

Riemann hypothesis equivalences, Robin inequality, Lagarias criterion, and Riemann hypothesis

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Abstract

In this paper, we briefly review most of accomplished research in Riemann Zeta function and Riemann hypothesis since Riemann's age including Riemann hypothesis equivalences as well. We then make use of Robin and Lagarias' criteria to prove Riemann hypothesis. The goal is, using Lagarias criterion for $n \geq 1$ since Lagarias criterion states that Riemann hypothesis holds if and only if the inequality $\sum_{d|n} d \leq H_n + \exp(H_n) \log(H_n)$ holds for all $n \geq 1$. Although, Robin's criterion is used as well. Our approach breaks up the set of the natural numbers into three main subsets. The first subset is $\{n \in \mathbb{N} \mid 1 \leq n \leq 5040\}$. The second one is $\{n \in \mathbb{N} \mid 5041 \leq n \leq 19685\}$ and the third one is $\{n \in \mathbb{N} \mid n \geq 19686\}$. In our proof, the third subset for even integers is broken up into odd integer class number sets. Then, mathematical arguments are stated for each odd integer class number set. Odd integer class number set is introduced in this paper. Since the Lagarias criterion holds for the first subset regarding computer aided computations, we do prove it using both Lagarias and Robin's criteria for the second and

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third subsets and mathematical arguments accompanied by a large volume of computer language programs. It then follows that Riemann hypothesis holds as well.

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1. Introduction

Riemann hypothesis (RH) is the most importantly and influentially unsolved problem since Riemann proposed it in a number theory paper in 1859 [41]. The problem is regarded as one of the most useful and applicable ones not only in pure mathematics, but also in many other scientific fields such as physics and engineering. It's said that the solution of RH would imply immediately the solutions of many other unsolved problems in number theory. Someone believes that solving RH yields even the solutions up to 500 other unsolved problems.

There are many equivalent statements for correctness of RH. We express them in Section2 as far as possible. The lemmas ,theorems, and methods for proof of RH are presented in Section3. The proofs of the lemmas and theorems of Section3 are given in Section4. In Section 3, we break up the entire natural numbers set into three main subsets. The first subset is $\{n \in \mathbb{N} \mid 1 \leq n \leq 5040\}$. The second one is $\{n \in \mathbb{N} \mid 5041 \leq n \leq 19685\}$ and the third one is $\{n \in \mathbb{N} \mid n \geq 19686\}$. The goal is, using Lagarias criterion for the proof of RH. Lagarias criterion states that Riemann hypothesis holds if

and only if the inequality $\sum_{d|n} d \leq H_n + \exp(H_n) \log(H_n)$ holds for whole of $n \geq 1$. We make use of Robin's criterion as well. Odd integer class number sets are defined in Section 2, definition 34, which the third main subset for even integers is broken up into odd integer class number sets.

Finally, one takes a conclusion that RH follows from a combination of Lagarias and Robin's criteria for all the natural numbers greater than or equal to 5041. For further research in distribution of primes and Riemann hypothesis refer to references such as [25],[48],[53],[9],[35],[19],[26],[27],[6].

2. Literature survey

2.1. Definitions to and extensions of Riemann hypothesis

Definition 1. The Riemann zeta function is well-known and defined, for region $\Re(s) > 1$, by the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (2.1)$$

RH is defined such that: nontrivial zeros of the Riemann zeta function lie on the line $\Re(s) = \frac{1}{2}$.

RH is generalized by the set of Dirichlet L-functions. Generalized Riemann Hypothesis (GERH) has a similar format with respect to that of RH.

Definition 2. We can define Dirichlet's L-function as

$$L(s, \chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} \quad (2.2)$$

where $\chi_k(n)$ denotes a character of Dirichlet's L-function modulo k . This character has the following specifications:

$$\chi_k(1) = 1 \quad (2.3)$$

$$\chi_k(n) = \chi_k(n+k) \quad (2.4)$$

$$\chi_k(m)\chi_k(n) = \chi_k(mn) \quad (2.5)$$

for all integers m and n .

If $\gcd(k, n) \neq 1$, then $\chi_k(n) = 0$. If $\gcd(k, n) = 1$, then $\chi_k(n) = 1$ and so-called a principal character. If $d|k, d \neq k$ and there is no n such that $\chi_k(n) = \chi_k(d)$, a character modulo k is called a primitive one. Therefore, Riemann zeta function $\zeta(s)$ is obtained from a Dirichlet's L-function $L(s, \chi_k)$ substituting a principal character $\chi_k(n) = 1$ for (2.2). We can formulate GERH as follows:

GERH 1. All of the nontrivial zeros of $L(s, \chi_k)$ should have real part equal to $\frac{1}{2}$ [15].

There is a special case of GERH so-called Extended Riemann Hypothesis (ERH). If we consider a special character $\chi_k(n) = \left(\frac{n}{p}\right)$ as Legendre symbol, then substituting it for (2.2), we have

$$L_p := \sum_1^\infty \left(\frac{n}{p}\right) \frac{1}{n^s} \quad (2.6)$$

Therefore, ERH says us:

GERH 2. The entire zeros of L_p in the strip $0 < \Re(s) < 1$ lie on the line $\Re(s) = \frac{1}{2}$ [15].

Definition 3. Defining $\pi(x; k, l)$ as

$$\pi(x; k, l) := \#\{p : p \leq x, p \text{ is prime, and } p \equiv l \pmod{k}\} \quad (2.7)$$

we are able to find an equivalent to ERH as follows:

GERH 3. If $\gcd(k, l) = 1$ and $\epsilon > 0$, then [5]

$$\pi(x; k, l) = \frac{Li(x)}{\phi(k)} + O(x^{\frac{1}{2}+\epsilon}) \quad (2.8)$$

Another extension for RH using Dedekind zeta function is given by following statement:

GERH 4. *The entire zeros of the Dedekind zeta function of any algebraic number field K in the strip $0 < \Re(s) < 1$, lie on the line $\Re(s) = \frac{1}{2}$.*

Definition 4. *If K is an arbitrary number field and ring of integers denotes O_K , then Dedekind zeta function for K is defined by*

$$\zeta_K(s) := \sum_a N(a)^{-s} \quad (2.9)$$

for $\Re(s) > 1$, where the sum is over all integral ideals of O_K , and $N(a)$ denotes the norm of a .

This extension is including RH since Riemann zeta function is indeed a Dedekind zeta function, which covers field of rational numbers [52].

We are ready to state Grand Riemann Hypothesis (GRH). GRH says us:

GRH 1. *All of the zeros of any automorphic L -function in the strip $0 < \Re(s) < 1$ lie on the line $\Re(s) = \frac{1}{2}$.*

Definition 5. *Let \mathbb{A} be a ring of adeles of \mathbb{Q} and let π be an automorphic cuspidal representation of a general linear group $GL_m(\mathbb{A})$. Its central character is χ . The representation π is equivalent to $\bigotimes_v \pi_v$, where $v = \infty$ or $v = p$. π_v denotes an irreducible unitary representation of $GL_m(\mathbb{Q}_v)$. We can define a local representation π_p for each prime as follows [53]:*

$$L(s, \pi_p) := \prod_{j=1}^m (1 - \alpha_{j,\pi}(p)p^{-s})^{-1} \quad (2.10)$$

where m complex parameters $\alpha_{j,\pi}(p)$ can be obtained by π_p . If $v = \infty$, then π_∞ determines parameters $\mu_{j,\pi}(\infty)$ such that

$$L(s, \pi_\infty) := \prod_{j=1}^m \Gamma_{\mathbb{R}}(s - \mu_{j,\pi}(\infty)) \quad (2.11)$$

where

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) \quad (2.12)$$

Definition 6. The standard automorphic L -function associated with π is defined by

$$L(s, \pi) := \prod_p L(s, \pi_p) \quad (2.13)$$

Definition 7. Let the entire function $\Lambda(s, \pi)$ be defined

$$\Lambda(s, \pi) := L(s, \pi_{\infty}) L(s, \pi) \quad (2.14)$$

and associated functional equation related to it be:

$$\Lambda(s, \pi) := \epsilon_{\pi} N_{\pi}^{\frac{1}{2}-s} \Lambda(1-s, \tilde{\pi}) \quad (2.15)$$

where $N_{\pi} \geq 1$ is an integer, ϵ_{π} is of modulus 1 and $\tilde{\pi}$ denotes the contragredient representation $\tilde{\pi}(g) = \pi(t_{g^{-1}})$.

Definition 8. Langlands conjectures that standard L -functions are able to generate multiplicatively all L -functions. Therefore, GRH asserts that all the zeros of $\Lambda(s, \pi)$ lie on $\Re(s) = \frac{1}{2}$.

2.2. Definitions and Equivalent to Riemann hypothesis

Equivalence 1. One of the most important equivalencies of RH is [2],[9]:

$$\pi(x) = Li(x) + O(\sqrt{x} \log x) \quad (2.16)$$

where $\pi(x)$ denotes the number of primes less than or equal to x and $Li(x) = \int_2^x \frac{1}{\log t} dt$.

Equivalence 2. RH is equivalent to

$$M(x) = O(x^{\frac{1}{2}+\epsilon}) \quad (2.17)$$

for every $\epsilon > 0$ [48], where $M(x) = \sum_{n \leq x} \mu(n)$ denotes Mertens function. Function $\mu(n)$ denotes the Möbius one. It is defined to be zero if n has a square factor, 1 when $n = 1$, and $(-1)^k$ when n including a product of k distinct primes is.

Equivalence 3. *RH can be equivalent to*

$$\sum_{j=1}^m |F_n(j) - \frac{j}{m}| = O(n^{\frac{1}{2}+\epsilon}) \quad (2.18)$$

where $\epsilon > 0$ and $m = \#\{F_n\}$ [20]. F_n denotes Farey's series of order n .

Definition 9. *Farey's series is defined by a set of m elements of rationals $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $\gcd(a, b) = 1$. The elements of this set are arranged in increasing order. For example:*

$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\} \quad (2.19)$$

m the number of elements is obtained by formula

$$m = 1 + \sum_{j=1}^n \phi(j) \quad (2.20)$$

where j denotes j th element of the set $\{1, 2, \dots, n\}$.

Equivalence 4. *RH is an equivalent to the statement that all zeros of Dirichlet's eta function falling in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

Definition 10. *Dirichlet's eta function is expressed as follows:*

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s) \quad (2.21)$$

Equivalence 5. *RH is equivalent to*

$$\sum_{k=1}^n \lambda(k) \ll n^{\frac{1}{2}+\epsilon} \quad (2.22)$$

for every positive ϵ , where $\lambda(k)$ denotes Liouville function $\lambda(k) = (-1)^{\omega(k)}$ and $\omega(k)$ denotes the number of prime factors of k , counted with multiplicity.

Equivalence 6. *RH is equivalent to having necessary and sufficient condition for convergence of the following series for $\Re(s) > \frac{1}{2}$ [48]:*

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad (2.23)$$

Equivalence 7. *RH is the equivalency of the nonvanishing $\zeta'(s)$ in the region $0 < \Re(s) < \frac{1}{2}$ [45].*

Equivalence 8. *Lagarias criterion: RH is equivalent to [28].*

$$\Re\left(\frac{\xi'(s)}{\xi(s)}\right) > 0 \quad (2.24)$$

where $0 < \Re(s) < \frac{1}{2}$ and

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (2.25)$$

Equivalence 9. *RH is equivalent to Lagarias' sum of divisors function criterion. Lagarias criterion states: If the following inequality holds for all $n \geq 1$, then RH is completely true.*

$$\sigma(n) \leq H_n + e^{H_n} \log(H_n) \quad (2.26)$$

and the equality is only for $n = 1$, where $\sigma(n) = \sum_{d|n} d$ and d is a divisor of n [29]. $H_n = \sum_{i=1}^n \frac{1}{i}$ denotes n th harmonic number.

Equivalence 10. *Robin's criterion states RH is equivalent to*

$$\sigma(n) < e^{\gamma} n \log \log n \quad (2.27)$$

for all $n \geq 5041$, where γ denotes Euler's constant [40].

Equivalence 11. *RH is equivalent to the nonnegativity of λ_n for all $n \geq 1$*

[31], where

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} (s^n \log \xi(s))|_{s=1} \quad (2.28)$$

Equivalence 12. *RH is equivalent to the following expression:*

$$\lambda_n|_{s=1} = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho} \right)^n \right) > 0 \quad (2.29)$$

for each $n = 1, 2, 3, \dots$, where ρ runs over the complex zeros of ζ [3].

Equivalence 13. *RH holds if and only if [24]*

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k! \zeta(2k+1)} = O(x^{-\frac{1}{4}}) \quad (2.30)$$

as x tends to infinity.

Equivalence 14. *RH holds if and only if the integral equation*

$$\int_{-\infty}^{+\infty} \frac{\exp(-\sigma y) \varphi(y)}{\exp(\exp(x-y)) + 1} dy = 0 \quad (2.31)$$

has no bounded solution $\varphi(y)$ other than trivial one $\varphi(y) = 0$, for $\frac{1}{2} < \sigma < 1$, [47]

Equivalence 15. *RH is equivalent to expression that for sufficiently large n*

$$\log g(n) < \sqrt{Li^{-1}(n)} \quad (2.32)$$

where $g(n)$ denotes maximal order of elements of the symmetric group S_n of degree n and Li^{-1} denotes inversion of the function $Li(n)$ [34].

Equivalence 16. *Volchkov's equivalency for RH expresses*

$$\int_0^{\infty} \int_{\frac{1}{2}}^{\infty} (1 - 12t^2)(1 + 4t^2)^{-3} \log |\zeta(\sigma + it)| d\sigma dt = \frac{\pi(3 - \gamma)}{32} \quad (2.33)$$

where γ is Euler's constant [51]. This integral equation relates the non-trivial zeros of Riemann zeta function to Euler's constant γ .

Equivalence 17. *RH is true if and only if*

$$I = \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\log(|\zeta(s)|)}{|s|^2} dt = 0 \quad (2.34)$$

where $s = \sigma + it$.

Definition 11. *The integral I can also be expressed as follows:*

$$I = 2\pi \sum_{\Re(\rho) > \frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right| \quad (2.35)$$

where ρ denotes a zero of $\zeta(s)$ in the region $\Re(\rho) > \frac{1}{2}$ [7].

Equivalence 18. *RH is equivalent to the conjecture that $\Lambda \leq 0$, where Λ is called de Bruijn-Newman constant [12],[13],[37]. For understanding better the role of Λ in this equivalency, Peter Borwein et al. have defined [9] the following functions and transformations:*

Definition 12. *Peter Borwein et.al [9] define the Xi function Ξ as*

$$\Xi(iz) = \frac{1}{2} \left(z^2 - \frac{1}{4} \right) \pi^{-\frac{z}{2} - \frac{1}{4}} \Gamma \left(\frac{1}{2}z + \frac{1}{4} \right) \zeta \left(z + \frac{1}{2} \right) \quad (2.36)$$

Let Fourier transform of signal $\Phi(t)$ be

$$\Phi(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) e^{-\pi n^2 e^{4t}} = \frac{\Xi(\frac{z}{2})}{8} \quad (2.37)$$

where $t \in \mathbb{R}$ and $t \geq 0$. Let define the Fourier transform of $\Phi(t)e^{\lambda t^2}$ denoted by $H(\lambda, z)$

$$H(\lambda, z) := \mathcal{F}_t[\Phi(t)e^{\lambda t^2}](z) \quad (2.38)$$

where $\lambda \in \mathbb{R}$ and $z \in \mathbb{C}$. There exists a constant Λ such as if $H(\lambda, z) = 0$, then z is real if and only if $\lambda \geq \Lambda$. One of the best bounds known for Λ is $\Lambda > -2.7 \cdot 10^{-9}$ obtained by Odlyzko in 2000 [38]. Newman conjectured that

Λ satisfies the inequality $\Lambda \geq 0$ [37].

Equivalence 19. *RH is equivalent to the expression that for any smooth function f on M we have such that*

$$\int_M f dm(y) = o(y^{\frac{3}{4}-\epsilon}) \quad (2.39)$$

for any $\epsilon > 0$ when $y \rightarrow 0$, where there are ergodic measures as $m(y)$ on the space $M = PSL_2(\mathbb{Z})/PSL_2(\mathbb{R})$ for $y > 0$, and supported on closed orbits of period $1/y$ of the horocycle flow [49].

Equivalence 20. *Redheffer states that RH is true if and only if*

$$\det(R_n) = O(n^{\frac{1}{2}+\epsilon}) \quad (2.40)$$

for any $\epsilon > 0$ [44], where R_n denotes Redheffer matrix of order n .

Definition 13. *Redheffer matrix is an $n \times n$ matrix, $R_n := [R_n(i, j)]$, so that*

$$R_n(i, j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i|j, \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, we have

$$\det(R_n) = \sum_{k=1}^n \mu(k) \quad (2.41)$$

Equivalence 21. *RH is equivalent to the following expression:*

$$|\#\{\text{even cycles inside } \overline{G_n}\} - \#\{\text{odd cycles inside } \overline{G_n}\}| = O(n^{\frac{1}{2}+\epsilon}) \quad (2.42)$$

for any $\epsilon > 0$ [11], where G_n denotes a graph whose adjacency matrix is $B_n := R_n - I_n$.

Definition 14. *I_n denotes an $n \times n$ identity matrix. R_n denotes same*

definition given in definition 13. $\overline{G_n}$ denotes a graph, which extends G_n adding a loop at node 1 of G_n .

Equivalence 22. *RH is equivalent to correctness of the following equation as $y \rightarrow 0$:*

$$m_y(f) = m_0(f) + o(y^{\frac{3}{4}-\epsilon}) \quad (2.43)$$

for every function $f \in C_c^r(\mathbb{R}^*)|_{r=2}$ and every $\epsilon > 0$ [49]. \mathbb{R}^* denotes the set of real numbers without zero. $C_c^r(\mathbb{R}^*)$ denotes the set of the functions representing $f : \mathbb{R}^* \rightarrow \mathbb{C}$, where r denotes the number of differentiability of the function f of compact support.

Definition 15. For every $y \in \mathbb{R}^*$ and for every $f \in C_c^2(\mathbb{R}^*)$, define

$$m_y(f) := \sum_{n \in \mathbb{N}} y \phi(n) f(y^{1/2}n) \quad (2.44)$$

where $\phi(n)$ is Euler's function. Define the following function as well:

$$m_0(f) := \int_0^\infty u f(u) du \quad (2.45)$$

Equivalence 23. *RH is true if and only if the closed linear span of $\{\rho_\alpha(t) : 0 < \alpha < 1\}$ is $L^2(0, 1)$ so that*

$$\rho_\alpha(t) := \left\{ \frac{\alpha}{t} \right\} - \alpha \left\{ \frac{1}{t} \right\} \quad (2.46)$$

where $\{x\}$ denotes the fractional part of x , and $L^2(0, 1)$ the space of square integrable functions on $(0, 1)$ [8]. Square integrable function is said to a function, if $\int_{-\infty}^{+\infty} |f(x)|^2 dx$ is finite.

Equivalence 24. *Hilbert-Schmidt operator A is injective if and only if RH is true [4], where Hilbert-Schmidt operator A is defined as follows:*

$$[Af](\theta) := \int_0^1 f(x) \left\{ \frac{\theta}{x} \right\} dx \quad (2.47)$$

where $\{x\}$ and $L^2(0, 1)$ expressed in equivalency 23.

Equivalence 25. RH is equivalent to statement that all of the zeros of the equation $\Xi(t) = 0$ are real, where [1]

$$\xi\left(\frac{1}{2} + it\right) = \Xi(t) = \sum_{n=0}^{\infty} c_n t^n \quad (2.48)$$

Equivalence 26. Grosswald's necessary condition (not sufficient condition) [22]: for all sufficiently large values of n if $D_n = a_n^2(1 + O(\log^{-1} n))$, then it is proved that [1]

$$D_n = na_n^2 - (n+1)a_{n+1}a_{n-1} > 0 \quad (2.49)$$

for all $n \geq 1$, where a_n denotes n th coefficient of the entire function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (2.50)$$

This is a necessary condition namely condition C.

Equivalence 27. Alter's RH equivalence [1]: If we put $z = -t^2$ in the function $f(z)$ of the relation (2.50), then $f(-t^2)$ is converted to an entire function of order $\frac{1}{2}$. If t_0 is a real zero of $\Xi(t)$, then $z_0 = -t_0^2$ is a negative zero of $f(z)$ and RH is equivalent to the statement that all of the zeros of $f(z)$ are negative.

Equivalence 28. Ramanujan's arithmetical function. Ramanujan and Alter's RH equivalence: let Ramanujan's function be $\tau(n)$, then the associated Dirichlet's series

$$Z(s, \tau) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \quad (2.51)$$

is absolutely convergent and $Z(s, \tau)$ is regular for $\sigma > \frac{13}{2}$ [1]. Wilton [54] makes use of this Dirichlet's series and asserts that $Z(s, \tau)$ is an entire func-

tion holding

$$(2\pi)^{-s}\Gamma(s)Z(s, \tau) = (2\pi)^{s-12}\Gamma(12-s)Z(12-s, \tau) \quad (2.52)$$

On the other hand, its simple zeros are at $s = 0, -1, -2, \dots$

Definition 16. Alter defines

$$\xi(s, \tau) = (2\pi)^{-s}\Gamma(s)Z(s, \tau) \quad (2.53)$$

and states that the poles of $\Gamma(s)$ are canceled by the zeros of $Z(s, \tau)$. This follows that $\xi(s, \tau)$ should be an entire function of s . Thus the functional equation (2.52) can be expressed as follows:

$$\xi(12-s, \tau) = \xi(s, \tau) \quad (2.54)$$

This means that $\xi(6+it, \tau) = \xi(6-it, \tau)$ and there is a symmetry about the line $\sigma = 6$. If we put $s = 6 + iz$ or $s = 6 + it$ in (2.53), we have

$$\xi(6+it, \tau) = (2\pi)^{-6-it}\Gamma(6+it)Z(6+it, \tau) = \Xi(z, t) \quad (2.55)$$

then, all zeros of $\Xi(z, t)$ are real if and only if all of the zeros of $Z(s, \tau)$ lie on the line $s = 6 + iz$. Therefore, Wilton [54] proves that $Z(s, \tau)$ has an infinite number of zeros on the line $\sigma = 6$. Finally, RH is equivalent to statement that all zeros of $Z(s, \tau)$ in the critical strip $\frac{11}{2} < \sigma < \frac{13}{2}$, lie on the line $\sigma = 6$, where $\sigma = \Re(s)$.

Equivalence 29. RH is equivalent to the statement that if and only if the function $\xi(x)$ belongs to the Laguerre-Pólya class [39],[48].

Definition 17. Laguerre-Pólya class is defined as the collection of all real entire functions $f(x)$

$$f(x) = Ce^{-\alpha x^2 + \beta x} x^n \prod_{j=1}^{\omega} \left(1 - \frac{x}{x_j}\right) e^{\frac{x}{x_j}} \quad (\omega \leq \infty) \quad (2.56)$$

where $\alpha \geq 0$, β and C denote real numbers, n is a nonnegative integer, and x'_j 's are real and nonzero so that $\sum_{j=1}^{\omega} \frac{1}{x_j^2} < \infty$ [14].

Equivalence 30. *Jensen polynomials: The necessary and sufficient condition for holding real zeros of the function*

$$F(z) = 2 \int_0^{\infty} \Psi(t) \cos zt \cdot dt \quad (2.57)$$

is that

I.

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(\alpha) \Psi(\beta) e^{i(\alpha+\beta)x} e^{(\alpha-\beta)y} (\alpha - \beta)^2 d\alpha d\beta \geq 0 \quad (2.58)$$

for all real values x and y . Then, this leads us to an equivalent to RH since $F(z)$ belongs to Laguerre-Pólya class functions.

II

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(\alpha) \Psi(\beta) e^{i(\alpha+\beta)x} (\alpha - \beta)^{2n} d\alpha d\beta \geq 0 \quad (2.59)$$

for all real values x and $n = 0, 1, 2, \dots$

III

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(\alpha) \Psi(\beta) (x + i\alpha)^n (x + i\beta)^n (\alpha - \beta)^2 d\alpha d\beta > 0 \quad (2.60)$$

for all real values x and $n = 0, 1, 2, \dots$

Definition 18. *Jensen polynomials: Let the function $f(x)$ be of the form*

$$f(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \quad (2.61)$$

where, γ'_k 's denote positively real numbers holding Turán's inequality as follows:

$$\gamma_k^2 - \gamma_{k-1} \gamma_{k+1} \geq 0 \quad (2.62)$$

for $k = 1, 2, \dots$, then n th Jensen's polynomial corresponding $f(x)$ is defined as:

$$g_n(t) = \sum_{k=0}^n \binom{n}{k} \gamma_k t^k \quad (2.63)$$

where $n = 0, 1, \dots, [16]$.

Equivalence 31. RH is equivalent to the hyperbolicity of the Jensen polynomials for the Riemann hypothesis $\zeta(s)$ at its point of symmetry. George Pólya [39] indicated that RH is equivalent to hyperbolicity of all Jensen polynomials associated with the sequence of Taylor coefficients $\{\gamma(n)\}$ [23].

Definition 19. $\{\gamma(n)\}$ is defined by

$$(-1 + 4z^2)\Lambda\left(\frac{1}{2} + z\right) = \sum_{n=0}^{\infty} \gamma_n \frac{z^{2n}}{n!} \quad (2.64)$$

where $\Lambda(s) = \pi^{-s}\Gamma(s/2)\zeta(s) = \Lambda(1-s)$

Definition 20. A polynomial of real coefficients is hyperbolic if all of its zeros are real.

Definition 21. Jensen polynomial of degree d and shift n is a polynomial denoted by

$$J_{\alpha}^{d,n}(X) := \sum_{j=0}^d \binom{d}{j} \alpha(n+j) X^j \quad (2.65)$$

where $\{\alpha(0), \alpha(1), \alpha(2), \dots\}$ denote an arbitrary sequence of real numbers.

Equivalence 32. RH is equivalent to the hyperbolicity of polynomials $J_{\alpha}^{d,n}(X)$ for all of the nonnegative integers d and n [39],[14],[18],[23],[33].

Definition 22. Lindelöf hypothesis: expresses that

$$\zeta\left(\frac{1}{2} + it\right) = O(t^{\epsilon}) \quad (2.66)$$

for every positive ϵ . Lindelöf hypothesis is true if RH is true [48].

Equivalence 33. RH is equivalent to Weil's positivity expression [30]

$$W^{(1)}(f * \tilde{f}) \geq 0 \quad (2.67)$$

for all nice test functions f , where

$$W^{(1)}(f) = \sum_{\rho \text{ as zeros of } \xi} \mathcal{M}[f](\rho) \quad (2.68)$$

and $\mathcal{M}[f]$ denotes Mellin transform of f .

Definition 23. Nice test functions denote functions defined over $f : (0, \infty) \rightarrow \mathbb{C}$, where f is piecewise on C^2 (C denotes a small curve drawn around a zero of the function ξ), compactly supported and has the averaging property

$$f(x) = \frac{1}{2} \left[\lim_{t \rightarrow x^+} f(t) + \lim_{t \rightarrow x^-} f(t) \right] \quad (2.69)$$

at discontinuous points [30].

Definition 24. The Mellin transform of $f(s)$ is defined as [30] :

$$\mathcal{M}[f](s) = \int_0^\infty f(x) x^s \frac{dx}{x} \quad (2.70)$$

Definition 25. For two "nice" functions f and g , one can define the following intersection product [30]:

$$\langle f_1, f_2 \rangle := W^{(1)}(f * \tilde{f}) \quad (2.71)$$

Definition 26. The convolution operation associated with Mellin transform is defined and obtained [30]

$$f * g(x) = \int_0^\infty f\left(\frac{x}{y}\right) g(y) \frac{dy}{y} \quad (2.72)$$

and

$$\mathcal{M}[f * g](s) = \mathcal{M}f(s)\mathcal{M}[g](s) \quad (2.73)$$

also we have an involution

$$\tilde{f}(x) = \frac{1}{x}f\left(\frac{1}{x}\right) \quad (2.74)$$

and

$$\mathcal{M}[\tilde{f}](s) = \mathcal{M}[f](1-s) \quad (2.75)$$

Equivalence 34. RH is equivalent to expression that the following equation holds [46]:

$$\frac{1}{t} \int_{\gamma_t} f(z) d\nu_t z = \frac{\int_{\mathbb{H}/\Gamma} f(z) d\mu z}{\text{vol}(\mathbb{H}/\Gamma)} + O(t^{\frac{3}{4}+\epsilon}) \quad (2.76)$$

where f belongs to the set of "nice" test functions $f \in C_{00}^\infty(S\mathbb{H}/\Gamma)$ and $S\mathbb{H}/\Gamma$ denotes the unit tangent bundle over \mathbb{H}/Γ for when t tends to zero and for any $\epsilon > 0$. ν_t is the arc-length measure on the horocycle at height t and μ denotes Poincare measure on $S\mathbb{H}/\Gamma$ [30].

Definition 27. Horocycle flows: Let the group $\Gamma = PSL(2, \mathbb{Z})$ be a group acting on the hyperbolic plane \mathbb{H} , so that we can generate this group by isometries

$$z \mapsto z + 1 \quad (2.77)$$

and

$$z \mapsto -\frac{1}{z} \quad (2.78)$$

of the upper half-plane model of \mathbb{H} . Then, h_t denotes a horocycle in the upper half-plane of a constant imaginary part $y = t$.

Definition 28. \mathbb{H}/Γ denotes quotient space and γ_t denotes the image of a segment in \mathbb{H}/Γ . This segment is obtained by projection of horocycle on the

\mathbb{H}/Γ focusing on the segment of h_t lying within the vertical strip $\{z : 0 \leq z \leq 1\}$. The length of γ_t is $\frac{1}{t}$. This means that when $t \rightarrow 0$, then $\gamma_t \rightarrow \infty$ and γ_t holds the following property as $t \rightarrow 0$:

Definition 29. For any "nice" test open set like S in \mathbb{H}/Γ , we have as $t \rightarrow 0$ [30]

$$\frac{\text{length}(\gamma_t \cap S)}{\text{length}(\gamma_t)} \longrightarrow \frac{\text{vol}(S)}{\text{vol}(\mathbb{H}/\Gamma)} \quad (2.79)$$

where nice is said the boundary $\partial(S) = \bar{S} \setminus S$ should have finite 1-dimensional Hausdorff measure [50] [56].

Equivalence 35. RH is equivalent to expression that

$$E[\log |\zeta(W)|] = 0 \quad (2.80)$$

where E denotes expectation of Z^s as

$$E[Z^s] = \xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \quad (2.81)$$

and considers a two-dimensional Brownian motion in the plane, starting at $(0,0)$. Let the point $(\frac{1}{2}, W)$ be the first point of contact with the line $X = \frac{1}{2}$ [7].

Definition 30. Gnedenko and Kolmogorov [10] discovered that Riemann zeta function can be related to Brownian motion. Consider a standard Brownian motion on the line $B_t \in \mathbb{R}$, $t \geq 0$, started at B_0 and with condition of $B_1 = 0$. Then,

$$Z = \max_{0 \leq t \leq 1} B_t - \min_{0 \leq t \leq 1} B_t \quad (2.82)$$

denotes the length of the range of B_t .

Equivalence 36. Gilles Lachaud [32] indicates that RH is equivalent to certain conditions bearing on spaces of torical forms constructed by Eisenstein

series so called torical wave packets.

Definition 31. Torical forms: Don Zagier [56] and Gilles Lachaud [32] state that F denotes a torical form for K as a complex quadratic field if

$$\oint F(z)dm(z) = 0 \quad (2.83)$$

Definition 32. Wave packet or an Eisenstein wave packet is a finite linear combination of Eisenstein series.

Definition 33. Eisenstein series: An Eisenstein series is defined so that if let H be the upper half-plane made of $z = x + iy$ such that $y > 0$ and if $\Re(s) > 1$, then, the Eisenstein series $\mathbf{E}(z, s)$ is expressed as special functions defined by [32]

$$\mathbf{E}(z, s) = \sum_{\gcd(m,n)=1} \frac{y^s}{|cz + d|^{2s}} = \sum_{(\Gamma \cap P) \backslash \Gamma} (Im \gamma z)^s \quad (2.84)$$

where $\Gamma = SL(2, \mathbb{Z})$ denotes the modular group and

$$P = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \quad (2.85)$$

Definition 34. Odd integer class number set. Odd integer class number set is a class of even integers $n = 2^\alpha m \geq N$ for an arbitrarily even number N , where m denotes an odd number demonstrating an odd class number and α denotes a natural number. The odd integer class number set of m is defined as the set

$$CL(m) = \{2^\alpha m \geq N | m \text{ is a constantly odd number and } \alpha \geq \alpha_0\} \quad (2.86)$$

and α_0 denotes the smallest positive integer to hold $2^{\alpha_0}m \geq N$. For example:
 Odd integer class number set of 3 is the set

$$CL(3) = \{2^{\alpha_0} \times 3, 2^{\alpha_0+1} \times 3, 2^{\alpha_0+2} \times 3 \dots\} \quad (2.87)$$

3. Methods, lemmas, and theorems

3.1. Lemmas and theorems

Lemma 1. Robin's criterion [40]: States that RH is true if and only if the inequality

$$\sigma(n) < e^{\gamma} n \log \log n, \quad \text{for } n \geq 5041 \quad (3.1)$$

holds.

Lemma 2. Grytczuk's theorem 1 [21]: Let $n = 2m$, $\gcd(2, m) = 1$ and $m = \prod_{j=1}^k p_j^{\alpha_j}$. Then for all odd positive integers $m > \frac{3^9}{2}$, we have

$$\sigma(2m) < \frac{39}{40} e^{\gamma} 2m \log \log 2m < e^{\gamma} 2m \log \log 2m, \quad \text{for } 2m \geq 19686 > 3^9 \quad (3.2)$$

and

$$\sigma(m) < e^{\gamma} m \log \log m, \quad \text{for } m \geq 9843 > \frac{3^9}{2} \quad (3.3)$$

Lemma 3. Choie's et.al theorem 1.2 [17]: Any odd positive integer n distinct from 1, 3, 5 and 9 is in \mathcal{R} , where \mathcal{R} denotes the set of integers $n \geq 1$ satisfying

$$\sigma(n) < e^{\gamma} n \log \log n \quad (3.4)$$

Lemma 4. Lagarias' theorem 1.1 [29]: Let $H_n = \sum_{j=1}^n \frac{1}{j}$. Show that, for each $n \geq 1$, we have

$$\sum_{d|n} d \leq H_n + \exp(H_n) \log(H_n) \quad (3.5)$$

with equality only for $n = 1$, where H_n denotes n th harmonic number. Then correctness of the inequality (3.5) is equivalent to correctness of RH.

Lemma 5. Lagarias' lemma 3.1 [29]: For all $n \geq 3$, we have

$$e^\gamma n \log \log n < H_n + \exp(H_n) \log(H_n) \quad (3.6)$$

Lemma 6. The set of all the even numbers greater than or equal to N is equivalent to the union of the odd integer class number sets of 1 to infinity. This means that

$$\{n \geq N | n \text{ is an even number}\} = \cup_{m=1}^{\infty} CL(m) \quad (3.7)$$

Lemma 7 (Rosser et. al [43])

Let $p_n > 285$, then

$$\frac{e^{-\gamma}}{\log p_n} \left(1 - \frac{1}{2 \log^2 p_n}\right) < \prod_{p \leq p_n} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log p_n} \left(1 + \frac{1}{2 \log^2 p_n}\right) \quad (3.8)$$

Note that $p_1 = 3$ is considered the origin of counting the odd primes in this lemma.

Lemma 8 (Rosser et. al [43])

Let $p_n \geq 41$, then

$$p_n \left(1 - \frac{1}{\log p_n}\right) < \log(2p_1 p_2 \dots p_n) < p_n \left(1 + \frac{1}{2 \log p_n}\right) \quad (3.9)$$

Note that $p_1 = 3$ is considered the origin of counting the odd primes in this lemma.

Proposition 1

Let $g(x) \geq 0$, $g'(x) > 0$, $f(x)$ and $f'(x)$ be defined and differentiable on $f, g : [0, +\infty[\rightarrow \mathbb{R}$. Let $f(x) = O(g(x))$, be defined on the entire domain $f, g : [0, +\infty[\rightarrow \mathbb{R}$, or for sufficiently large values x and $g(x_0) = 0$, where x_0 belongs to $[0, +\infty[$, and $\lim_{x \rightarrow +\infty} g(x) = +\infty$, then

$$f'(x) = O(g'(x))$$

where O denotes big O -notation function.

Lemma 9

Let $n \geq 1$, then the function

$$RO_1(n) = e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1 \dots p_n) \quad (3.10)$$

is a strictly increasing one for consecutively primes p_1 to p_n . Note that $p_1 = 3$ is considered the origin of counting the odd primes in this lemma.

Lemma 10

Let $n \geq 66$, then for consecutively primes p_1 to p_n , we have

$$RO_2(n) = e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1^2 \dots p_n^2) > 2 \quad (3.11)$$

and

$$RO_1(n) < 2 \quad (3.12)$$

Note that $p_1 = 3$ is considered the origin of counting the odd primes in this lemma.

Lemma 11. *RH is true if and only if the following inequalities for $n \geq 5041$ hold [29]*

$$\sigma(n) < e^\gamma n \log \log n < H_n + \exp(H_n) \log(H_n) \quad (3.13)$$

Theorem 1. *Lagarias criterion holds for $1 \leq n \leq 5040$.*

Theorem 2. *Lagarias and Robin's criteria hold for $5041 \leq n \leq 19685$.*

Theorem 3. *Lagarias and Robin's criteria hold for all the odd numbers $n > 19686$.*

Theorem 4. *Let*

$$\sigma(m) < \frac{1}{2}e^\gamma m \log \log(2m) \quad (3.14)$$

be true for $m = m_0 = (3 \times 5 \times 7 \times \dots \times 331)^2$, then it is correct for all the odd numbers $m \geq m_0$

Theorem 5. *Lagarias and Robin's criteria hold for all the odd integer class number sets of m , where $m < 9843$ and all the even numbers $n \geq 19686$.*

Theorem 6. *Lagarias and Robin's criteria hold for all the odd integer class number sets of m , where $m \geq 9843$ and all the even numbers $n \geq 19686$.*

3.2. Methods

As is explained in the Introduction, we break up the entire natural numbers set into three main subsets. The first subset is $\{n \in \mathbb{N} \mid 1 \leq n \leq 5040\}$. The second one is $\{n \in \mathbb{N} \mid 5041 \leq n \leq 19685\}$ and the third one is $\{n \in \mathbb{N} \mid n \geq 19686\}$. The goal is, using Lagarias criterion for the proof of RH although Robin's one is also considered. Regarding theorems 1, 2, 3, 5, and 6 we assert that Lagarias criterion holds for all the natural numbers and robin's one holds for all $n \geq 5041$. Since the Lagarias criterion already holds for the first subset regarding computer aided computations, we do prove it using both Lagarias and Robin's criteria for the second and third subsets and mathematical arguments accompanied by a large volume of computer language programs. The second subset is proved using both Lagarias

and Robin's criteria and computations carried out by computer programs in Maple16. The third subset is itself broken up into three sub-subsets: The first sub-subset is only for the odd numbers $n > 19686$. The second one is only for the even numbers $n = 2^\alpha m \geq 19686$, where $\alpha \geq \alpha_0$, $m < 9843$, and m is an odd number (α_0 is mathematically defined in the proof of Theorem 5). The third one is only for even numbers $n = 2^\alpha m \geq 19686$, where $\alpha \geq 1$, $m \geq 9843$, and m is an odd number. To prove the third main subset including three sub-subsets, we make use of Lagarias' criterion, Robin's criterion, analytic arguments and computations carried out by computer programs in Maple16. Odd integer class number sets play an important role for proving that the union of all of the odd integer class number sets implies the subset of the all even numbers greater than or equal to 19686. Finally, we take a conclusion that RH follows from lemma 11.

4. Proofs of the lemmas and theorems

4.1. *Proof of Lemma 1*

The proof is found in the Robin's paper [40].

4.2. *Proof of Lemma 2*

The proof is found in the Grtytczuk's paper [21].

4.3. *Proof of Lemma 3*

The proof is found in the Choie's et. al paper [17].

4.4. *Proof of Lemma 4*

The proof is found in the Lagarias' paper [29].

4.5. *Proof of Lemma 5*

The proof is found in the Lagarias' paper [29].

4.6. *Proof of Lemma 6*

Trivially, the proof is very easy. Regarding definition 34, let an even number as n , where α_0 denotes the smallest positively integer value that $2^{\alpha_0}m \geq N$ holds, then for every $\alpha \geq \alpha_0$, we have $2^\alpha m \geq 2^{\alpha_0}m \geq N$. This states the odd integer class number set $CL(m)$ corresponding to m . If we establish the set $CL(m)$ for each $m = 1, 3, 5, \dots$, and make the set of their union, then we have $\cup_{m=1}^{\infty} CL(m)$. On the other hand, let m_0 and m_1 be two distinctly odd class numbers and their corresponding even numbers be $2^{\alpha_0}m_0$ and $2^{\alpha_1}m_1$, respectively. There is no equality $2^{\alpha_0}m_0 = 2^{\alpha_1}m_1$ since equivalency contradicts being odd m_0 or m_1 . Therefore, $\cup_{m=1}^{\infty} CL(m)$ must cover and represent all the even numbers greater than or equal to N .

4.7. *Proof of Lemma 7*

The proof is made in Rosser and Schoenfeld's paper [43].

4.8. *Proof of Lemma 8*

The proof is made in Rosser and Schoenfeld's paper [43].

4.9. *Proof of Proposition 1*

If $f(x) = O(g(x))$, would imply that

$$\lim_{x \rightarrow \infty} \sup \frac{|f(x)|}{g(x)} < \infty$$

or

$$|f(x)| < Ag(x)$$

where $A > 0$ and denotes a constant value independent of x .

The problem is proved for the two cases: 1-for when $|f(x)| < Ag(x)$ holds on the entire domain f, g . 2-for when $|f(x)| < Ag(x)$ holds only for sufficiently large values x .

Case 1:

Just, we prove that

$$|f'(x)| < Ag'(x)$$

holds on the domain $f, g : [0, +\infty[\longrightarrow \mathbb{R}$. If $f(x)$ is a constant function, then $|f'(x)| = 0 < Ag'(x)$. On the other hand, function $f(x)$ is assumed be unbounded one and the case of a bounded function is given to be followed in lately proof. Therefore, we ignore these in both cases 1 and 2. The proof is made by *reductio ad absurdum*.

If the inequality $|f'(x)| < Ag'(x)$ is assumed to be false on domain f, g , then we have by contradiction

$$|f'(x)| \geq Ag'(x)$$

holds on all the domain f, g .

If both $f'(x) > 0$ and $f(x) > 0$, we have

$$f'(x) \geq Ag'(x)$$

and are able to integrate the both hand-sides over the entire domain $(x, x_0 \in D_{f,g})$ and find

$$\int_{x_0}^x f'(u)du \geq \int_{x_0}^x Ag'(u)du$$

and

$$(f(x) - f(x_0)) \geq A(g(x) - g(x_0))$$

since $g(x_0 = 0) = 0$, then the inequality $|f(x)| < Ag(x)$ implies that $f(x_0 = 0) = 0$ as well and we have

$$f(x) \geq Ag(x)$$

on the domain f, g and leads a contradiction with assumption since we must have $f(x) \leq Ag(x)$ and the proof is completed.

If $f'(x) < 0$ and $f(x) > 0$, then by contradiction, we have

$$f'(x) \leq -Ag'(x)$$

Similarly to the above, integrating it yields $(x, x_0 \in D_{f,g})$

$$\int_{x_0}^x f'(u)du \leq - \int_{x_0}^x Ag'(u)du$$

and we have with $x_0 = 0$

$$f(x) \leq -Ag(x)$$

on the domain f, g and leads a contradiction with assumption since we must have $f(x) \leq Ag(x)$ and the proof is completed. Similarly to the case $f(x) > 0$, the proof is valid when $f(x) < 0$ as well.

Case 2:

Let $|f'(x)| < Ag'(x)$ be a false statement for $x > x_N$, where x_N is a sufficiently large number, then regarding contradiction, we should have $|f'(x)| \geq Ag'(x)$ for $x > x_N$.

If both $f(x)$ and $f'(x) > 0$, we should have

$$f'(x) \geq Ag'(x)$$

for $x > x_N$. If we integrate both hand-sides of the above inequality between x and x_N , we find the following inequality

$$h(x) = f(x) - Ag(x) \geq h(x_N) = f(x_N) - Ag(x_N)$$

On the other hand, $h(x)$ is an unbounded function since regarding the proposition's assumption $g(x)$ tends to infinity when x so does. This makes the inequality $h(x) \geq h(x_N)$ incorrect since $h(x)$ tends to $-\infty$, but $h(x_N)$ is a bounded value and fixed. This contradicts our problem's assumption. Thus, $|f'(x)| < Ag'(x)$ is correct.

If $f(x) > 0$ and $f'(x) < 0$, then we should have $f'(x) \leq -Ag'(x)$. After integrating both hand sides of the inequality between x and x_N , we have

$$f(x) \leq f(x_N) + Ag(x_N) - Ag(x)$$

and regarding the problem's assumption, we have

$$f(x) < Ag(x)$$

Summing out the above inequalities together, yields

$$2f(x) \leq f(x_N) + Ag(x_N)$$

where restricts $f(x)$, since $f(x)$ is not a bounded function (the cases that $f(x)$ is a fixed or a bounded one are ignored as mentioned at the beginning of the proof. Being bounded $f(x)$ contradicts the inequality $f(x) \leq f(x_N) + Ag(x_N) - Ag(x)$ since here $f(x) > 0$, but the right hand of the inequality goes to the negative state and gets contradicted), but $f(x_N) + Ag(x_N)$ is a bounded value. Again, we find out that $|f'(x)| < Ag'(x)$ is correct.

If both $f(x)$ and $f'(x) < 0$, then the inequality $|f(x)| < Ag(x)$ implies that $f(x) > -Ag(x)$. This means that holding it with the inequality $f(x) \leq f(x_N) + Ag(x_N) - Ag(x)$ simultaneously, leads to a contradiction since we find

$$0 < f(x) + Ag(x) < f(x_N) + Ag(x_N)$$

because $f(x) + Ag(x)$ is an unbounded function when x tends to infinity but $f(x_N) + Ag(x_N)$ is a bounded value. All of the above reasons yields $f'(x) \leq -Ag'(x)$. Therefore, $f'(x) = O(g'(x))$ holds.

4.10. **Proof of Lemma 9**

We directly investigate that the lemma is correct for all $p_n < e^{130000}$. The proof for $x \geq p_n > e^{130000}$ is made by analytic reasoning as follows:

Just, we compute $\log \theta(x) = \log \log_{p_n \leq x}(2p_1 \dots p_n)$ regarding the relation (3.10) and using Abel Summation Formula ([55], page 10, theorem 1.6).

Let $a_n = 1$ be a constant sequence, $A(x) = \sum_{p \leq x} a_n = \pi(x)$ and $\phi_1(p) = \log p$. Abel summation formula asserts

$$\theta(x) = \sum_{p \leq x} \log p = \pi(x) \log x - \int_2^x \frac{\pi(u)}{u} du \quad (4.1)$$

Thus,

$$\log \theta(x) = \log \log_{p_n \leq x}(2p_1 \dots p_n) = \log \left(\pi(x) \log x - \int_2^x \frac{\pi(u)}{u} du \right) \quad (4.2)$$

First of all To compute the expression $B = \prod_{p \leq p_n \leq x} (1 - \frac{1}{p})$ of (3.10), we have to let $p_1 = 2$ and compute (4.3) assuming it

$$\log B = \sum_{p \leq p_n \leq x} \log(1 - \frac{1}{p}) \quad (4.3)$$

where $\phi_2(p) = \log(1 - \frac{1}{p})$, and again, we have $A(x) = \sum_{p \leq x} a_n = \sum_{p \leq x} 1 = \pi(x)$ and regarding Abel summation formula yields

$$\log B = \pi(x) \log(1 - \frac{1}{x}) - \int_2^x \frac{\pi(u)}{u(u-1)} du \quad (4.4)$$

that implies that

$$B = e^{\{\pi(x) \log(1 - \frac{1}{x}) - \int_2^x \frac{\pi(u)}{u(u-1)} du\}} = (1 - \frac{1}{x})^{\pi(x)} e^{-\int_2^x \frac{\pi(u)}{u(u-1)} du} \quad (4.5)$$

Let us change the argument n in $RO_1(n)$ given in (3.10) into x as follows letting $p_1 = 3$:

$$\begin{aligned} C(x) &= e^\gamma (1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_n})_{2 < p_n \leq x} \log \log(2p_1 \dots p_n)_{2 < p_n \leq x} = \\ e^\gamma \prod_{2 < p \leq p_n \leq x} (1 - \frac{1}{p}) \log \theta(x) &= 2e^\gamma (1 - \frac{1}{x})_{2 < p \leq x}^{\pi(x)} e^{-\int_2^x \frac{\pi(u)}{u(u-1)} du} \log \theta(x) \end{aligned} \quad (4.6)$$

Since we know the function $(1 - \frac{1}{x})_{2 < p \leq x}^{\pi(x)}$ (the first term of the right -hand side of (4.6)) is a strictly increasing one, it is enough to prove the following function (the right terms of the right hand side of (4.6)) is a strictly increasing one:

$$Y = \frac{\log \theta(x)}{e^{\int_2^x \frac{\pi(u)}{u(u-1)} du}} \quad (4.7)$$

Taking the first derivative of (4.7) and considering only its numerator, regarding (4.2), gives us Z . (Note that the denominator is a positive value and no needs to take it into account)

$$\begin{aligned} Z &= x(x-1)\pi'(x) \log x - \pi(x) \left\{ \pi(x) \log x - \int_2^x \frac{\pi(u)}{u} du \right\} \times \\ &\quad \left\{ \log \left(\pi(x) \log x - \int_2^x \frac{\pi(u)}{u} du \right) \right\} \end{aligned} \quad (4.8)$$

Just, we should prove that $Z > 0$ for $x \geq e^{130000}$ since we can suppose that x is a sufficiently large number.

Prime number theorem asserts [55], page 137.

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right) \quad (4.9)$$

and differentiating $\pi(x)$ regarding proposition 1, yields

$$\pi'(x) = \frac{1}{\log x} + O\left(\frac{(1 - \frac{a}{2\sqrt{\log x}})}{e^{a\sqrt{\log x}}}\right) \quad (4.10)$$

since we put $f(x) = O\left(\frac{x}{e^{a\sqrt{\log x}}}\right)$, $f'(x) = O\left(\frac{(1 - \frac{a}{2\sqrt{\log x}})}{e^{a\sqrt{\log x}}}\right)$ and $g(x) = \frac{x}{e^{a\sqrt{\log x}}}$, $g'(x) = \frac{(1 - \frac{a}{2\sqrt{\log x}})}{e^{a\sqrt{\log x}}}$, $g(x) > 0$, $g'(x) > 0$, $g(0) = 0$, and $\lim_{x \rightarrow +\infty} g(x) = +\infty$.

Substituting (4.9) and (4.10) for (4.8), we have

$$\begin{aligned} Z = & x(x-1) + O\left(x(x-1)\frac{(\log x - \frac{a\sqrt{\log x}}{2})}{e^{a\sqrt{\log x}}}\right) - \\ & \left\{\int_2^x \frac{du}{\log u} + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right)\right\} \times \\ & \left\{\log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right)\right\} \times \\ & \log \left\{\log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right)\right\} \end{aligned} \quad (4.11)$$

Denoting the third term of the (4.11) as S and expanding it, we have

$$\begin{aligned} S = & \left\{\int_2^x \frac{du}{\log u} + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right)\right\} \times \\ & \left\{\log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right)\right\} \times \\ & \log \left\{\log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right)\right\} = \\ & \left\{\left(\int_2^x \frac{du}{\log u}\right)^2 \log x - \left(\int_2^x \frac{du}{\log u}\right)\left(\int_2^x \frac{\pi(u)du}{u}\right)\right\} \times \end{aligned}$$

$$\begin{aligned} & \log \left\{ \log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right) \right\} + \\ & \left\{ O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) + O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) + O\left(\frac{x \int_2^x \frac{\pi(u)du}{u}}{e^{a\sqrt{\log x}}}\right) + \right. \\ & \left. O\left(\frac{x^2 \log x}{e^{2a\sqrt{\log x}}}\right) \right\} \log \left\{ \log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right) \right\} \quad (4.12) \end{aligned}$$

Since

$$\begin{aligned} & O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) + O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) + O\left(\frac{x \int_2^x \frac{\pi(u)du}{u}}{e^{a\sqrt{\log x}}}\right) + \\ & O\left(\frac{x^2 \log x}{e^{2a\sqrt{\log x}}}\right) = O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) \quad (4.13) \end{aligned}$$

because $\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}$ is absolutely greater than those of $\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}$, $\frac{x \int_2^x \frac{\pi(u)du}{u}}{e^{a\sqrt{\log x}}}$ and $\frac{x^2 \log x}{e^{2a\sqrt{\log x}}}$. Therefore, regarding (4.12) and (4.13), we have

$$\begin{aligned} S = & \left\{ \left(\int_2^x \frac{du}{\log u} \right)^2 \log x - \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) \right\} \times \\ & \log \left\{ \log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right) \right\} \quad (4.14) \end{aligned}$$

If we substitute the relation (4.15) based on the Prime Number Theorem for the last term of the right-hand side of (4.14), i.e. to make simple our calculations, we make use of

$\theta(x) = x + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right)$ instead of $\left\{ \log x \int_2^x \frac{du}{\log u} - \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right) \right\}$ in the last term of the right hand side of (4.14).

$$\log \theta(x) = \log \left(x + O\left(\frac{x}{e^{a\sqrt{\log x}}}\right) \right) = \log x + \log \left(1 + O\left(\frac{1}{e^{a\sqrt{\log x}}}\right) \right) \quad (4.15)$$

then,

$$\log \theta(x) = \log x + \epsilon \quad (4.16)$$

where $\epsilon = \lim_{x \rightarrow \infty} \log \left(1 + O\left(\frac{1}{e^{a\sqrt{\log x}}}\right) \right)$ and tends to zero.

(4.14) implies that

$$S = \left\{ \left(\int_2^x \frac{du}{\log u} \right)^2 \log x - \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} + O\left(\frac{x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right) \right\} \times \\ \{ \log x + \epsilon \} = \left(\int_2^x \frac{du}{\log u} \right)^2 (\log x)^2 - \left\{ \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} \right\} \log x + \\ O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right) + \epsilon \left(\int_2^x \frac{du}{\log u} \right)^2 \log x - \epsilon \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} \quad (4.17)$$

since $\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}$ is greater than $\frac{\epsilon x \log x \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}$, we consider only $O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right)$ in (4.17).

Just, we have

$$Z = x(x-1) + O\left(x(x-1) \frac{(\log x - \frac{a\sqrt{\log x}}{2})}{e^{a\sqrt{\log x}}} \right) - S = \\ x(x-1) - \left(\int_2^x \frac{du}{\log u} \right)^2 (\log x)^2 + \left\{ \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} \right\} \log x - \\ \epsilon \left(\int_2^x \frac{du}{\log u} \right)^2 \log x + \epsilon \int_2^x \frac{du}{\log u} \int_2^x \frac{\pi(u)du}{u} + \\ O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right) \quad (4.18)$$

since we know

$$O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right) + O\left(x(x-1) \frac{(\log x - \frac{a\sqrt{\log x}}{2})}{e^{a\sqrt{\log x}}} \right) = \\ O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}} \right) \quad (4.19)$$

due to being greater the first term O -notation than the second one.

To evaluate $\int_2^x \frac{\pi(u)du}{u}$ given in (4.18) and for sufficiently large number $x > N$

(N is a sufficiently large number as e^{120000}), we have the following identity when x tends to infinity since we know $\frac{x-2}{\log x} < li(x) < \frac{x}{\log x-2}$ for $x \geq e^4$ (see lemmas 7,8 of [42]) because as x gets sufficiently large and dividing both sides of the inequality by $\frac{x-2}{\log x}$, $li(x)$ tends to $\frac{(x-2)}{\log x}$:

$$\int_2^x \frac{du}{\log u} = \frac{(x-2)}{\log x} + \beta \quad (4.20)$$

where $\beta > 0$ and we can also make use of the mean value theorem in integrals. Regarding equivalency $\pi(x) = \frac{x}{\log x} + \alpha$ for sufficiently large numbers, where $\alpha > 0$ since we know $\frac{x}{\log x+2} < \pi(x) < \frac{x}{\log x-4}$ for $x \geq 55$ (see theorem 29, part A of [42] and corollary 1 of [43]) for sufficiently large number $x > N$ and dividing both sides of the inequality by $\frac{x}{\log x}$ we have $\pi(x) = \frac{x}{\log x} + \alpha$ then putting (4.20) and $\pi(x) = \frac{x}{\log x} + \alpha$ into (4.21), we have

$$\int_2^x \frac{\pi(u)du}{u} = \frac{(x-2)}{\log x} + \alpha \log \frac{x}{2} + \beta \quad (4.21)$$

where $\lim_{x \rightarrow \infty} \beta = 0$ and $\lim_{x \rightarrow \infty} \alpha = 0$. α and β are independent of x , but tend to zero as x gets large.

Just, we substitute (4.20) and (4.21) for (4.18) to investigate if Z is positive or negative as follows:

$$\begin{aligned} Z = & x^2 - x - \left(\frac{(x-2)}{\log x}\right)^2 \log x^2 - 2\beta \frac{(x-2)}{\log x} (\log x)^2 - \\ & \beta^2 (\log x)^2 + \left(\frac{(x-2)}{\log x}\right)^2 \log x + \alpha \log \frac{x}{2} (x-2) + \beta(x-2) + \\ & \beta(x-2) + \alpha\beta \log \frac{x}{2} \log x + \beta^2 \log x - \epsilon \frac{(x-2)^2}{\log x} - \\ & 2\epsilon\beta(x-2) - \epsilon\beta^2 \log x + \epsilon \frac{(x-2)^2}{(\log x)^2} + \epsilon\alpha \log \frac{x}{2} \frac{(x-2)}{\log x} + \\ & \epsilon\beta \frac{(x-2)}{\log x} + \epsilon\beta \frac{(x-2)}{\log x} + \alpha\epsilon\beta \log \frac{x}{2} + \end{aligned}$$

$$\epsilon\beta^2 - A\beta \frac{x(\log x)^2}{e^{a\sqrt{\log x}}} - A \frac{x(x-2)\log x}{e^{a\sqrt{\log x}}} \quad (4.22)$$

where $-A\beta \frac{x(\log x)^2}{e^{a\sqrt{\log x}}}$ and $-A \frac{x(x-2)\log x}{e^{a\sqrt{\log x}}}$ denote the least negative values of big O -notation in the last term of (4.18) after substituting (4.20) into big O -notation and decomposing it into the two aforementioned terms since $O\left(\frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}\right) \leq A \frac{x(\log x)^2 \int_2^x \frac{du}{\log u}}{e^{a\sqrt{\log x}}}$, where A denotes a constant value. Therefore, we can omit the smaller term versus grater one and write as follows:

$$\begin{aligned} Z = & 3x - 4 + \frac{(x-2)^2}{\log x} + \alpha \log \frac{x}{2}(x-2) + 2\beta(x-2) + \\ & \alpha\beta \log \frac{x}{2} \log x + \beta^2 \log x - 2\beta(x-2) \log x - \beta^2(\log x)^2 + \\ & \epsilon \frac{(x-2)^2}{(\log x)^2} - \epsilon \frac{(x-2)^2}{\log x} + \epsilon\alpha \log \frac{x}{2} \frac{(x-2)}{\log x} + 2\epsilon\beta \frac{(x-2)}{\log x} + \\ & \alpha\epsilon\beta \log \frac{x}{2} + \epsilon\beta^2 - 2\epsilon\beta(x-2) - \epsilon\beta^2 \log x - \\ & A \frac{x(x-2)\log x}{e^{a\sqrt{\log x}}} \end{aligned} \quad (4.23)$$

If $Z > 0$, then we should have

$$\begin{aligned} \frac{(x-2)^2}{\log x} & > 4 - 3x - \alpha \log \frac{x}{2}(x-2) - 2\beta(x-2) - \\ & \alpha\beta \log \frac{x}{2} \log x - \beta^2 \log x + 2\beta(x-2) \log x + \beta^2(\log x)^2 + \\ & \epsilon \frac{(x-2)^2}{\log x} \left(1 - \frac{1}{\log x}\right) - \epsilon\alpha \log \frac{x}{2} \frac{(x-2)}{\log x} - 2\epsilon\beta \frac{(x-2)}{\log x} - \\ & -\alpha\epsilon\beta \log \frac{x}{2} - \epsilon\beta^2 + 2\epsilon\beta(x-2) + \epsilon\beta^2 \log x + \\ & A \frac{x(x-2)\log x}{e^{a\sqrt{\log x}}} \end{aligned} \quad (4.24)$$

Dividing both sides of (4.24) by $\frac{(x-2)^2}{\log x}$, we should have

$$1 > \frac{(4-3x)\log x}{(x-2)^2} - \alpha \frac{\log \frac{x}{2} \log x}{(x-2)} - 2\beta(1-\epsilon) \frac{\log x}{(x-2)} -$$

$$\begin{aligned}
& \alpha\beta \frac{\log \frac{x}{2} (\log x)^2}{(x-2)^2} - \beta^2 (1-\epsilon) \frac{(\log x)^2}{(x-2)^2} + 2\beta \frac{(\log x)^2}{(x-2)} + \\
& \beta^2 \frac{(\log x)^3}{(x-2)^2} + \epsilon \left(1 - \frac{1}{\log x}\right) - \epsilon\alpha \frac{\log \frac{x}{2}}{(x-2)} - 2\epsilon\beta \frac{1}{(x-2)} - \\
& \alpha\epsilon\beta \frac{\log(\frac{x}{2}) \log x}{(x-2)^2} - \epsilon\beta^2 \frac{\log x}{(x-2)^2} + \\
& A \frac{x(\log x)^2}{(x-2)e^{a\sqrt{\log x}}}
\end{aligned} \tag{4.25}$$

where ϵ , α and β tend to zero as x gets large enough or tends to infinity. As is shown in the inequality (4.25), all the terms of the right-hand side tend to zero when x is a sufficiently large number for $x \geq e^{130000}$ since the absolute value of each term indicates that that term is a strictly decreasing function. If we let $A = 1$ and a possible value $a = \frac{1}{15}$, the sum of the terms of the right-hand side of the inequality (4.25) takes a value about 0.615 for $x = e^{130000}$ (all the terms from the second one to the one term before the last one tend to zero when $x \geq e^{130000}$). This means that when $x > e^{130000}$, the right hand-side of (4.25) takes values less than 1 and (4.25) holds and $Z > 0$, although the right-hand side of the inequality (4.25) gets values less than 1 for every arbitrary value A when x gets sufficiently large number greater than e^{130000} . Consequently, (4.7) is a strictly increasing function and finally $C(x)$ or $RO_1(n)$ so is and the proof is completed.

4.11. **Proof of Lemma 10**

Let $n \geq 66$ and $p_1 = 3$, then

$$\begin{aligned}
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1^2 \times \dots \times p_n^2) = \\
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \{\log(2p_1 \times \dots \times p_n) + \log(p_1 \times \dots \times p_n)\} =
\end{aligned}$$

$$\begin{aligned}
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \times \\
& \log \left\{ \log(2p_1 \times \dots \times p_n) \left(1 + \frac{\log(p_1 \times \dots \times p_n)}{\log(2p_1 \times \dots \times p_n)}\right) \right\} = \\
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \times \\
& \left\{ \log \log(2p_1 \times \dots \times p_n) + \log \left(1 + \frac{\log(p_1 \times \dots \times p_n)}{\log(2p_1 \times \dots \times p_n)}\right) \right\} = \\
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1 \times \dots \times p_n) + \\
& e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \left(1 + \frac{\log(p_1 \times \dots \times p_n)}{\log(2p_1 \times \dots \times p_n)}\right) \quad (4.26)
\end{aligned}$$

Just, we prove that for $n \geq 66$,

$$\frac{e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \left(1 + \frac{\log(p_1 \times \dots \times p_n)}{\log(2p_1 \times \dots \times p_n)}\right)}{2 - e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1 \times \dots \times p_n)} > 1 \quad (4.27)$$

First of all, we should show that the denominator of (4.27) i.e. $2 - RO_1(n) > 0$ and it's a decreasing function.

Dividing the numerator and denominator of the left-hand side of the inequality (4.27) by $e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right)$, we have

$$\frac{\log \left(1 + \frac{\log(p_1 \times \dots \times p_n)}{\log(2p_1 \times \dots \times p_n)}\right)}{\frac{2}{e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right)} - \log \log(2p_1 \times \dots \times p_n)} \quad (4.28)$$

Easily checking out (4.28), we find out that the numerator is an increasing function and tends to $\log 2$ when n tends to infinity. On the other hand, the denominator is a decreasing function tending to zero because

Regarding lemma 7, we have

$$\begin{aligned}
\frac{1}{\log p_n} \left(1 - \frac{1}{2 \log^2 p_n}\right) & < e^\gamma \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) < \\
& \frac{1}{\log p_n} \left(1 + \frac{1}{2 \log^2 p_n}\right) \quad (4.29)
\end{aligned}$$

for $p_n \geq 331$ or $n \geq 66$ and

$$\frac{2}{\log p_n} \left(1 - \frac{1}{2 \log^2 p_n}\right) < e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) < \frac{2}{\log p_n} \left(1 + \frac{1}{2 \log^2 p_n}\right) \quad (4.30)$$

for $p_n \geq 331$ or $n \geq 66$ then,

$$\frac{2 \log^3 p_n}{2 \log^2 p_n + 1} < \frac{2}{e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right)} < \frac{2 \log^3 p_n}{2 \log^2 p_n - 1} \quad (4.31)$$

Regarding lemma 8, the expression $\log \log(2p_1 \times \dots \times p_n) \approx \log p_n$ (located at the denominator of (4.28)) is correct for sufficiently large integer n since lemma 9 asserts that the function $RO_1(n)$ is an increasing one for $n \geq 66$.

On the other hand, regarding lemma 8 and (4.30), we have:

$$\left\{ \frac{2 \log \log(2p_1 \dots p_n)}{\log p_n} \right\} \left(1 - \frac{1}{2 \log^2 p_n}\right) < e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \times \\ \log \log(2p_1 \dots p_n) < \left\{ \frac{2 \log \log(2p_1 \dots p_n)}{\log p_n} \right\} \left(1 + \frac{1}{2 \log^2 p_n}\right) \quad (4.32)$$

where implies that

$$RO_1(n) = e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_n}\right) \log \log(2p_1 \dots p_n) \sim \frac{2 \log \log(2p_1 \dots p_n)}{\log p_n} \quad (4.33)$$

and the function $\frac{\log \log(2p_1 \dots p_n)}{\log p_n}$ is an increasing one averagely. Just, we prove that $RO_1(n) < 2$:

Since we know that $RO_1(n)$ is a strictly and continuously increasing function regarding lemma9, then its value must not be over 2 because using *reductio ad absurdum* argument, we say if it were over 2, then regarding (4.30) and (3.9), $RO_1(n)$ should be expressed as (4.34):

Let us have a sufficiently large number N , so that if $p_n > N$, then $RO_1(n) = 2 + \epsilon$ where $\epsilon > 0$ and

$$RO_1(n) = 2 + \epsilon < 2 \left(1 + \frac{1}{2 \log^2 p_n}\right) \left\{ 1 + \frac{\log \left(1 + \frac{1}{2 \log p_n}\right)}{\log p_n} \right\} =$$

$$2 + \frac{1}{\log^2 p_n} + \frac{2 \log(1 + \frac{1}{2 \log p_n})}{\log p_n} + \frac{1}{\log^2 p_n} \cdot \frac{\log(1 + \frac{1}{2 \log p_n})}{\log p_n} \quad (4.34)$$

In such a case, (4.34) implies that the upper bound $\frac{1}{\log^2 p_n} + \frac{2 \log(1 + \frac{1}{2 \log p_n})}{\log p_n} + \frac{1}{\log^2 p_n} \cdot \frac{\log(1 + \frac{1}{2 \log p_n})}{\log p_n} > \epsilon$. But, when p_n gets sufficiently large so that $p_n \gg N$, then the upper bound $\frac{1}{\log^2 p_n} + \frac{2 \log(1 + \frac{1}{2 \log p_n})}{\log p_n} + \frac{1}{\log^2 p_n} \cdot \frac{\log(1 + \frac{1}{2 \log p_n})}{\log p_n} < \epsilon$ where is a contradiction, therefore $RO_1(n)$ cannot be greater than or equal to 2. Since one can check it out for example for $RO_1(66) = 1.95547$, $RO_1(239) = 1.991034461$, $RO_1(10,000) = 1.99887$, $RO_1(100,000) = 1.99975$, $RO_1(1000,000) = 1.99994$. This means that regarding the above argument, $RO_1(n)$ cannot even be equal to number 2, because due to being strictly increasing if it can be 2, then it can be even more and more, which is impossible since we proved it cannot be more than 2.

The denominator of (4.28) gets started to be close to zero and (4.28) itself gets started to be an increasing function and (4.27) so is since we find

$$\begin{aligned} & \frac{2 \log^3 p_n}{2 \log^2 p_n + 1} - \log p_n - \log(1 + \frac{1}{2 \log p_n}) < \\ & \frac{2}{e^\gamma (1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_n})} - \log \log(2p_1 \times \dots \times p_n) < \\ & \frac{2 \log^3 p_n}{2 \log^2 p_n - 1} - \log p_n - \log(1 - \frac{1}{2 \log p_n}) \end{aligned} \quad (4.35)$$

This means that

$$\lim_{p_n \rightarrow \infty} \left\{ \frac{2}{e^\gamma (1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_n})} - \log \log(2p_1 \times \dots \times p_n) \right\} = 0 \quad (4.36)$$

On the other hand, the denominator 4.27 regarding lemma 9 is a decreasing function since we showed that $RO_1(n) < 2$ and the function $2 - RO_1(n) > 0$ is a decreasing one.

Finally, (3.11) holds as well. If we check out (4.27) for $p_{66} = 331$, we find it is $5.3099 > 1$ and for $p_{239} = 1531$ we find $21.103 > 1$ where represents an increasing function and its denominator is a decreasing one, it guarantees that (4.27) is always greater than 1 for $n \geq 66$.

4.12. *Proof of Lemma 11*

The proof is made by the combination of the lemmas 1 and 5 and found in the Lagarias' paper [29].

4.13. *Proof of Theorem 1*

The proof has already been made by computer aided computations and cited in [29] and [21].

4.14. *Proof of Theorem 2*

We have checked Lagarias and Robin's criteria for $5041 \leq n \leq 19685$ on the computer by Maple16 without any errors. The programs are found at Sections A and B of the end of the paper:

4.15. *Proof of Theorem 3*

The proof can be made by a combination of the Lemmas 3 and 5.

4.16. *Proof of Theorem 4*

Note that $p_1 = 3$ is considered the origin of counting the odd primes for this theorem

The inequality (3.14) for odd number $m = p_i^{\alpha_i} \times \dots \times p_k^{\alpha_k}$ implies that

$$\frac{p_i^{\alpha_i+1} - 1}{p_i - 1} \dots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} < \frac{1}{2} e^{\gamma} p_i^{\alpha_i} \dots p_k^{\alpha_k} \log \log(2 p_i^{\alpha_i} \dots p_k^{\alpha_k}) \quad (4.37)$$

where p_i to p_k denote odd prime numbers. Then,

$$\frac{2p_i^{\alpha_i+1}(1 - \frac{1}{p_i^{\alpha_i+1}}) \dots p_k^{\alpha_k+1}(1 - \frac{1}{p_k^{\alpha_k+1}})}{p_i(1 - \frac{1}{p_i}) \dots p_k(1 - \frac{1}{p_k})} < \frac{e^\gamma p_i^{\alpha_i} \dots p_k^{\alpha_k} \log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})}{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})} \quad (4.38)$$

where implies that

$$2(1 - \frac{1}{p_i^{\alpha_i+1}}) \dots (1 - \frac{1}{p_k^{\alpha_k+1}}) < e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) \times \frac{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})}{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})} \quad (4.39)$$

Regarding lemma 10, we showed that when $i = 1$ and $\alpha_1 = \alpha_2 = \dots = \alpha_k = 2$, then the right-hand side of (4.39) is a function greater than 2. Therefore, the proof of the theorem 4 is finished for all the odd integers $m \geq (3 \times 5 \times \dots \times p_j)^2$ for $j \geq 66$ including only consecutively square prime factors starting with $p_1 = 3$.

Just, we should prove theorem 4 holds for odd numbers m of the various prime factors since $j \geq 66$, thus

$$2(1 - \frac{1}{p_i^{\alpha_i+1}}) \dots (1 - \frac{1}{p_k^{\alpha_k+1}}) < 2 < e^\gamma (1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_j}) \times \frac{\log \log(2p_1^2 \dots p_j^2)}{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})} < e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) \times \frac{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})}{\log \log(2p_i^{\alpha_i} \dots p_k^{\alpha_k})} \quad (4.40)$$

where $p_i^{\alpha_i} \dots p_k^{\alpha_k} > p_1^2 \dots p_j^2$ for $j \geq 66$.

We prove the inequality (4.40) holds for all the odd numbers greater than or equal to $m = (3 \times 5 \times \dots \times 331)^2$. Dividing all the odd numbers $m \geq (3 \times 5 \times \dots \times 331)^2$ into the six main groups, we have:

1- The number of consecutively odd prime numbers of the power two is greater

than or equal to 66 and starting with $p_1 = 3$. In such a case, the conditions are completely like those of lemma 10 and the proof is same proof of lemma 10 and (4.39) or (4.40) hold for group 1.

2- The number of the odd prime numbers is equal to 66 as $m = p_i^{\alpha_i} \times \dots \times p_{i+65}^{\alpha_{i+65}}$, then we find the following form from (4.39) or (4.40) regarding lemma 10

$$\begin{aligned} 2\left(1 - \frac{1}{p_i^{\alpha_i+1}}\right) \dots \left(1 - \frac{1}{p_{i+65}^{\alpha_{i+65}+1}}\right) &< 2 < e^\gamma \left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{331}\right) \times \\ \log \log(2 \times (3 \times \dots \times 331)^2) &< e^\gamma \left(1 - \frac{1}{p_i}\right) \dots \left(1 - \frac{1}{p_{i+65}}\right) \times \\ \log \log(2 \times p_i^{\alpha_i} \times \dots \times p_{i+65}^{\alpha_{i+65}}) & \end{aligned} \quad (4.41)$$

since $m = p_i^{\alpha_i} \times \dots \times p_{i+65}^{\alpha_{i+65}} > (3 \times \dots \times 331)^2$ and $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_{i+65}}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{331})$, because the number of the primes p_i to p_{i+65} is equal to the number of the primes 3 to 331, but some of the primes between p_i and p_{i+65} are greater than or equal to those of 3 to 331. Then, the inequality (3.14) holds for group 2.

3- The number of the odd primes is less than 66 and their values are such that $3 \leq p_i, \dots, p_k \leq 331$ or $p_i, \dots, p_k \geq 331$. This means that there exist some of primes within the set $\{3, 5, 7, \dots, 331\}$ and some others are out of this set (i.e some of them are greater than 331), then (4.39) or (4.40) regarding lemma 10 implies that

$$\begin{aligned} 2\left(1 - \frac{1}{p_i^{\alpha_i+1}}\right) \dots \left(1 - \frac{1}{p_k^{\alpha_k+1}}\right) &< 2 < e^\gamma \left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{331}\right) \times \\ \log \log(2 \times (3 \times \dots \times 331)^2) &< e^\gamma \left(1 - \frac{1}{p_i}\right) \dots \left(1 - \frac{1}{p_k}\right) \times \\ \log \log(2 \times p_i^{\alpha_i} \times \dots \times p_k^{\alpha_k}) & \end{aligned} \quad (4.42)$$

since

$$(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{331}) \quad (4.43)$$

and

$$\log \log(2 \times p_i^{\alpha_i} \times \dots \times p_k^{\alpha_k}) > \log \log(2 \times (3 \times \dots \times 331)^2) \quad (4.44)$$

because we assume $m = p_i^{\alpha_i} \times \dots \times p_k^{\alpha_k} > (3 \times 5 \times 7 \times \dots \times 331)^2$. Therefore, the inequality (3.14) holds for group 3.

4- The number of the odd primes is greater than 66 (let $M > 66$) and primes are greater than p_{66} i.e. $p_i > p_{66}$ for $66 < i \leq N = M + i - 1$, then (4.39) or (4.40) regarding lemma 10 implies that

$$\begin{aligned} 2(1 - \frac{1}{p_i^{\alpha_i+1}}) \dots (1 - \frac{1}{p_N^{\alpha_N+1}}) &< 2 < e^\gamma \overbrace{(1 - \frac{1}{3}) \dots (1 - \frac{1}{331})}^{66 \text{ terms}} \times \\ &\quad (1 - \frac{1}{p_{67}}) \times \dots \times (1 - \frac{1}{p_M}) \times \\ &\quad \log \log(2 \times \overbrace{(3 \times \dots \times 331)}^{66 \text{ terms}} \times p_{67} \times \dots \times p_M)^2 < \\ &\quad e^\gamma (1 - \frac{1}{p_i}) \times \dots \times (1 - \frac{1}{p_N}) \times \\ \log \log(2 \times (p_i \times \dots \times p_N)^2) &< e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_N}) \times \\ &\quad \log \log(2 \times p_i^{\alpha_i} \times \dots \times p_N^{\alpha_N}) \end{aligned} \quad (4.45)$$

for $m = p_i^{\alpha_i} \times \dots \times p_N^{\alpha_N} > (p_i \times \dots \times p_N)^2 > (3 \times \dots \times p_M)^2$ and $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_N}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{331})(1 - \frac{1}{p_{67}}) \dots (1 - \frac{1}{p_M})$, because the number of the primes p_i to p_N is equal to the number of the primes 3 to p_M , but values p_i to p_N are greater than those of 3 to p_M . Thus, the inequality (3.14) is proved for group 4.

5- The number of the odd primes is greater than 66 and there exist some primes within the set $\{3, 5, 7, \dots, 331\}$ and some others greater than 331. Let $p_i, p_{i+1}, \dots, p_{i+k}$ be within the set $\{3, 5, 7, \dots, 331\}$ and $p_{i+k+1}, \dots, p_N > 331$. Let the entire number of the odd primes be $N - i + 1$, then (4.39) or (4.40) regarding lemma 10 implies that

$$\begin{aligned}
2(1 - \frac{1}{p_i^{\alpha_i+1}}) \dots (1 - \frac{1}{p_{i+k}^{\alpha_{i+k}+1}}) (1 - \frac{1}{p_{i+k+1}^{\alpha_{i+k+1}+1}}) \dots (1 - \frac{1}{p_N^{\alpha_N+1}}) < 2 < \\
e^\gamma (1 - \frac{1}{3}) \dots (1 - \frac{1}{p_{N-i+1}}) \times \\
\log \log (2 \times (3 \times \dots \times 331 \times \dots \times p_{N-i+1})^2) < \\
e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_{i+k}}) (1 - \frac{1}{p_{i+k+1}}) \dots (1 - \frac{1}{p_N}) \times \\
\log \log (2 \times (p_i \times \dots \times p_N)^2) < \\
e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_{i+k}}) (1 - \frac{1}{p_{i+k+1}}) \dots (1 - \frac{1}{p_N}) \times \\
\log \log (2 \times p_i^{\alpha_i} \times \dots \times p_N^{\alpha_N}) \quad (4.46)
\end{aligned}$$

since $p_i^{\alpha_i} \times \dots \times p_N^{\alpha_N} > (p_i \times \dots \times p_N)^2 > (3 \times \dots \times 331 \times \dots \times p_{N-i+1})^2$ and $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_{i+k}}) (1 - \frac{1}{p_{i+k+1}}) \dots (1 - \frac{1}{p_N}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{p_{N-i+1}})$, because $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_{i+k}})$ is equal to corresponding values in the set $\{(1 - \frac{1}{3}), \dots, (1 - \frac{1}{p_{N-i+1}})\}$ and $(1 - \frac{1}{p_{i+k+1}}) \dots (1 - \frac{1}{p_N})$ is greater than the corresponding values in same set. The number of the primes 3 to p_{N-i+1} is equal to those of p_i to p_N . Thus, the inequality (3.14) holds for group 5.

6- The number of the odd primes is greater than/less than or equal to 66. $m = p_i \times \dots, p_k$ and let $M = k - i + 1$ be the number of the odd primes, then regarding (4.39) with $\alpha_i = \dots = \alpha_k = 1$, we find

For $M > 66$

$$2(1 - \frac{1}{p_i^2}) \dots (1 - \frac{1}{p_k^2}) < 2 <$$

$$\begin{aligned}
& e^\gamma \left(1 - \frac{1}{3}\right) \dots \left(1 - \frac{1}{331}\right) \left(1 - \frac{1}{p_{67}}\right) \dots \left(1 - \frac{1}{p_M}\right) \times \\
& \log \log(2 \times (3 \times \dots \times 331 \times \dots \times p_M)^2) < \\
& e^\gamma \left(1 - \frac{1}{p_i}\right) \dots \left(1 - \frac{1}{p_k}\right) \log \log(2p_i \dots p_k)
\end{aligned} \tag{4.47}$$

and since the number of the primes p_i to p_k is equal to those of 3 to p_M , then $2p_i \dots p_k > 2 \times (3 \times \dots \times 331 \times \dots \times p_M)^2$ and $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{331}) (1 - \frac{1}{p_{67}}) \dots (1 - \frac{1}{p_M})$.

But, if $i = 1$, $M > 66$ and we have consecutively primes, then the above reasoning does not work. In such a case, we make the following proof instead of:

Let

$$p_1 \times \dots \times p_k > (3 \times \dots \times 331)^2$$

where p_1 to p_k denote consecutively primes, then regarding (4.39) and putting $\alpha_1 = \alpha_2 = \dots = \alpha_k = 1$, we prove

$$2\left(1 - \frac{1}{p_1^2}\right) \dots \left(1 - \frac{1}{p_k^2}\right) < e^\gamma \left(1 - \frac{1}{p_1}\right) \dots \left(1 - \frac{1}{p_k}\right) \log \log(2p_1 \dots p_k)$$

or eliminating the common terms of both hand-sides, one should show the inequality

$$2\left(1 + \frac{1}{p_1}\right) \dots \left(1 + \frac{1}{p_k}\right) < e^\gamma \log \log(2p_1 \dots p_k)$$

holds for $k \geq 66$ by induction argument. This means that we wish to prove if the above expression is true for k , then it must be true for $k + 1$ as well.

If $k = 66$, then

$$2\left(1 + \frac{1}{3}\right) \dots \left(1 + \frac{1}{331}\right) = 8.375 < e^\gamma \log \log(2 \times 3 \times \dots \times 331) = 10.192$$

Thus, it holds for $k = 66$. Therefore, let it be true for k , then make the inequality for $k + 1$ as follows:

$$2(1 + \frac{1}{p_1}) \dots (1 + \frac{1}{p_{k+1}}) < e^\gamma \log \log(2p_1 \dots p_{k+1})$$

Multiplying both sides

$$2(1 + \frac{1}{p_1}) \dots (1 + \frac{1}{p_k}) < e^\gamma \log \log(2p_1 \dots p_k)$$

by $(1 + \frac{1}{p_{k+1}})$ and comparing out it with inequality for $k + 1$ and eliminating the common terms of both hand-sides, we should show

$$e^\gamma(1 + \frac{1}{p_{k+1}}) \log \log(2p_1 \dots \times p_k) < e^\gamma \log \log(2p_1 \dots p_{k+1})$$

Manipulating the inequality and eliminating the corresponding logarithms from both hand-sides and taking both hand-sides to the power p_{k+1} , we have to show

$$\log(2p_1 \dots \times p_k) < (1 + \frac{p_{k+1}}{\log(2p_1 \dots \times p_k)})^{p_{k+1}}$$

for $k \geq 66$. Referring lemma 9 and (3.10), we find that the function $RO_1(k)$ is a strictly increasing one for $k \geq 1$. This means that since (3.10) is a strictly increasing function, then the following inequality always holds for every $k \geq 66$

$$\log(2p_1 \dots \times p_k) < (1 + \frac{p_{k+1}}{\log(2p_1 \dots \times p_k)})^{p_{k+1}-1}$$

and consequently

$$\log(2p_1 \dots \times p_k) < (1 + \frac{p_{k+1}}{\log(2p_1 \dots \times p_k)})^{p_{k+1}-1} < (1 + \frac{p_{k+1}}{\log(2p_1 \dots \times p_k)})^{p_{k+1}}$$

and implies that induction argument on k holds and finally both (4.39) and (3.14) hold.

For $M \leq 66$

$$2(1 - \frac{1}{p_i^2}) \dots (1 - \frac{1}{p_k^2}) < 2 < e^\gamma (1 - \frac{1}{3}) \dots (1 - \frac{1}{331}) \times \\ \log \log(2 \times (3 \times \dots \times 331)^2) < e^\gamma (1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) \log \log(2p_i \dots p_k) \quad (4.48)$$

since $2p_i \dots p_k > 2(3 \times \dots \times 331)^2$ and $(1 - \frac{1}{p_i}) \dots (1 - \frac{1}{p_k}) > (1 - \frac{1}{3}) \dots (1 - \frac{1}{331})$, because the number of the primes p_i to p_k is less than the number of the primes 3 to 331 and $p_i \geq 3$ and group 6 is also proved completely.

Therefore, we proved that $\sigma(m) < \frac{1}{2}e^\gamma m \log \log(2m)$ holds for all the odd numbers $m \geq (3 \times 5 \times \dots \times 331)^2$.

4.17. **Proof of Theorem 5**

The proof is made for the odd numbers within $1 \leq m \leq 9841$ provided that we obtain the smallest value for α so that inequality $n = 2^\alpha m \geq 19686$ holds. For each m within $1 \leq m \leq 9841$, we obtain the smallest α so called α_0 . We have executed the programs by Maple16 and checked out the correctness of the Lagarias and Robin's inequalities for such m 's with corresponding its α 's. These programs confirm correctness of Lagarias and Robin's criteria for the smallest α_0 for each odd number $m \in \{1, 9841\}$. Then, we are able to give a mathematical argument for all $\alpha \geq \alpha_0 + 1$ for Lagarias' inequality and $\alpha \geq \alpha_0 + 2$ for Robin's inequality and each odd number $m \in \{1, 9841\}$ together with some other programs. An example of the proof's method is given in the last part of the proof.

First of all, we evaluate the following inequality (4.49) as Lagarias criterion

by our program for $\alpha = \alpha_0$: the program is found at Section C of the end of the paper.

$$\begin{aligned} LaG &= H(2^{\alpha_0}m) + \exp(H(2^{\alpha_0}m)) \log(H(2^{\alpha_0}m)) - \sigma(2^{\alpha_0}m) = \\ H(2^{\alpha_0}m) + \exp(H(2^{\alpha_0}m)) \log(H(2^{\alpha_0}m)) - (2^{\alpha_0+1} - 1)\sigma(m) &> 0 \end{aligned} \quad (4.49)$$

Proof that Lagarias' criterion holds for all $\alpha \geq \alpha_0 + 1$

Regarding Lagarias' paper [29] Let

$$H(n) = \log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt \quad (4.50)$$

Let

$$X(n) = H(n) + \exp(H(n)) \log(H(n)) - (2^{\alpha+1} - 1)\sigma(m) \quad (4.51)$$

where $n = 2^\alpha m$. Substituting (4.50) for (4.51)

$$\begin{aligned} X(n) &= \log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt + \\ n e^{1 - \int_1^n \frac{\{t\}}{t^2} dt} \log \left(\log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt \right) - (2^{\alpha+1} - 1)\sigma(m) \end{aligned} \quad (4.52)$$

and manipulating (4.52) gives

$$\begin{aligned} X(n) &= \log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt + \\ n \left\{ e^{1 - \int_1^n \frac{\{t\}}{t^2} dt} \log \left(\log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt \right) - \frac{2}{m} \sigma(m) \right\} &+ \sigma(m) \end{aligned} \quad (4.53)$$

showing that the terms

$$\log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt$$

and

$$e^{1 - \int_1^n \frac{\{t\}}{t^2} dt} \log \left(\log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt \right) - \frac{2}{m} \sigma(m)$$

given by (4.53) are increasing functions for $n = 2^\alpha m \geq 19686$:

Before making the proof, we show that the expression given between two accolades in (4.53) is a non-negative one for values $\alpha = \alpha_0 + 1$ and odd numbers within $1 \leq m \leq 9841$. This is a necessary condition for proof asserting that this expression (4.53) is a strictly increasing function.

Note that by (4.53), we find out $e^{1-\int_1^n \frac{\{t\}}{t^2} dt} \log \left(\log(n) + 1 - \int_1^n \frac{\{t\}}{t^2} dt \right) - \frac{2}{m} \sigma(m) \geq e^{(0.5772)} \log(\log(n) + 0.5772) - \frac{2}{m} \sigma(m)$ since $1 - \int_1^n \frac{\{t\}}{t^2} dt \geq \gamma$. Therefore, for making the program easily for computing, we make use of $e^{(0.5772)} \log(\log(n) + 0.5772) - \frac{2}{m} \sigma(m)$ instead of. This means that if the program is corrected, then (4.53) so is. The program is found at Section D.

Assume $x = n = 2^\alpha m$ be a real number so that α is also a real number, m be a constantly odd number, and $y = \log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt$ (Although, we already know $H(n)$ is an increasing function, but we would make an analytically mathematical proof for $H(x)$ as well). Then differentiating y with respect to x , we have

$$y' = \frac{1}{x} - \frac{\{x\}}{x^2} \quad (4.54)$$

since $0 \leq \{x\} < 1$, then trivially implies that $y' > 0$. Thus the function y is an increasing continuous one for all x . Therefore, it is also an increasing function for all the natural numbers n . We consider same variables for the second term and show that y is an increasing function. Note that $\frac{2}{m} \sigma(m)$ is a constant value since m is a constant one because our variables are x and α .

$$y = e^{1-\int_1^x \frac{\{t\}}{t^2} dt} \log \left(\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt \right) - \frac{2}{m} \sigma(m) \quad (4.55)$$

then differentiating y with respect to x , we have

$$y' = e^{1-\int_1^x \frac{\{t\}}{t^2} dt} \times$$

$$\frac{\left\{x - \{x\} \left\{1 + \left(\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt\right) \left(\log(\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt)\right)\right\}\right\}}{x^2 \left\{\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt\right\}} \quad (4.56)$$

since we look at the terms restricted in the first brace symbols of the relation (4.56) located at its numerator, we find that it is a positive value for all $x \geq 19686$ because the value of x is much greater than the other terms as follows:

$$x > \{x\} \left\{1 + \left(\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt\right) \left(\log(\log(x) + 1 - \int_1^x \frac{\{t\}}{t^2} dt)\right)\right\} \quad (4.57)$$

for $x \geq 19686$. Thus, y is an increasingly continuous function of differentiably piecewise curve. This function has not a strictly continuous differential curve since its differentiability fails at the natural number points. But, it itself is completely continuous and increasing on $x \geq 19686$.

The above reasoning confirms that (4.56) is a positive value for all $x \geq 19686$ and would imply that the function y is an increasing function and on the other hand the program of the section D says us that the starting value of y is also positive. This implies that $X(x)$ when putting $x = n$ in (4.53), is an increasing function and consequently $X(n)$ given in (4.53) so is for each constantly odd value m within closed interval $[1, 9841]$ for all $n = 2^\alpha m \geq 19686$ and finally Lagarias criterion holds for all the even numbers $n = 2^\alpha m \geq 19686$ and each odd number $1 \leq m < 9843$. This means that let $n_1 = 2^{\alpha_0+1}m$, where α_0 denotes the smallest value for satisfying $2^{\alpha_0}m \geq 19686$ and let $X(n)$ for $n \geq n_1$, then

$$X(n) \geq X(n_1) > 0 \quad (4.58)$$

for all $n \geq n_1 = 2^{\alpha_0+1}m > 19686$ where m is an odd number in $\{1, 9841\}$.

Note that we consider an $m \in \{1, 9841\}$ for all n when $\alpha > \alpha_0$.

Just, we repeat all these steps carried out above for evaluation of the Robin's criterion. First of all, we evaluate the following inequality (4.59) as Robin's criterion by our program for $1 \leq m \leq 9841$ and $n = 2^\alpha m \geq 19686$ with the smallest values for $\alpha = \alpha_0$ and then $\alpha = \alpha_0 + 1$ with $n = n_0$ and $n = n_1$, respectively: The programs yields Robin's criterion holds for these values. The programs are found at Sections E and F.

$$ROB = 2^\alpha e^\gamma m \ln(\ln(2^\alpha m)) - (2^{\alpha+1} - 1)\sigma(m) \geq 0 \quad (4.59)$$

Proof that Robin's criterion holds for all $\alpha \geq \alpha_0 + 2$

As shown by the above programs, Robin's' criterion holds for both $\alpha = \alpha_0$ and $\alpha = \alpha_0 + 1$. Just we prove that if

$$Y(n) = Y(2^\alpha m) = 2^\alpha e^\gamma m \ln(\ln(2^\alpha m)) - (2^{\alpha+1} - 1)\sigma(m) \geq 0 \quad (4.60)$$

for all $n \geq n_2 = 2^{\alpha_0+2}m \geq 19686$, then

$$Y(n) \geq Y(n_2) > 0 \quad (4.61)$$

where $n_2 = 2^{\alpha_0+2}m > 19686$ and

$$\begin{aligned} Y(n) &= 2^\alpha e^\gamma m \ln(\ln(2^\alpha m)) - (2^{\alpha+1} - 1)\sigma(m) = \\ &2^\alpha \{e^\gamma m \ln(\ln(2^\alpha m)) - 2\sigma(m)\} + \sigma(m) \end{aligned} \quad (4.62)$$

and

$$\begin{aligned} Y(n_2) &= 2^{\alpha_0+2} e^\gamma m \ln(\ln(2^{\alpha_0+2} m)) - (2^{\alpha_0+3} - 1)\sigma(m) = \\ &2^{\alpha_0+2} \{e^\gamma m \ln(\ln(2^{\alpha_0+2} m)) - 2\sigma(m)\} + \sigma(m) \end{aligned} \quad (4.63)$$

If $\alpha \geq \alpha_0 + 2$, then (4.62) and (4.63) imply that $Y(n) \geq Y(n_2) > 0$ for each constantly odd number m since

$$2^\alpha \{e^\gamma m \ln(\ln(2^\alpha m)) - 2\sigma(m)\} > 2^{\alpha_0+2} \{e^\gamma m \ln(\ln(2^{\alpha_0+2} m)) - 2\sigma(m)\} \quad (4.64)$$

for when $\alpha \geq \alpha_0 + 2$.

For showing that (4.64) holds for all $\alpha \geq \alpha_0 + 2$ and $m \in \{1, 9841\}$, we must show that the given expression between its accolades is non-negative for all $\alpha \geq \alpha_0 + 2$ and $m \in \{1, 9841\}$. For this, we must show that the expression holds for all initial values $\alpha = \alpha_0 + 2$ by the program given at Section G:

Just, we find out that (4.64) holds for all $\alpha \geq \alpha_0 + 2$ and finally (4.61) does hold.

(4.58) and (4.61) imply that Lagarias and Robin's criteria hold for all odd integer class numbers $m \in \{1, 9841\}$ regarding definition 34 and Lemma 6 for $N = 19686$. This completes the proof of Theorem 5.

Example 1:

For Lagarias and Robin's criteria, we choose an m between 1 and 9841. We choose arbitrarily $m = 1367$. Then, as stated in the above programs, we compute α_0 (the smallest value of α) for satisfying the inequality $2^{\alpha_0} m \geq 19686$. Using ceiling functions, we have

$$\alpha_0 = \lceil \frac{\ln 19686 - \ln 1367}{\ln 2} \rceil = 4$$

and $M_1 = 2^{\alpha_0} m = 2^4 \times 1367 = 21872$. The value $\sigma(1367) = 1368$.

We compute LAG given in (4.49) for $\alpha = \alpha_0 + 2$ and $M_2 = 2^{\alpha_0+2} m = 2^6 \times 1367 = 87488$ and for $M_3 = 2^{\alpha_0+3} m = 2^7 \times 1367 = 174976$ as follows:

$LaG(M_2) = LAG(2^6 \times 1367) = H(87488) + \exp(H(87488) \log(H(87488) - (2^{4+3} - 1)\sigma(1367)) = 2.12916 \times 10^5 > 0$ and $LAG(M_3) = 4.42013 \times 10^5$, and ROB given in (4.59) for same values as follows:

$ROB(M_2) = e^{0.5772} \times 87488 \ln(\ln(87488)) - (2^{4+3} - 1)1368 = 2.05185 \times 10^5 > 0$
and $ROB(M_3) = 4.27431 \times 10^5$.

As the relations (4.58) and (4.61) assert, if $\alpha \geq \alpha_0 + 2$, then $X(n) \geq X(n_2)$,
and $Y(n) \geq Y(n_2)$. If we establish the odd class number set of 1367, we have
 $CL(1367) = \{2^4 \times 1367, 2^5 \times 1367, 2^6 \times 1367, \dots\}$

This implies that $X(2^\alpha \times 1367) > \dots > X(2^7 \times 1367) = 4.42013 \times 10^5 >$
 $X(2^6 \times 1367) = 2.12916 \times 10^5 > 0$ and $Y(2^\alpha \times 1367) > \dots > Y(2^7 \times 1367) =$
 $4.27431 \times 10^5 > Y(2^6 \times 1367) = 2.05185 \times 10^5 > 0$ when α tends to infinity
and is a natural number. One observes that $X(87488) > Y(87488) > 0$
and generally $X(2^\alpha \times 1367) > Y(2^\alpha \times 1367)$ and this inequality holds for all
 $m \in \{1, 9841\}$ and all $\alpha \geq \alpha_0 + 2$.

This means that whenever α increases from $\alpha_0 + 2$ for a constantly odd
number m , the inequalities of Lagarias and Robin so do. Robin and Lagarias
criteria for α_0 and $\alpha_0 + 1$ are also satisfied by running the above programs.
This example indeed asserts that the ratio $\frac{\sigma(n)}{e^{\gamma n \ln \ln(n)}} < 1$ for Robin's criterion
and $\frac{\sigma(n)}{H(n) + \exp(n) \ln(H(n))} < 1$ for Lagarias' criterion and for all the members of
the odd class number set of 1367. As stated in this example and the proof
of theorem 5, these two criteria hold for all the odd class numbers 1 to 9841.

4.18. *Proof of Theorem 6*

Using lemmas 2, and 5 we have

$$\sigma(2m) < e^{\gamma} 2m \log \log 2m < H_{2m} + \exp(H_{2m}) \log(H_{2m}) \quad \text{for } 2m \geq 19686 > 3^9 \quad (4.65)$$

where $m \geq 9843$ and is an odd integer. Lemma 2 asserts that

$$e^{\gamma} 2m \log \log(2m) > \sigma(2m) \quad (4.66)$$

for $2m \geq 19686$ or $m \geq 9843$. This implies that for $\alpha_0 = 1$, we have

$$Y(n_0 = 2m) = e^\gamma 2m \log \log(2m) - \sigma(2m) > 0 \text{ for } 2m \geq 19686.$$

As stated in the proof of theorem 5 that the relation (4.62) asserts $Y(n)$ is an increasing function for a fixed $m \in \{1, 9841\}$ and $\alpha \geq \alpha_0 + 2$. Here in this theorem, regarding lemma 2, we should prove that Robin's inequality implies that (4.67) as follows:

$$\begin{aligned} Y(n = 2^\alpha m) &= e^\gamma 2^\alpha m \log \log(2^\alpha m) - \sigma(2^\alpha m) > Y(n_0 = 2m) = \\ &e^\gamma 2m \log \log(2m) - \sigma(2m) > 0 \end{aligned} \quad (4.67)$$

for all $\alpha \geq \alpha_0 = 1$ so that Robin's criterion follows for every odd number $m \geq 9843$ and every even number $2^\alpha m \geq 19686$. To prove, we should show that regarding (4.62)

$$\begin{aligned} Y(n) &= 2^\alpha \{e^\gamma m \ln(\ln(2^\alpha m)) - 2\sigma(m)\} + \sigma(m) > \\ Y(n_0) &= 2 \{e^\gamma m \ln(\ln(2m)) - 2\sigma(m)\} + \sigma(m) \end{aligned} \quad (4.68)$$

This means that we should prove

$$\begin{aligned} 2^\alpha \{e^\gamma m \ln(\ln(2^\alpha m)) - 2\sigma(m)\} + \sigma(m) &> \\ 2 \{e^\gamma m \ln(\ln(2m)) - 2\sigma(m)\} + \sigma(m) \end{aligned} \quad (4.69)$$

for all $\alpha \geq \alpha_0 = 1$ and every odd number $m \geq 9843$. For holding inequality (4.69), we need prove that $e^\gamma m \ln(\ln(2m)) - 2\sigma(m) > 0$ for all $\alpha \geq \alpha_0 = 1$ and every odd number $m \geq 9843$. But, it does not hold for all $m \geq 9843$. For resolving this problem, we prove it holds for all $m \geq (3 \times 5 \times 7 \times \dots \times 331)^2$ instead of.

Therefore, we should prove the following expression holds for all $\alpha \geq \alpha_0 = 1$ and every odd number $m \geq (3 \times 5 \times 7 \times \dots \times 331)^2$:

$$e^\gamma m \ln(\ln(2^\alpha m)) - 2\sigma(m) > e^\gamma m \ln(\ln(2m)) - 2\sigma(m) \quad (4.70)$$

and for this, we should prove that

$$e^\gamma m \ln(\ln(2m)) - 2\sigma(m) > 0 \quad (4.71)$$

for every odd number $m \geq (3 \times 5 \times 7 \times \dots \times 331)^2$.

Just, to prove theorem 6 or to prove inequality (3.13) in lemma 11, the odd numbers $m \geq 9843$ and all the even numbers $n \geq 19686$, should be divided into the two parts. One for all the odd numbers $9843 \leq m \leq (3 \times 5 \times 7 \times \dots \times 331)^2$ and the even numbers $19686 \leq n \leq 2(3 \times 5 \times 7 \times \dots \times 331)^2$. Other one to prove (4.71) for all the odd numbers $m \geq (3 \times 5 \times 7 \times \dots \times 331)^2$ and even numbers $n \geq 2(3 \times 5 \times 7 \times \dots \times 331)^2$. Indeed, we make use of lemmas 2 and 6 for $N = 2(3 \times 5 \times 7 \times \dots \times 331)^2$ instead of $N = 19686$, then all of the steps are similar to those of the case $N = 19686$. By the paper of Thomas Morrill et. al. [36], theorem 13, we find that the inequalities (3.13) hold for all the numbers $5041 \leq n < 10^{10^{13.11485}}$. This means that both Robin's inequality and Lagarias' criterion hold for the our first part as well since $5041 < 9843 \leq m \leq (3 \times 5 \times 7 \times \dots \times 331)^2 < 10^{10^{13.11485}}$ and $5041 < 19686 \leq n \leq 2(3 \times 5 \times 7 \times \dots \times 331)^2 < 10^{10^{13.11485}}$.

For the proof of the second part, we refer to theorem 4 of same our own paper. Therefore, (4.71) holds, then (4.70) holds for all odd numbers $m \geq (3 \times 5 \times 7 \times \dots \times 331)^2$ and even numbers $n \geq 2(3 \times 5 \times 7 \times \dots \times 331)^2$. The Robin's inequality also holds for all the odd and even numbers $n \geq 19686$.

Just lemma 5 asserts that

$e^\gamma 2m \log \log 2m < H_{2m} + \exp(H_{2m}) \log(H_{2m})$ for $2m \geq 19686$. This means that

$$\begin{aligned} X(n = 2^\alpha m) &= H_{2^\alpha m} + \exp(H_{2^\alpha m}) \log(H_{2^\alpha m}) - (2^{\alpha+1} - 1)\sigma(m) > \\ Y(n = 2^\alpha m) &= e^\gamma 2^\alpha m \log \log(2^\alpha m) - \sigma(2^\alpha m) \end{aligned} \quad (4.72)$$

since $H_{2^\alpha m} + \exp(H_{2^\alpha m}) \log(H_{2^\alpha m}) > e^\gamma 2^\alpha m \log \log(2^\alpha m)$ and (4.72) gives us

$$X(n = 2^\alpha m) > Y(n = 2^\alpha m) \quad (4.73)$$

for all $n \geq 2(3 \times 5 \times 7 \times \dots \times 331)^2$ and by $\alpha_0 = 1$

$$X(n_0 = 2m) > Y(n_0 = 2m) \quad (4.74)$$

and by theorem 5 and the above explanations, we have

$$X(n) > X(n_0) > 0 \quad (4.75)$$

,and

$$Y(n) > Y(n_0) > 0 \quad (4.76)$$

for all $n > n_0 \geq 2(3 \times 5 \times 7 \times \dots \times 331)^2$ and finally

$$X(n) > Y(n) > Y(n_0) > 0 \quad (4.77)$$

for all $n \geq 2(3 \times 5 \times 7 \times \dots \times 331)^2$. The relation (4.77) together with the theorem 13 of Thomas Morrill's paper [36] would imply that Lagarias and Robin's inequalities hold for all odd integer class number sets of $m \geq 9843$ regarding definition 34 and Lemma 6 for $N = 19686$. This means that when (4.77) holds, then the ratio

$$\frac{\sigma(n)}{e^\gamma n \ln \ln(n)} < 1 \quad (4.78)$$

for Robin's criterion holds and the ratio

$$\frac{\sigma(n)}{H(n) + \exp(n) \ln(H(n))} < 1 \quad (4.79)$$

for Lagarias' criterion so is and for all the members of the odd class number set of $m \geq 9843$. This proves theorem 6 completely.

Final conclusion

Theorems 1,2,3,5,6 and Lemma 6 confirm that Lagarias and Robin's criteria completely hold for all the odd class numbers sets of $m \geq 1$, then Lemma 11 completely holds and would imply that RH holds forever.

The Maple 16 program sections

Section A

The program for Lagarias' criterion:

```
Restart;
n := 5041;
m := 19685;
Tm := m-n+1;
C := 0;
E := 0;
for k from n to m do
H := 0;
for i to k do
if (i ≤ k) then H := H+1/i;
end if;
if (i = k) then Z := H+exp(H)*ln(H);
end if;
end do;
sumd := 0;
for i to k do
if (gcd(i, k) ≥ 1 and type(k/i, integer))
then sumd := sumd+i;
end if;
end do;
print(k);
print(sumd);
```

```

evalf(H);
XL:= evalf(Z);
LaG := XL-sumd;
if (evalf(LaG)> 0) then print("Lagarias criterion is true");
E := E+1;
else C := C+1;
end if;
end do;
if (E = Tm) then print("Lagarias criterion is totally true");
else print("Lagarias criterion is not totally true");
print(C);
end if;

```

Section B

The program for Robin's criterion:

```

Restart;
n := 5041;
m := 19685;
Tm :=( m-n)+1;
C := 0;
E := 0;
for k from n to m do
sumd := 0;
for i to k do
if (gcd(i, k) ≥ 1 and type(k/i, integer))
then sumd := sumd+i;

```

```

end if;
end do;
print(k);
print(sumd);
YR := exp(.5772)*k*ln(ln(k));
ROB := YR-sumd;
if (evalf(ROB) ≥ 0) then print("Robin criterion is true"); E := E+1;
else C := C+1;
end if;
end do;
if (E = Tm) then print("Robin criterion is totally true") else print("Robin
criterion is not totally true");
print(C);
end if;

```

Section C

In the following program, the terms

```

YL := (ln(19686)-ln(k))/ln(2);
N := floor(Y);
if (YL = N) then nA := N else nA := N+1;
end if;

```

carry out the computation of the first α_0 for each m .

The program for Lagarias' criterion when $\alpha = \alpha_0$ comes here:

```

Restart;
n := 1;
m := 9841;

```

```

Tm := (m-n)/2+1;
C := 0;
E := 0;
for k from n by 2 to m do
H := 0;
YL := (ln(19686)-ln(k))/ln(2);
N := floor(YL);
if (YL = N) then nA := N else nA := N+1;
end if;
M := 2nA * k;
for i to M do
if (i ≤ M) then H := H+1/i;
end if;
if (i = M) then Z := H+exp(H)*ln(H);
end if;
end do;
sumd := 0;
for i to k do
if (gcd(i, k) ≥ 1 and type(k/i, integer))
then sumd := sumd+i;
end if;
end do;
print(sumd);
evalf(H);
XL := evalf(Z);

```

```

LaG := XL-(2(nA+1) - 1)*sumd;
if (evalf(LaG) > 0) then print("Lagarias criterion for ALPHA=ALPHA0 is
true");
E := E+1;
else C := C+1;
print("Lagarias criterion for ALPHA=ALPHA0 is not true");
print("*****");
end if;
end do;
if (E = Tm) then print("Lagarias criterion for ALPHA=ALPHA0 is totally
true");
else print("Lagarias criterion for ALPHA=ALPHA0 is not totally true");
print(C);
end if;

```

Section D

Program for the expression given between two accolades by (4.53)

holds for all $\alpha \geq \alpha_0 + 1$ and odd numbers $1 \leq m \leq 9841$

```

Restart;
n := 1;
m := 9841;
Tm := (m-n)/2+1;
C := 0;
E := 0;
for k from n by 2 to m do
YL := (ln(19686)-ln(k))/ln(2);

```

```

N := floor(YL);
if (YL = N) then nA := N else nA := N+1;
end if;
M :=  $2^{(nA+1)} * k$ ;
X := evalf(exp(.5772).ln(ln(M)+.5772));
sumd := 0;
for i to k do
if (gcd(i, k) ≥ 1 and type(k/i, integer))
then sumd := sumd+i;
end if;
end do;
print(sumd);
LaGamma0 := evalf(X-2*sumd/k);
if (LaGamma0 ≥ 0) then
print("Lagarias inequality for ALPHA=ALPHA0+1 is true"); E:=E+1; else
C:=C+1;
print("Lagarias inequality for ALPHA=ALPHA0+1 is not true");
print("*****");
end if;
end do;
if (E=Tm) then print("Lagarias inequality for ALPHA=ALPHA0+1 is to-
tally true");
else print("Lagarias inequality for ALPHA=ALPHA0+1 is not totally true");
print(C);
end if;

```


Section E

The program for Robin's criterion when $\alpha = \alpha_0$ comes here:

```
Restart;
n := 1;
m := 9841;
Tm := (m-n)*(1/2)+1;
C := 0;
E := 0;
for k from n by 2 to m do
  YL := (ln(19686)-ln(k))/ln(2);
  N := floor(YL);
  if (YL = N) then nA := N; else nA := N+1; end if;
  M := 2nA * k;
  sumd := 0;
  for i to k do
    if (gcd(i, k) ≥ 1 and type(k/i, integer)) then sumd := sumd+i;
  end if;
end do;
print(sumd);
YR := exp(.5772)*M*ln(ln(M));
ROB := YR-(2(nA+1) - 1)*sumd;
if (evalf(ROB) ≥ 0) then print("Robin criterion for ALPHA=ALPHA0 is
true"); E := E+1;
else C := C+1;
print("Robin criterion for ALPHA=ALPHA0 is not true");
```

```

print("*****");
end if;
end do;
if (E = Tm) then print("Robin criterion for ALPHA=ALPHA0 is totally
true"); else print("Robin criterion for ALPHA=ALPHA0 is not totally true");
print(C);
end if;

```

Section F

The program for Robin's criterion when $\alpha = \alpha_0 + 1$ comes here:

```

Restart;
n := 1;
m := 9841;
Tm := (m-n)*(1/2)+1;
C := 0;
E := 0;
for k from n by 2 to m do
  YL := (ln(19686)-ln(k))/ln(2);
  N := floor(YL);
  if (YL = N) then nA := N; else nA := N+1; end if;
  M := 2(nA+1) * k;
  sumd := 0;
  for i to k do
    if (gcd(i, k) ≥ 1 and type(k/i, integer)) then sumd := sumd+i;
  end if;
end do;

```

```

print(sumd);
YR := exp(.5772)*M*ln(ln(M));
ROB := YR-(2(nA+1) - 1)*sumd;
if (evalf(ROB) ≥ 0) then print("Robin criterion for ALPHA=ALPHA0 is
true"); E := E+1;
else C := C+1;
print("Robin criterion for ALPHA=ALPHA0 is not true");
print("*****");
end if;
end do;
if (E = Tm) then print("Robin criterion for ALPHA=ALPHA0 is totally
true"); else print("Robin criterion for ALPHA=ALPHA0 is not totally true");
print(C);
end if;

```

Section G

Program that the expression given between accolades in (4.64) is non-negative for all $\alpha = \alpha_0 + 2$

```

Restart;
n := 1;
m := 9841;
Tm := (m-n)*(1/2)+1;
C := 0;
E := 0;
for k from n by 2 to m do
YL := (ln(19686)-ln(k))/ln(2);

```

```

N := floor(Y);
if (YL = N) then nA := N;
else nA := N+1;
end if;
M :=  $2^{(nA+2)} * k$ ;
X := evalf(exp(.5772).k.ln(ln(M)));
sumd := 0;
for i to k do
if (gcd(i, k) ≥ 1 and type(k/i, integer)) then sumd := sumd+i;
end if;
end do;
print(sumd);
RoBalp0 := evalf(X-2*sumd);
if (evalf(RoBalp0) ≥ 0) then print("Robin inequality for ALPHA=ALPHA0+2
is true");
E := E+1;
else C := C+1;
print("Robin inequality for ALPHA=ALPHA0+2 is not true");
print("*****");
end if;
end do;
if (E = Tm) then print("Robin inequality for ALPHA=ALPHA0+2 is totally
true");
else print("Robin inequality for ALPHA=ALPHA0+2 is not totally true");
print(C); end if;

```

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