$ (or Minkowski space $\mathbf{R}^3_1$), in which the curvilinear Lebesgue integral is defined for any two points of the subset along any smooth path connecting them.

Let us introduce the function

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\vec{r},\vec{n}),$$

showing the "degree of incompleteness" of the space at the point $\vec{r}$ toward the direction of the unit vector $\vec{n}$. The function takes values in the interval [0, 1], with a value of 1 corresponding to a complete space. Assuming sufficient smoothness of the function $\boldsymbol{\Lambda}$, the length of the curve $\vec{r} = \vec{r}\,(t)$, given parametrically in the incomplete space $\overline{\mathbf{R}}^3$, can be expressed through the usual Riemann integral by the formula:

$$l = \int \boldsymbol{\Lambda}(\vec{r},\frac{\dot{\vec{r}}}{\dot{r}})\,\left|\overrightarrow{dr}\right|$$

We will consider the distance between two points in $\overline{\mathbf{R}}^3$ as the smallest value of the integral for all curves connecting points **a** and **b:**

$$l(a,b) = min \int \boldsymbol{\Lambda}(\vec{r},\frac{\dot{\vec{r}}}{\dot{r}})\,\left|\overrightarrow{dr}\right|$$

which leads to the variational problem for the integrand and the function $\boldsymbol{\Lambda}$.

Some properties of the incomplete space $\overline{\mathbf{R}}^3$

1. If along some curve between two points all the points of the complete space are discarded, or only a countable number of them are left, the distance between these points will be zero (as a minimum of the lengths of all possible distances), and it is natural to identify such points with each other. The same will happen to all points on the boundary of a continuous region thrown out of the complete space entirely (but it will no longer be an dense set in $\mathbf{R}^3$),

2. The notion of distance thus introduced turns our space into a metric space, i.e. axioms 1 - 3 of metric space will be satisfied (taking into account the identification of points whose distance is zero),

3. The definition of areas and volumes in such a space can be introduced in a way opposite to the way it is done in classical analysis. A common technique is to reduce area and volume integrals to repeated integrals (double and triple). In our case, we can take the repeated integrals as the initial definition. In this case, the requirement of independence of the area and volume values from the way of coordinate partitioning of the measured subset will lead to some restrictive conditions on the form of the function $\boldsymbol{\Lambda}(\vec{\boldsymbol{r}},\vec{\boldsymbol{n}})$, similar to the calibration conditions in physics.

2. Applications of incomplete space in physics.

A complete Cartesian space $\mathbf{R}^3$, Minkowski space or Riemannian space with metric tensor$\mathbf{g}_{\mathbf{ik}}$ is used as a model of space r space-time in physical theories.

Suppose that the incompleteness of the space-time continuum is due to the presence of gravitating bodies in it. The incompleteness is described by the function $\boldsymbol{\Lambda}$**,** which now depends also on the distribution of masses of gravitating bodies. The value of the function will asymptotically tend to 1 away from the gravitating bodies, and decrease as we approach them. The laws governing the dynamics in such a space will be:

1. Galileo's principle (dynamical principle or Newton's first law): in the absence of external forces, bodies move uniformly and linearly, and travel equal distances in equal time intervals (distances in the sense of the "metric" we have introduced). Gravity, as in GR, is not a force, and only affects the completeness of time-space. Thus, the "force of attraction" is the gradient of the incompleteness function,

2. Trajectory (variational) principle: bodies move along the shortest path in an incomplete Minkowski space $\overline{\mathbf{R}_1^3}$ with the notion of distance introduced earlier.

3. Derivation of general formulas of motion of a particle in a gravitational field

Lagrangian of a free particle of mass m in spherical coordinates

$$\mathrm{L} = -mc\int ds = -mc\int\sqrt{c^2 - \rho^2\left(\frac{d\varphi}{dt}\right)^2 - \ sin^2\theta(\frac{d\varphi}{dt})^2 - \left(\frac{d\rho}{dt}\right)^2}\,dt$$

Let us postulate for a particle in the gravitational field the following form of the square of the interval

$$ds^2 = (c^2dt^2 - \rho^2d\theta^2 - \rho^2sin^2\theta d\varphi^2 - d\rho^2) * \Lambda^2(\rho, M) \qquad \text{(I)}$$

where the function **Λ(ρ,M)** is a "measure of the incompleteness of space" caused by the presence of a gravitating body **M** located at the origin.

Let us perform the necessary calculations in a general form without specifying a particular kind of function **Λ(ρ,M)**

Let us write out the Hamilton-Jacobi equation for an interval of the form (I) in polar coordinates (Kepler problem):

$$\frac{1}{\Lambda^2c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \frac{1}{\Lambda^2}\left(\frac{\partial S}{\partial\rho}\right)^2 - \frac{1}{\Lambda^2\rho^2}\left(\frac{\partial S}{\partial\varphi}\right)^2 = m^2c^2$$

We will look for the solution in the form $S = -Et + \mathfrak{M}\varphi + S_\rho(\rho)$ with constants E and M and radial part of action S_ρ, and then

$$\frac{\partial S}{\partial t} = -E; \quad \frac{\partial S}{\partial\varphi} = \mathfrak{M}; \quad \frac{\partial S}{\partial\rho} = \frac{dS_\rho}{d\rho}$$

Substituting, we obtain

$$\frac{E^2}{\Lambda^2c^2} - \frac{1}{\Lambda^2}\left(\frac{dS_\rho}{d\rho}\right)^2 - \frac{\mathfrak{M}^2}{\Lambda^2\rho^2} = m^2c^2$$

And then

$$S_\rho = \int\sqrt{\frac{E^2}{c^2} - \frac{\mathfrak{M}^2}{\rho^2} - m^2c^2\Lambda^2}\ d\rho$$

Since $\frac{\partial S}{\partial E} = 0,\ t = \frac{\partial S_\rho}{\partial E}$, or

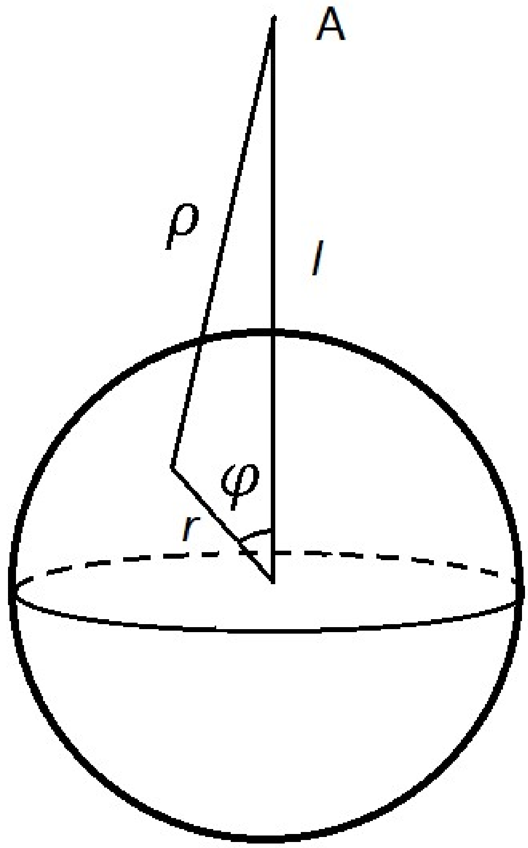
$$t = \int \frac{E d\rho}{c^2 \sqrt{\frac{E^2}{c^2} - \frac{\mathfrak{M}^2}{\rho^2} - m^2 c^2 \Lambda^2}}$$

And finally

$$\frac{d\rho}{dt} = \sqrt{c^2 - \frac{\mathfrak{M}^2 c^4}{E^2 \rho^2} - \frac{m^2 c^6}{E^2} \Lambda^2} \qquad \text{(II)}$$

4. Determining the form of function Λ

To determine the form of function **Λ**, let us ask how the addition of the "fields" of several gravitating bodies should take place in the formalism we are developing. Since the values of the function **Λ** are in the interval [0, 1] and larger masses should correspond to smaller values of **Λ**, it is logical to assume that the result of the joint action of several massive bodies should be the product of their incompleteness functions. This simple condition allows us to determine exactly the form of the function **Λ(ρ)**. Let us require, as in the classical Newtonian theory, that the gravitational field of a sphere of constant density is equal to the field of a material point with the same mass placed in the center of the sphere - not only asymptotically at infinity, but also at any distance from the center of the ball outside it.



Consider a sphere with radius **R** and constant density **σ**. We will be interested in the field of the sphere at the point A at a distance ***l*** from its center (***l* > *R***); for simplicity of calculations, the point A will lie on the Z axis of the spherical coordinate system.

Volume element of a sphere in spherical coordinates

$$dV = r^2 \sin\varphi \, dr \, d\theta \, d\varphi$$

Distance ***ρ*** from a point inside the sppere with coordinates (r,φ,θ) to the point A by the cosine theorem for an arbitrary triangle

$$\rho^2 = r^2 + l^2 - 2rl\cos\varphi$$

The contribution of the volume element ΔV to the "measure of incompleteness" of space at point A is equal to

$$\Delta\Lambda = \Lambda(\Delta m, \rho), \text{ где } \Delta m = \sigma \Delta V$$

The total field of all parts of the sphere is expressed by an infinite product over all elements of its volume and must be, by virtue of the assumption, equal to

$$\prod_{\Delta m} \Lambda(\Delta m, \rho) = \Lambda(\mathrm{M}, l)$$

where $\mathrm{M} = \sum \Delta m$- total mass of the sphere.

Let's pass from the infinite product to the infinite sum by putting

$$\Lambda(\Delta m, \rho) = e^{\lambda(\Delta m, \rho)}, \quad \prod \Lambda(\Delta m, \rho) = e^{\sum \lambda(\Delta m, \rho)}$$

and then the required equality will take the form

$$e^{\sum \lambda(\Delta m, \rho)} = e^{\lambda(\mathrm{M}, l)} \qquad \text{(III)}$$

Using the smallness of Δm, we write

$$\lambda(\Delta m, \rho) = \left.\frac{\partial \lambda}{\partial m}\right|_{m=0} \Delta m = \mu(\rho) \Delta m$$

and then the sum in the exponent of degree on the left side in (III) can be transformed to an integral in spherical coordinates

$$\sum \lambda(\Delta m, \rho) =$$

$$= \sigma \int_0^R dr \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \, r^2 \sin\varphi \, \mu(\rho) =$$

$$= 2\pi\sigma \int_0^r dr \int_0^{\pi} r^2 \mu(r^2 + l^2 - 2rl \cos\varphi) \sin\varphi \, d\varphi =$$

$$= 2\pi\sigma \int_0^R dr \int_0^{\pi} r^2 \mu(r^2 + l^2 - 2rl \cos\varphi) \, d(-\cos\varphi) =$$

$$= 2\pi\sigma \int_0^R \int_0^{\pi} \frac{r^2}{2rl} \mu(r^2 + l^2 - 2rl \cos\varphi) \, d(r^2 + l^2 - 2rl \cos\varphi) =$$

$$= \frac{\pi\sigma}{l} \int_0^R r dr \int_{(r-l)^2}^{(r+l)^2} \mu(x) dx$$

Here we substitute $x = r^2 + l^2 - 2\, rl \cos\varphi$ and recalculated the integration limits according to it.

Let $N(x)$ be primal function for $\mu(x)$, then the integral is transformed to

$$\frac{\pi\sigma}{l}\int_0^R [N((r+l)^2) - N((r-l)^2)]\, rdr = \lambda(M,l)$$

It is easy to see that this equality is satisfied at

$$N((r+l)^2) - N((r-l)^2) = 4\alpha r$$

where α is some constant. Indeed, we have in this case

$$\frac{\pi\sigma}{l}\int_0^R [N((r+l)^2) - N((r-l)^2)]\, rdr = \alpha\frac{4\pi\sigma}{l}\int_0^R r^2 dr = \alpha\frac{4\pi R^3}{3}\frac{\sigma}{l} = \frac{\alpha M}{l}$$

(we have the volume of the sphere, which, multiplied by its density, gives its total mass $M = \frac{4}{3}\pi R^3\sigma$)

and function is $N(x) = 2\alpha\sqrt{x}$, then $\mu(x)$

$$\mu(x) = M'(x) = \frac{\alpha}{\sqrt{x}} = \frac{\alpha}{\sqrt{r^2 + l^2 - 2rl\cos\varphi}} = \frac{\alpha}{\rho}$$

i.e., in the form it completely coincides with the right-hand side for the point mass field M. From here we also see that the dependence of the function **λ** in the exponent of degree on the mass is linear:

$$\lambda(M,l) = \frac{\alpha M}{l}$$

and then

$$\Lambda(M,\rho) = e^{\frac{\alpha M}{\rho}} \qquad \text{(IV)}$$

To determine the constant α, let us consider the simplest one-dimensional motion - the fall of a material point on a gravitational center.

Substituting (IV) into (II), taking into account one-dimensionality of the problem, we obtain

$$\frac{d\rho}{dt} = \sqrt{c^2 - \frac{m^2c^6}{E^2}e^{\frac{2\alpha M}{\rho}}}$$

Let's find the acceleration as a function of coordinate ρ

$$\frac{d^2\rho}{dt^2} = \frac{1}{2\sqrt{c^2 - \frac{m^2c^6}{E^2}e^{\frac{2\alpha M}{\rho}}}}\left(-\frac{m^2c^6}{E^2}\right)\left(-\frac{2\alpha M}{\rho^2}\right)e^{\frac{2\alpha M}{\rho}}\frac{d\rho}{dt}$$

or, substituting $\frac{d\rho}{dt}$

$$\frac{d^2\rho}{dt^2} = \frac{m^2c^6}{E^2}\frac{\alpha M}{\rho^2}\, e^{\frac{2\alpha M}{\rho}}$$

To coincide with Newton's law of gravity

$$F = m\frac{d^2\rho}{dt^2} = -\frac{GMm}{\rho^2}$$

in the first approximation it is enough to put

$$\alpha = -\frac{G}{c^2}$$

(We have also considered the formula for the energy of a particle starting motion from rest at infinity$E = mc^2$)

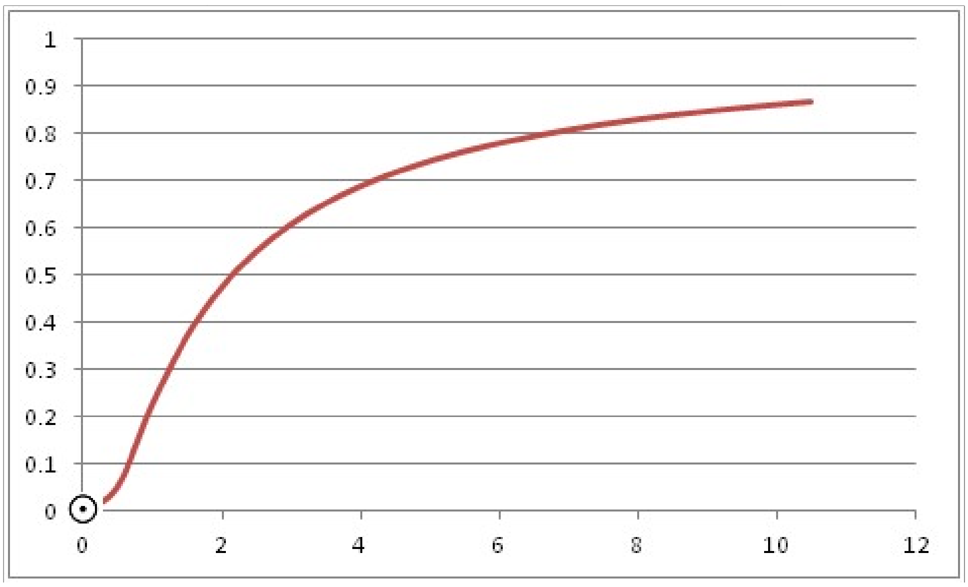
or, introducing the gravitational radius of the central body,

$$\rho_g = \frac{2GM}{c^2}$$

finally we obtain the form of the incompleteness function

$$\Lambda(M,\rho) = e^{-\frac{\rho_g}{2\rho}} \qquad \text{(V)}$$

The ratio $\frac{\rho_g}{\rho}$ is fortunately very small for most problems. For example, for Mercury, whose orbital precession was one of the proofs of the General Theory of Relativity, this ratio is equal to ~$5*10^{-8}$



The graph shows a view of the function **Λ** for a material point with mass equal to the mass of the Sun. The horizontal line marks kilometers (the gravitational radius of the Sun is ~ 3 km)

5. Determination of the angular dependence of the function Λ

We require that the form of the interval function in our formalism coincides to terms of the first order of $\frac{\rho_g}{\rho}$ with the Schwarzschild metric. For this condition to be satisfied, it is enough to put

$$\Lambda(M, \rho, n_t, n_\rho) = e^{-\frac{\rho_g}{2\rho}\left(n_t^2 + \frac{1}{1-\frac{\rho_g}{\rho}} n_\rho^2\right)}$$

The unit direction vector in Minkowski space-time is simply a 4-velocity vector, the projections we need (or rather, their squares) in spherical coordinates are:

$$n_t^2 = \frac{c^2}{c^2 - \rho^2(sin^2\theta\ (\frac{d\varphi}{dt})^2\ +\ (\frac{d\theta}{dt})^2) -\ (\frac{d\rho}{dt})^2}$$

$$n_\rho^2 = \frac{(\frac{d\rho}{dt})^2}{c^2 - \rho^2(sin^2\theta\ (\frac{d\varphi}{dt})^2\ +\ (\frac{d\theta}{dt})^2) -\ (\frac{d\rho}{dt})^2}$$

The decomposition by $\frac{\rho_g}{\rho}$ will give us the necessary corrections to the flat space-time interval:

$$ds^2 = (\ c^2dt^2 - \rho^2(sin^2\theta\ d\varphi^2 + d\theta^2) - d\rho^2) * \mathbf{\Lambda^2}\ \approx$$

$$\approx (\ c^2dt^2 - \rho^2(sin^2\theta\ d\varphi^2 + d\theta^2) - d\rho^2) * \\ * \left(1 - \frac{\rho_g}{\rho}\frac{c^2 + \frac{1}{1-\frac{\rho_g}{\rho}}\left(\frac{d\rho}{dt}\right)^2}{c^2 - \rho^2\left(sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2\right) - \left(\frac{d\rho}{dt}\right)^2}\right)$$

$$= c^2dt^2 - \rho^2(sin^2\theta\ d\varphi^2 + d\theta^2) - d\rho^2 - c^2\frac{\rho_g}{\rho}\ dt^2 - \frac{\frac{\rho_g}{\rho}}{1-\frac{\rho_g}{\rho}}d\rho^2 =$$

$$= (1 - \frac{\rho_g}{\rho})\ c^2dt^2 - \rho^2(sin^2\theta\ d\varphi^2 + d\theta^2)\ - \frac{d\rho^2}{1-\frac{\rho_g}{\rho}}$$

In the stationary case we return to the previous form (V) of the function $\mathbf{\Lambda}(\rho)$**.**

6. Calculation of physical phenomena in suggested formalism

Since in terms of the first order of $\frac{\rho_g}{\rho}$ the metric of the incomplete Minkowski space-time coincides with the Schwarzschild metric of curved space-time, we obtain exactly the same numerical values of the observed effects of GR, such as

- gravitational time dilation near gravitating bodies,
- precession of elliptical orbits,
- the deflection of a light ray in a gravitational field of a large mass

Differences in the terms of next orders of $\frac{\rho_g}{\rho}$ are of interest.

7. Elimination of divergences of classical electrodynamics

One of the ideas that inspired the present work was an attempt to eliminate the divergences of classical electrodynamics.

In the construction of quantum electrodynamics, the stumbling block was divergence, which Richard Feynman was able to circumvent with the help of renormalization mechanism. The problem is that elementary particles should be considered as point ones, since by definition they cannot have any internal structure, and then any quantity of the form 1/r, such as the electric charge field potential, will give in zero an infinite value, a divergence.

The proposed formalism also allows to get rid of divergences of classical electrodynamics. The integral on the volume of 3-dimensional space should be added a multiplier $\mathbf{\Lambda^3}$which effectively negates the infinity at the origin from the multiplier $\frac{1}{\rho^2}$ (or any other power of $1/\rho$).

Let us calculate the total energy of the electron determined by its electric field

$$U = \frac{1}{8\pi} \int E^2 dV$$

In the developed formalism for a point particle

$$U = \frac{1}{8\pi} \int E^2 \Lambda^3(m, \rho) \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

For an electron, the field intensity it generates

$$E = \frac{e}{\rho^2}$$

And then

$$U = \frac{e^2}{2}\int_0^\infty \frac{1}{\rho^2}\Lambda^3(M,\rho)d\rho$$

Thus we have the integral

$$U = \frac{e^2}{2}\int_0^\infty \frac{1}{\rho^2} e^{-\frac{3\rho_g}{2\rho}} d\rho = \frac{e^2}{3\rho_g}\int_0^\infty e^{-\frac{3\rho_g}{2\rho}} d\left(-\frac{3\rho_g}{2\rho}\right) = \frac{e^2}{3\rho_g}$$

It is not clear whether the gravitational radius of the electron can be considered physical - this value is many orders of magnitude smaller than the Planck length.

The field energy of the electron:

4.76*10^{13} erg – if Planck length is used as ρ_g

5.62*10^{35} erg – if true gravitational radius of the electron is used

Which is slightly greater than the total rest energy of the electron equal to 8*10^{-7} erg

Acknowledgments

I would like to thank

Tikhonenkov Igor E., itikhonen@hotmail.com,

PhD in physics,

Physicist & algorithm developer,

Fourier optics, Israel,

for reviewing and helping with this paper.