# On Dedekind Function for the Riemann Hypothesis

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*To my mother*

**Abstract.** The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. On the one hand, the Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of $n$, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind $\Psi$ function where $q \mid n$ means the prime $q$ divides $n$. We require the properties of primorial numbers, that is to say the primorial of order $n$ as $N_n = 2 \cdot \ldots \cdot q_n$. On the other hand, Solé and Planat criterion states that the Riemann hypothesis is true if and only if the inequality $\zeta(2) \cdot \frac{\Psi(N_n)}{N_n} > e^{\gamma} \cdot \log \theta(q_n)$ holds for all prime numbers $q_n > 3$, where $\theta(x)$ is the Chebyshev function and $\zeta(x)$ is the Riemann zeta function. In this note, using both inequalities on primorial and prime numbers, we prove that the Riemann hypothesis is true.

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## 1. Introduction

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers $q$ that are less than or equal to $x$, where log is the natural logarithm.

**Proposition 1.1.** *For $x > 1$* [9, Theorem 4 (3.15) pp. 70]*:*

$$\theta(x) < x \cdot \left(1 + \frac{1}{2 \cdot \log x}\right).$$

We know the following inequality on natural logarithm:

**Proposition 1.2.** *For $x > -1$* [5, pp. 1]*:*

$$\log(1 + x) \leq x.$$

Leonhard Euler profoundly studied the Riemann zeta function (1734) [1].

**Proposition 1.3.** *For $s > 1$, we can formulate the Riemann zeta function as* [1, (1) pp. 1070]*:*

$$\zeta(s) = \prod_{k=1}^{\infty} \frac{q_k^s}{q_k^s - 1},$$

*where $q_k$ is the kth prime number (We also use the notation $q_n$ to denote the nth prime number). By definition, we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

*where $n$ denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

*where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc. In addition, it is well-known the value of the Apéry's constant*

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \prod_{k=1}^{\infty} \frac{q_k^3}{q_k^3 - 1} \approx 1.20205$$

*which is an irrational number.*

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right) = \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constants $B$ and $H$ (1874) [6]. We define $H = \gamma - B$ such that $B \approx 0.26149$ is the Meissel-Mertens constant [6].

**Proposition 1.4.** *We have* [2, Lemma 2.1 (1) pp. 359]*:*

$$\sum_{k=1}^{\infty} \left( \log \left( \frac{q_k}{q_k - 1} \right) - \frac{1}{q_k} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows [7, pp. 379]:

$$u(x) = \sum_{q > x} \left( \log \left( \frac{q}{q-1} \right) - \frac{1}{q} \right).$$

**Proposition 1.5.** *We have* [7, (11) pp. 379]*:*

$$0 < u(x) \leq \frac{1}{2 \cdot (x-1)}.$$

On the sum of the reciprocals of all prime numbers not exceeding $x$, we have:

**Proposition 1.6.** *For* $x \geq 2278383$ [3, Theorem 5.6 (1) pp. 243]*:*

$$-\frac{0.2}{\log^3 x} \leq \sum_{q \leq x} \frac{1}{q} - B - \log \log x \leq \frac{0.2}{\log^3 x}$$

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind $\Psi$ function, where $q \mid n$ means the prime $q$ divides $n$.

**Definition 1.7.** We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

A natural number $N_n$ is called a primorial number of order $n$ precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ is satisfied.

**Proposition 1.8.** $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann hypothesis is true* [10, Theorem 4.2 pp. 5].

Unconditionally on Riemann hypothesis, we know that:

**Proposition 1.9.** *For all prime numbers* $q_n > 5$ [2, Theorem 1.1 pp. 358]*:*

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) < e^{\gamma} \cdot \log \theta(q_n).$$

As usual $\sigma(n)$ is the sum-of-divisors function of $n$

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer $d$ divides $n$. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

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**Proposition 1.10.** *Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of $n$ as a product of prime numbers $q_1 < \ldots < q_r$ with natural numbers $a_1, \ldots, a_r$ as exponents. Then* [4, Lemma 1 pp. 2],

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

**Definition 1.11.** We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n.$$

The Ramanujan's Theorem states that if the Riemann hypothesis is true, then the previous inequality holds for large enough $n$. Next, we have the Robin's Theorem:

**Proposition 1.12.** $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true* [8, Theorem 1 pp. 188].

Unconditionally on Riemann hypothesis, we also know that:

**Proposition 1.13.** *For all natural numbers $n \geq 5$* [2, Lemma 3.2 pp. 363]*:*

$$\prod_{q \mid N_n} \frac{q}{q-1} < e^{\gamma} \cdot \log \log N_n^2.$$

Putting all together yields a proof for the Riemann hypothesis.

## 2. Central Lemma

The following is a key Lemma.

**Lemma 2.1.** *Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order $n$ for some prime $q_n > 2278383$. Then:*

$$\frac{e^{\gamma}}{\zeta(2)} \leq R(N_n^2) \cdot \frac{\sigma(N_n^2)}{\Psi(N_n^2)}.$$

*Proof.* We know that

$$R(N_n^2) \cdot \frac{\sigma(N_n^2)}{\Psi(N_n^2)} = \frac{f(N_n^2)}{\log \log N_n^2}$$

by properties of these special functions. That would be

$$\frac{e^{\gamma}}{\zeta(2)} \cdot \log \log N_n^2 \leq f(N_n^2)$$

which is

$$\gamma - \log(\zeta(2)) + \log \log \log N_n^2 \leq \log f(N_n^2).$$

So, we would have

$$\gamma - \log(\zeta(2)) + \log \log \left(2 \cdot \theta(q_n)\right) \leq \left(\sum_{i=1}^{n} \log \left(\frac{q_i}{q_i - 1}\right)\right) + \sum_{i=1}^{n} \log \left(1 - \frac{1}{q_i^3}\right)$$

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by Proposition 1.10. In addition, we have

$$\begin{aligned}
\log\log\left(2\cdot\theta(q_n)\right) &\leq \log\log\left(2\cdot q_n\cdot\left(1+\frac{1}{2\cdot\log q_n}\right)\right)\\
&= \log\left(\log q_n+\log\left(1+1+\frac{1}{\log q_n}\right)\right)\\
&= \log\left((\log q_n)\cdot\left(1+\frac{\log\left(1+1+\frac{1}{\log q_n}\right)}{\log q_n}\right)\right)\\
&= \log\log q_n+\log\left(1+\frac{\log\left(1+1+\frac{1}{\log q_n}\right)}{\log q_n}\right)\\
&\leq \log\log q_n+\frac{\log\left(1+1+\frac{1}{\log q_n}\right)}{\log q_n}\\
&\leq \log\log q_n+\frac{1}{\log q_n}+\frac{1}{\log^2 q_n}
\end{aligned}$$

by Propositions 1.1 and 1.2. Besides, we deduce

$$\begin{aligned}
\sum_{i=1}^{n}\log\left(1-\frac{1}{q_i^3}\right) &= -\sum_{i=1}^{n}\log\left(\frac{q_i^3}{q_i^3-1}\right)\\
&\geq -\sum_{i=1}^{\infty}\log\left(\frac{q_i^3}{q_i^3-1}\right)\\
&= -\log(\zeta(3))
\end{aligned}$$

by Proposition 1.3. Consequently, we obtain that

$$\gamma-\log(\zeta(2))+\log\log q_n+\frac{1}{\log q_n}+\frac{1}{\log^2 q_n}\leq\left(\sum_{i=1}^{n}\log\left(\frac{q_i}{q_i-1}\right)\right)-\log(\zeta(3))$$

which is

$$\gamma-\log(\zeta(2))+\log\log q_n+\frac{1}{\log q_n}+\frac{1}{\log^2 q_n}\leq\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)+H-u(q_n)-\log(\zeta(3))$$

by Proposition 1.4. We would obtain

$$B-\log(\zeta(2))+\log\log q_n+\frac{1}{\log q_n}+\frac{1}{\log^2 q_n}+\frac{1}{2\cdot(q_n-1)}+\log(\zeta(3))\leq\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)$$

by Propositions 1.4 and 1.5. Therefore, the inequality

$$B-\log(\zeta(2))+\log\log q_n+\frac{1}{\log q_n}+\frac{1}{\log^2 q_n}+\frac{1}{2\cdot(q_n-1)}+\log(\zeta(3))\leq\left(\sum_{i=1}^{n}\frac{1}{q_i}\right)$$

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is satisfied since
$$-\frac{0.2}{\log^3 q_n} \geq -\log(\zeta(2)) + \frac{1}{\log q_n} + \frac{1}{\log^2 q_n} + \frac{1}{2 \cdot (q_n - 1)} + \log(\zeta(3))$$
holds for $q_n > 2278383$ by Proposition 1.6. □

## 3. Main Insight

This is the main insight.

**Lemma 3.1.** $\mathsf{Dedekind}(q_n)$ *holds for every prime* $q_n > 2278383$ *whenever* $\mathsf{Robin}(N_n^2)$ *holds.*

*Proof.* Under the assumption that $\mathsf{Robin}(N_n^2)$ holds, we check that
$$f(N_n^2) < e^\gamma \cdot \log\log N_n^2.$$
That is the same as
$$f(N_n^2) < (e^\gamma \cdot \log\log N_n) \cdot \frac{\log\log N_n^2}{\log\log N_n}.$$
We know that
$$\frac{\log\log N_n^2}{\log\log N_n} = \frac{R(N_n)}{R(N_n^2)}.$$
Moreover, we verify that
$$(e^\gamma \cdot \log\log N_n) > \frac{\Psi(N_n^2)}{N_n^2}$$
by Proposition 1.9. Putting all together, we show that
$$R(N_n) > R(N_n^2) \cdot \frac{\sigma(N_n^2)}{\Psi(N_n^2)}.$$
Hence, it is enough to prove that
$$\frac{e^\gamma}{\zeta(2)} \leq R(N_n^2) \cdot \frac{\sigma(N_n^2)}{\Psi(N_n^2)}$$
holds according to the Lemma 2.1. □

## 4. Main Theorem

This is the main theorem.

**Theorem 4.1.** *The Riemann hypothesis is true.*

*Proof.* We could check by a numerical computation that $\mathsf{Dedekind}(q_n)$ holds for every prime $2278383 \geq q_n > 3$. Furthermore, we can prove that $\mathsf{Dedekind}(q_n)$ holds for every prime $q_n > 2278383$ as a direct consequence of Proposition 1.13 and Lemma 3.1. Certainly, we deduce that
$$f(N_n^2) < \prod_{q \mid N_n} \frac{q}{q-1}$$

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by Proposition 1.10. Finally, we can state that $\mathsf{Dedekind}(q_n)$ holds for all primes $q_n > 3$. By Proposition 1.8, we conclude that the Riemann hypothesis is true. □

## 5. Conclusion

The Riemann hypothesis's importance remains from its deep connection to the distribution of prime numbers, which are essential in many computational and theoretical aspects of mathematics. Understanding the distribution of prime numbers is crucial for developing efficient algorithms and improving our understanding of the fundamental structure of numbers. Besides, the Riemann hypothesis stands as a testament to the power and allure of mathematical inquiry. It challenges our understanding of the fundamental structure of numbers, inspiring mathematicians to push the boundaries of their field and seek ever deeper insights into the universe of mathematics. Indeed, the Riemann hypothesis has far-reaching implications for mathematics, with potential applications in cryptography, number theory, and even particle physics. Certainly, a proof of the hypothesis would not only provide a profound insight into the nature of prime numbers but also open up new avenues of research in various mathematical fields.

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