

# On the sum of reciprocals of primes

Young Deuk Kim  
Faculty of Liberal Education  
Seoul National University  
Seoul 08826, Korea  
(yk274@snu.ac.kr)

January 8, 2024

## Abstract

Suppose that  $y > 0$ ,  $0 \leq \alpha < 2\pi$  and  $0 < K < 1$ . Let  $P^+$  be the set of primes  $p$  such that  $\cos(y \ln p + \alpha) > K$  and  $P^-$  the set of primes  $p$  such that  $\cos(y \ln p + \alpha) < -K$ . In this paper we prove  $\sum_{p \in P^+} \frac{1}{p} = \infty$  and  $\sum_{p \in P^-} \frac{1}{p} = \infty$ .

2020 Mathematics Subject Classification ; 11N05.

## 1 Introduction

Let  $P$  be the set of primes and  $\mathbb{N}$  be the set of natural numbers. In 1737, Euler[2] proved the sum of reciprocals of primes is divergent.

$$\sum_{p \in P} \frac{1}{p} = \infty$$

**Definition 1.1.** Suppose that  $y > 0$ ,  $0 \leq \alpha < 2\pi$  and  $0 < K < 1$ . Let

$$P^+(y, \alpha, K) = \{p \in P \mid \cos(y \ln p + \alpha) > K\}$$

and

$$P^-(y, \alpha, K) = \{p \in P \mid \cos(y \ln p + \alpha) < -K\}.$$

We write  $P^+$  and  $P^-$  for the sake of simplicity.

Throughout this paper we always assume that  $y > 0$ . In this paper we prove

**Theorem 1.2.**

$$\sum_{p \in P^+} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in P^-} \frac{1}{p} = \infty.$$

## 2 Proof of Theorem 1.2

We will use the prime number theorem in the proof of Theorem 1.2.

**Prime Number Theorem** ([1, 3]). *Let  $\pi(x)$  be the number of primes less than or equal to  $x$ . Then*

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x/\ln x} = 1.$$

**Lemma 2.1.** *Recall that  $y > 0$ . Let  $0 \leq \gamma < 2\pi$ . There are at most two primes  $p$  such that*

$$y \ln p = 2n\pi + \gamma$$

*for some  $n \in \mathbb{N} \cup \{0\}$ .*

*Proof.* Suppose that there exist three distinct primes  $p_1 < p_2 < p_3$  and  $\ell, m, n \in \mathbb{N} \cup \{0\}$  such that

$$y \ln p_1 = 2\ell\pi + \gamma, \quad y \ln p_2 = 2m\pi + \gamma, \quad y \ln p_3 = 2n\pi + \gamma. \quad (1)$$

We will get a contradiction. From eq. (1), we have

$$y(\ln p_2 - \ln p_1) = 2(m - \ell)\pi, \quad y(\ln p_3 - \ln p_1) = 2(n - \ell)\pi. \quad (2)$$

Notice that  $\ell < m < n$ . Let  $m - \ell = h$  and  $n - \ell = k$ . From eq. (2), we have

$$\frac{\ln p_3 - \ln p_1}{\ln p_2 - \ln p_1} = \frac{k}{h}.$$

Therefore

$$h(\ln p_3 - \ln p_1) = k(\ln p_2 - \ln p_1)$$

and hence

$$\left(\frac{p_3}{p_1}\right)^h = \left(\frac{p_2}{p_1}\right)^k.$$

Thus

$$p_1^k p_3^h = p_1^h p_2^k.$$

This contradicts to the uniqueness of prime factorization.  $\square$

**Definition 2.2.** Recall that  $y > 0$  and  $0 < K < 1$ . Let  $\beta$  be the number such that

$$\cos \beta = K, \quad 0 < \beta < \frac{\pi}{2}.$$

For each  $n \in \mathbb{N} \cup \{0\}$ , let

$$\begin{aligned} A_n &= \{p \in P \mid 2n\pi - \beta < y \ln p + \alpha \leq 2n\pi + \beta\}, \\ B_n &= \{p \in P \mid (2n+1)\pi - \beta < y \ln p + \alpha \leq (2n+1)\pi + \beta\} \end{aligned}$$

and

$$A = \bigcup_{n=0}^{\infty} A_n, \quad B = \bigcup_{n=0}^{\infty} B_n.$$

## Proof of Theorem 1.2

Notice that  $P^+ \subset A$  and  $P^- \subset B$ . From Lemma 2.1, we know that  $A - P^+$  has at most two elements and  $B - P^-$  also has at most two elements. Therefore it is enough to show that

$$\sum_{p \in A} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in B} \frac{1}{p} = \infty.$$

Recall that  $y > 0$ . By the prime number theorem, there exists  $M > 0$  such that if  $x > M$  then

$$e^{-\frac{\beta}{2y}} \frac{x}{\ln x} \leq \pi(x) \leq e^{\frac{\beta}{2y}} \frac{x}{\ln x}. \quad (3)$$

From Definition 2.2, we have

$$A_n = \left\{ p \in P \mid e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} < p \leq e^{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}} \right\}$$

and

$$B_n = \left\{ p \in P \mid e^{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}} < p \leq e^{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}} \right\}.$$

Notice that  $A_1, B_1, A_2, B_2, \dots$  are mutually disjoint. There exists  $N \in \mathbb{N}$  such that if  $n > N$  then

$$e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} > M.$$

From now on, we assume that  $n > N$ . By eq. (3), we can find the lower bounds of the number of elements of  $A_n$  and  $B_n$ . We have

$$\begin{aligned} |A_n| &\geq e^{-\frac{\beta}{2y}} \frac{e^{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}}}{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}} - e^{\frac{\beta}{2y}} \frac{e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}}}{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} \\ &= \frac{ye^{\frac{2n\pi}{y} + \frac{\beta-2\alpha}{2y}}}{2n\pi + \beta - \alpha} - \frac{ye^{\frac{2n\pi}{y} - \frac{\beta+2\alpha}{2y}}}{2n\pi - \beta - \alpha} \end{aligned} \quad (4)$$

and

$$\begin{aligned} |B_n| &\geq e^{-\frac{\beta}{2y}} \frac{e^{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}}}{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}} - e^{\frac{\beta}{2y}} \frac{e^{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}}}{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}} \\ &= \frac{ye^{\frac{(2n+1)\pi}{y} + \frac{\beta-2\alpha}{2y}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{\frac{(2n+1)\pi}{y} - \frac{\beta+2\alpha}{2y}}}{(2n+1)\pi - \beta - \alpha}. \end{aligned} \quad (5)$$

Notice that if  $p \in A_n$  then

$$\frac{1}{p} \geq e^{-\frac{2n\pi}{y} - \frac{\beta-\alpha}{y}} \quad (6)$$

and if  $p \in B_n$  then

$$\frac{1}{p} \geq e^{-\frac{(2n+1)\pi}{y} - \frac{\beta-\alpha}{y}}. \quad (7)$$

From eq. (4) and (6), we have

$$\begin{aligned}
\sum_{p \in A_n} \frac{1}{p} &\geq \left( \frac{ye^{\frac{2n\pi}{y} + \frac{\beta-2\alpha}{2y}}}{2n\pi + \beta - \alpha} - \frac{ye^{\frac{2n\pi}{y} - \frac{\beta+2\alpha}{2y}}}{2n\pi - \beta - \alpha} \right) e^{-\frac{2n\pi}{y} - \frac{\beta-\alpha}{y}} \\
&= \frac{ye^{-\frac{\beta}{2y}}}{2n\pi + \beta - \alpha} - \frac{ye^{-\frac{3\beta}{2y}}}{2n\pi - \beta - \alpha} \\
&= y \frac{(2n\pi - \beta - \alpha)e^{-\frac{\beta}{2y}} - (2n\pi + \beta - \alpha)e^{-\frac{3\beta}{2y}}}{(2n\pi - \alpha)^2 - \beta^2} \\
&= y \frac{(2n\pi - \alpha) \left( e^{-\frac{\beta}{2y}} - e^{-\frac{3\beta}{2y}} \right) - \beta \left( e^{-\frac{\beta}{2y}} + e^{-\frac{3\beta}{2y}} \right)}{(2n\pi - \alpha)^2 - \beta^2} \\
&= \frac{2cn - d}{(2n\pi - \alpha)^2 - \beta^2}.
\end{aligned}$$

where

$$c = y\pi \left( e^{-\frac{\beta}{2y}} - e^{-\frac{3\beta}{2y}} \right) > 0 \quad (8)$$

and

$$d = y\alpha \left( e^{-\frac{\beta}{2y}} - e^{-\frac{3\beta}{2y}} \right) + y\beta \left( e^{-\frac{\beta}{2y}} + e^{-\frac{3\beta}{2y}} \right).$$

Similarly from eq. (5) and (7), we have

$$\begin{aligned}
\sum_{p \in B_n} \frac{1}{p} &\geq \left( \frac{ye^{\frac{(2n+1)\pi}{y} + \frac{\beta-2\alpha}{2y}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{\frac{(2n+1)\pi}{y} - \frac{\beta+2\alpha}{2y}}}{(2n+1)\pi - \beta - \alpha} \right) e^{-\frac{(2n+1)\pi}{y} - \frac{\beta-\alpha}{y}} \\
&= \frac{ye^{-\frac{\beta}{2y}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{-\frac{3\beta}{2y}}}{(2n+1)\pi - \beta - \alpha} \\
&= y \frac{((2n+1)\pi - \beta - \alpha)e^{-\frac{\beta}{2y}} - ((2n+1)\pi + \beta - \alpha)e^{-\frac{3\beta}{2y}}}{(2n\pi + \pi - \alpha)^2 - \beta^2} \\
&= y \frac{((2n+1)\pi - \alpha) \left( e^{-\frac{\beta}{2y}} - e^{-\frac{3\beta}{2y}} \right) - \beta \left( e^{-\frac{\beta}{2y}} + e^{-\frac{3\beta}{2y}} \right)}{(2n\pi + \pi - \alpha)^2 - \beta^2} \\
&= \frac{c(2n+1) - d}{(2n\pi + \pi - \alpha)^2 - \beta^2}.
\end{aligned}$$

Recall eq. (8). Since  $c > 0$ , we have

$$\sum_{p \in A} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \sum_{p \in A_n} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \frac{2cn - d}{(2n\pi - \alpha)^2 - \beta^2} = \infty$$

and

$$\sum_{p \in B} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \sum_{p \in B_n} \frac{1}{p} \geq \sum_{n=N+1}^{\infty} \frac{c(2n+1) - d}{(2n\pi + \pi - \alpha)^2 - \beta^2} = \infty.$$

Thus

$$\sum_{p \in A} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in B} \frac{1}{p} = \infty.$$

□

## References

- [1] W. Dittrich, *Reassessing Riemann's Paper: On the Number of Primes Less Than a Given Magnitude*, Springer, 2021.
- [2] L. Euler, "Variae observationes circa series infinitas" [Various observations concerning infinite series]. *Commentarii Academiae Scientiarum Petropolitanae* **9**(1737), **1744** 160–188.
- [3] Roland van der Veen and Jan van de Craats, *The Riemann Hypothesis A Million Dollar Problem*, Anneli Lax New Mathematical Library, MAA Press, 2015.