NUMERICAL COMPUTATIONS ON PRIME NUMBERS

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To my mother

ABSTRACT. Let $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $N_n = 2 \cdot \ldots \cdot q_n$ be the primorial of order n. We state that if the inequality $R(N_{n+1}) < R(N_n)$ holds for all primes q_n (greater than some threshold), then the Riemann hypothesis is true and the Cramér's conjecture is false. In this note, we prove that the previous inequality always holds for all sufficiently large primes q_n. This manuscript was based on numerical computations which confirm and support the truthfulness of this mathematical result.

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. The hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, there have been several developments that have brought us closer to a proof of the Riemann hypothesis. There are many approaches to the Riemann hypothesis based on analytic number theory, algebraic geometry, non-commutative geometry, etc.

The Riemann zeta function $\zeta(s)$ is a function under the domain of complex numbers. It has zeros at the negative even integers: These are called the trivial zeros. The zeta function is also zero for other values of s, which are called nontrivial zeros. The Riemann hypothesis is concerned with the locations of these nontrivial zeros. Bernhard Riemann conjectured that the real part of every nontrivial zero of the Riemann zeta function is $\frac{1}{2}$.

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The Riemann hypothesis's importance remains from its deep connection to the distribution of prime numbers, which are essential in many computational and theoretical aspects of mathematics. Understanding the distribution of prime numbers is crucial for developing efficient algorithms and improving our understanding of the fundamental structure of numbers. Besides, the Riemann hypothesis stands as a testament to the power and allure of mathematical inquiry. It challenges our understanding of the fundamental structure of numbers, inspiring mathematicians to push the boundaries of their field and seek ever deeper insights into the universe of mathematics.

A prime gap is the difference between two successive prime numbers. The nth prime gap is the difference between the (n + 1)st and the nth prime numbers, i.e. $q_{n+1} - q_n$. The Cramér's conjecture states that $q_{n+1} - q_n = O((\log q_n)^2)$, where O is big O notation and log is the natural logarithm. This conjecture was formulated by the Swedish mathematician Harald Cramér in 1936. Nowadays, many mathematicians believe that the Cramér's conjecture is false.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x. We use the following inequalities:

Proposition 1.1. *For every $x > 1$* [9, Theorem 4 (3.15) pp. 70]*:*

$$\theta(x) < \left(1 + \frac{1}{2 \cdot \log x}\right) \cdot x.$$

Proposition 1.2. *For all $x \geq 3$* [8, (5) pp. 378]*:*

$$\log \log \theta(x) \leq \log \log x + \frac{\theta(x) - x}{x \cdot \log x}.$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [1].

Proposition 1.3. *We define* [1, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number (We also use the notation q_n to denote the nth prime number). By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right)$$
$$= \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Franz Mertens discovered some important results about the constants B and H (1874) [7]. We define $H = \gamma - B$ such that $B \approx 0.26149$ is the Meissel-Mertens constant [7].

Proposition 1.4. *We have* [2, Lemma 2.1 (1) pp. 359]*:*

$$\sum_{k=1}^{\infty} \left(\log \left(\frac{q_k}{q_k - 1} \right) - \frac{1}{q_k} \right) = \gamma - B = H.$$

Proposition 1.5. *On the sum of the reciprocals of all prime numbers not exceeding x, we have for $x \geq 2278383$* [4, Theorem 5.6 (1) pp. 243]*:*

$$-\frac{0.2}{\log^3 x} \leq \sum_{q \leq x} \frac{1}{q} - B - \log \log x \leq \frac{0.2}{\log^3 x}.$$

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n.

Definition 1.6. We say that $\mathsf{Dedekind}(q_n)$ holds provided that

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) \geq \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(q_n).$$

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} q_k.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$. $\mathsf{Dedekind}(q_n)$ holds if and only if $R(N_n) \geq \frac{e^{\gamma}}{\zeta(2)}$ is satisfied.

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Proposition 1.7. *Unconditionally on Riemann hypothesis, we know that* [10, Proposition 3 pp. 3]*:*

$$\lim_{n \to \infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.8. *The inequality $R(N_n) > R(N_{n+1})$ is violated for infinitely many n's under the assumption that the Cramér's conjecture is true* [3, Proposition 4 pp. 5]*,* [3, Proposition 7 pp. 7]*.*

Proposition 1.9. *For all prime numbers $q_n > 5$* [2, Theorem 1.1 pp. 358]*:*

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) < e^{\gamma} \cdot \log \theta(q_n).$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [5]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [6]:

$$f(x) = \Omega_R(g(x)) \text{ as } x \to \infty \text{ if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$
$$f(x) = \Omega_L(g(x)) \text{ as } x \to \infty \text{ if } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_{\pm}(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as:

$$f(x) = \Omega_+(g(x)) \text{ if } \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [8, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Finally, we have the Nicolas Theorem:

Proposition 1.10. *If the Riemann hypothesis is false then there exists a real b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$* [8, Theorem 3 (c) pp. 376]*:*

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Putting all together yields two breakthrough results on prime numbers.

2. Auxiliary Lemmas

The following are helpful Lemmas.

Lemma 2.1.

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k}-\log\left(1+\frac{1}{q_k}\right)\right)=\log(\zeta(2))-H.$$

Proof. We obtain that

$$\begin{aligned}
\log(\zeta(2))-H &= \log\left(\prod_{k=1}^{\infty}\frac{q_k^2}{q_k^2-1}\right)-H\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k^2}{(q_k^2-1)}\right)\right)-H\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k^2}{(q_k-1)\cdot(q_k+1)}\right)\right)-H\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)+\log\left(\frac{q_k}{q_k+1}\right)\right)-H\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)-\log\left(\frac{q_k+1}{q_k}\right)\right)-H\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)-\log\left(1+\frac{1}{q_k}\right)\right)-\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)-\frac{1}{q_k}\right)\\
&=\sum_{k=1}^{\infty}\left(\log\left(\frac{q_k}{q_k-1}\right)-\log\left(1+\frac{1}{q_k}\right)-\log\left(\frac{q_k}{q_k-1}\right)+\frac{1}{q_k}\right)\\
&=\sum_{k=1}^{\infty}\left(\frac{1}{q_k}-\log\left(1+\frac{1}{q_k}\right)\right)
\end{aligned}$$

by Propositions 1.3 and 1.4. □

Lemma 2.2. *The inequality* $\sqrt{1.25}\geq R(N_n)>1$ *holds for all primes* $q_n>10^8$.

Proof. The inequality $R(N_n)>1$ is the same as

$$\sum_{q\leq q_n}\log\left(1+\frac{1}{q}\right)>\log\log\theta(q_n)$$

after of applying the logarithm. That would be

$$\sum_{q\leq q_n}\frac{1}{q}>\log\log\theta(q_n)+\sum_{q\leq q_n}\left(\frac{1}{q}-\log\left(1+\frac{1}{q}\right)\right).$$

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By Lemma 2.3, we check that

$$\sum_{q \leq q_n} \frac{1}{q} > \log\log\theta(q_n) + \log(\zeta(2)) - H.$$

By Proposition 1.5, we have

$$\log\log q_n + B - \frac{0.2}{\log^3 q_n} > \log\log\theta(q_n) + \log(\zeta(2)) - H.$$

By Proposition 1.4, we know that

$$\gamma - \log(\zeta(2)) - \frac{0.2}{\log^3 q_n} > \log\log\theta(q_n) - \log\log q_n.$$

By Proposition 1.2, we notice that

$$\gamma - \log(\zeta(2)) - \frac{0.2}{\log^3 q_n} > \frac{\theta(q_n) - q_n}{q_n \cdot \log q_n}.$$

In this way, we obtain that

$$\left(\gamma - \log(\zeta(2)) - \frac{0.2}{\log^3 q_n}\right) \cdot \log q_n > \frac{1}{2 \cdot \log q_n}$$

since

$$\theta(q_n) - q_n = \left(1 + \frac{1}{2 \cdot \log q_n}\right) \cdot q_n - q_n = \frac{q_n}{2 \cdot \log q_n}$$

by Proposition 1.1. Indeed, the inequality

$$\left(\gamma - \log(\zeta(2)) - \frac{0.2}{\log^3 q_n}\right) \cdot \log q_n > \frac{1}{2 \cdot \log q_n}$$

trivially holds since

$$\left(\gamma - \log(\zeta(2)) - \frac{0.2}{\log^3 q_n}\right) \cdot \log q_n > 1.46330$$

and

$$1 > \frac{1}{2 \cdot \log q_n}$$

for all primes $q_n > 10^8$. Moreover, we know that $\sqrt{1.25} \geq R(N_n)$ because of

$$\frac{\log 1.25}{2} \gg \left(\gamma - \log(\zeta(2)) + \frac{0.2}{\log^3 q_n}\right)$$

by Proposition 1.5 for all primes $q_n > 10^8$. Here, the symbol $\gg$ means "much greater than". $\square$

Lemma 2.3. *For* $1.25 \geq y > x > 1$*:*

$$\frac{y}{x} \geq \frac{\log y}{\log x}.$$

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Proof. We rewrite the above expression as

$$(\log(y) - \frac{y}{x} \cdot \log(x)) \leq 0.$$

We need to show the function

$$f(x, y) = (\log(y) - \frac{y}{x} \cdot \log(x))$$

implies that $f(x, y) \leq 0$ holds for $1.25 \geq y > x > 1$. Let's plot the function $f(x, y)$ (This is the Python code):

```
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(1, 4, 0.1)
y = np.arange(1, 4, 0.1)

z = np.zeros((len(x), len(y)))

for i in range(len(x)):
    for j in range(len(y)):
        z[i, j] = np.log(y[j]) - (y[j] / x[i]) * np.log(x[i])

plt.imshow(z, extent=(1, 4, 1, 4), cmap='plasma')
plt.colorbar()
plt.xlabel('x')
plt.ylabel('y')
plt.title('f(x, y) = (log(y) - (y/x) log(x))')
plt.show()
```

which creates the following graph, where we notice our desired result. ☐

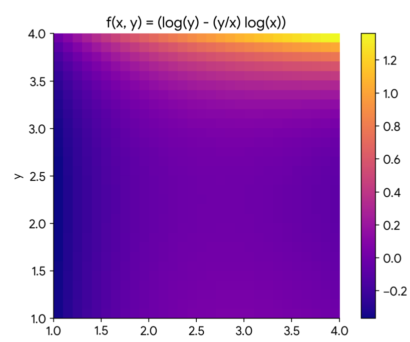

FIGURE 1. Plot of the function $f(x, y) = (\log(y) - \frac{y}{x} \cdot \log(x))$.

3. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. Nevertheless, there

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exist some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 3.1. *If the Riemann hypothesis is false, then there exist infinitely many prime numbers q_n such that* $\mathsf{Dedekind}(q_n)$ *fails (i.e.* $\mathsf{Dedekind}(q_n)$ *does not hold).*

Proof. The function g is defined as [10, Theorem 4.2 pp. 5]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We claim that $\mathsf{Dedekind}(q_n)$ fails whenever there exists some real number $x_0 \geq 5$ for which $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ and q_n is the greatest prime number such that $q_n \leq x_0$. It was proven the following bound [10, Theorem 4.2 pp. 5]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}.$$

By Proposition 1.10, if the Riemann hypothesis is false, then there is a real number $0 < b < \frac{1}{2}$ such that there exist infinitely many numbers x for which $\log f(x) = \Omega_+(x^{-b})$. Actually Nicolas proved that $\log f(x) = \Omega_{\pm}(x^{-b})$, but we only need to use the notation Ω_+ under the domain of the real numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{R}, \exists y \in \mathbb{R}\ (y > y_0)\colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the value of k is not necessary in the statement above. In this way, if the Riemann hypothesis is false, then there exist infinitely many wide apart numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{1}{\sqrt{x_0}} > \frac{2}{x_0}$ for $x_0 \geq 5$, then it would be infinitely many wide apart real numbers x_0 such that $\log g(x_0) > 0$. In addition, if $\log g(x_0) > 0$ for some real number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. The reason is because of the equality of the following terms:

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

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4. New Criterion

This is a new Criterion for the Riemann hypothesis.

Lemma 4.1. *The Riemann hypothesis is true whenever for each large enough prime number q_n, there exists another prime $q_{n'} > q_n$ such that*

$$R(N_{n'}) \leq R(N_n).$$

Proof. By Lemma 3.1, if the Riemann hypothesis is false and the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for each large enough prime number q_n, then there exists an infinite subsequence of natural numbers n_i such that

$$R(N_{n_{i+1}}) \leq R(N_{n_i}),$$

$q_{n_{i+1}} > q_{n_i}$ and $\mathsf{Dedekind}(q_{n_i})$ fails. By Proposition 1.7, this is a contradiction with the fact that

$$\liminf_{n\to\infty} R(N_n) = \lim_{n\to\infty} R(N_n) = \frac{e^{\gamma}}{\zeta(2)}.$$

By definition of the limit inferior for any positive real number ε, only a finite number of elements of $R(N_n)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This contradicts the existence of such previous infinite subsequence and thus, the Riemann hypothesis must be true. □

5. Main Insight

This is the main insight.

Theorem 5.1. *The inequality $R(N_n) > R(N_{n+1})$ holds for all primes q_n (greater than some threshold).*

Proof. For all primes q_n (greater than some threshold), we need to prove that the inequality

$$R(N_{n+1}) < R(N_n)$$

is satisfied. Numerical computations confirm the inequality holds for all $10^8 \geq q_n \geq 5$. For an arbitrary prime number $q_n > 10^8$, we can assume that

$$R(N_n) < R(N_{n-1})$$

to show that

$$R(N_{n+1}) < R(N_n)$$

using a proof by complete induction. In this way, we have

$$R(N_{n+1}) \cdot R(N_{n-1}) < R(N_n) \cdot R(N_{n-1})$$

which is

$$R(N_{n+1}) \cdot R(N_{n-1}) \leq R(N_n)^2$$

under our assumption. We would have

$$\log R(N_{n+1}) \cdot R(N_{n-1}) \leq \log R(N_n)^2$$

and

$$\frac{\log R(N_{n+1}) \cdot R(N_{n-1})}{\log R(N_n)} \leq 2$$

since

$$\log R(N_n) > 0$$

by Lemma 2.2. By Lemma 2.3, we can see that

$$\frac{\log R(N_{n+1}) \cdot R(N_{n-1})}{\log R(N_n)} \leq \frac{R(N_{n+1}) \cdot R(N_{n-1})}{R(N_n)}.$$

Certainly, we know that

$$1.25 \geq R(N_{n+1}) \cdot R(N_{n-1}) > R(N_n) > 1$$

jut putting together the assumptions

$$1 > \frac{R(N_n)}{R(N_{n-1})}$$

and

$$\sqrt{1.25} \geq R(N_{n+1}) > 1$$

by Lemma 2.2. Finally, we obtain the following inequality:

$$\frac{R(N_{n+1}) \cdot R(N_{n-1})}{R(N_n)} \leq 2$$

which is

$$\frac{R(N_{n+1})}{R(N_n)} \leq \frac{2}{e^{\gamma}}$$

by Proposition 1.9 since

$$R(N_{n-1}) < e^{\gamma}.$$

That is the same as

$$\frac{\log \theta(q_n)}{\log \theta(q_{n+1})} \leq \frac{2}{e^{\gamma}} \cdot \left(1 - \frac{1}{q_{n+1}+1}\right).$$

For $q_n > 10^8$, we can verify that

$$\left(1 - \frac{1}{q_{n+1}+1}\right) > \left(1 - \frac{1}{10^8}\right).$$

However, we know the inequality

$$\frac{\log \theta(q_n)}{\log \theta(q_{n+1})} \leq \frac{2}{e^{\gamma}} \cdot \left(1 - \frac{1}{10^8}\right)$$

trivially holds because of

$$\frac{\log \theta(q_n)}{\log \theta(q_{n+1})} < 1$$

and

$$\frac{2}{e^{\gamma}} \cdot \left(1 - \frac{1}{10^8}\right) > \frac{2}{1.7811} \cdot \left(1 - \frac{1}{10^8}\right) > 1.12290$$

and therefore, the proof is done. $\square$

6. Main Theorem

This is the main theorem.

Theorem 6.1. *The Riemann hypothesis is true and the Cramér's conjecture is false.*

Proof. By Lemma 4.1, the Riemann hypothesis is true if for all primes q_n (greater than some threshold), the inequality

$$R(N_{n'}) \leq R(N_n)$$

is satisfied for some prime $q_{n'} > q_n$. Therefore, the Riemann hypothesis is true by Theorem 5.1. We also know the the Cramér's conjecture is false as a consequence of Proposition 1.8 and Theorem 5.1. □

7. Conclusion

On the one hand, the Riemann hypothesis has far-reaching implications for mathematics, with potential applications in cryptography, number theory, and even particle physics. Certainly, a proof of the hypothesis would not only provide a profound insight into the nature of prime numbers but also open up new avenues of research in various mathematical fields. On the other hand, our proof of the untruthfully Cramér's conjecture could spur considerable advances in number theory as well.

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