ONE GENERALIZATION OF NATURAL NUMBERS AND DERIVED THEOREMS

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Abstract. This paper is supplements of two previous papers. Mainly we show one generalization of natural numbers, and the fact that the identity on natural numbers in the one of the papers is derived from that structure.

1. Introduction

This paper is supplements of two previous papers [1, 2]. On the paper [1, Sect.4], we suggest that we can think about similar objects of natural numbers by decomposing them. In this paper, first we show one generalization of natural numbers from their analysis, and second, we show how the main theorem, the identity on natural numbers, in the paper [2] is derived from the structure in that generalization.

Other contents are almost supplements of the paper [2].

2. One generalization of natural numbers

Definition 2.1.

$X := \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3 \times \cdots$

$\mathbb{P}_i$ is any subset of a power set of any set P_i. $\mathbb{P}_i$ has an operation, and it is a unital magma, which means $\mathbb{P}_i$ is closed and has an identity element.

This operation corresponds to the multiple in natural numbers. P_i corresponds to a prime number and $\mathbb{P}_i$ corresponds to the powers of a prime number. An operation expands to X naturally. Also, we can define *gcd* on X basing on the intersection between the elements of $\mathbb{P}_i$.

Definition 2.2. *Y has an operation, and it is a magma, which means Y is closed and does not have an identity element.*

This operation corresponds to the addition in natural numbers.

Definition 2.3.

$f : X \to Y$

f is a mapping on X to Y, and they are not necessarily finite or infinite. The entire of these three definitions are the generalization of natural numbers. It means we can make natural numbers by adding appropriate conditions to the above.

Especially, if we add explanation on Definition 2.1, we can make natural numbers with setting P_i and $\mathbb{P}_i$ as followings.

$P_i = \{a_1, a_2, a_3, \cdots\}$

$\mathbb{P}_i = \{\boldsymbol{e}, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \cdots\}$

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then setting an operation as corresponding $\mathbb{P}_i$ to the powers of a prime number $\{1, p, p^2, p^3, \cdots\}$.

When we think the following two set A and B, we can describe the identity, the main theorem in the paper [2] as $A = B$.

Definition 2.4. *We denote $a, b, c \in X$ and $d, e \in Y$. For any c, $ab = c \wedge f(a) = d + e \wedge gcd(f^{-1}(d), f^{-1}(e)) = \boldsymbol{e}$ is satisfied.*

$$A = \{\ (d, e, b) \mid \textit{Conditions above and } c \textit{ takes some area.}\ \}$$

Definition 2.5. *We denote $b, c \in X$ and $d, e \in Y$. For any c, $f(c) = d + e \wedge gcd(f^{-1}(d), f^{-1}(e)) = b$ is satisfied.*

$$B = \{\ (f(\frac{f^{-1}(d)}{b}), f(\frac{f^{-1}(e)}{b}), b) \mid \textit{Conditions above and } c \textit{ takes some area.}\ \}$$

On natural numbers, f is one to one correspondence, the area taken by c *and* $f(c)$ is $\{1, 2, 3, \cdots, n\}$, and the set A describes the left side of Theorem 2.1 in the paper [2] and the set B describes the right side.

In general $A = B$ does not hold. One simple counterexample is here:

$$X = \mathbb{P}_1 = \{\boldsymbol{e}, \{x_1\}, \{x_1, x_2\}\} = \{\boldsymbol{e}, \boldsymbol{x_1}, \boldsymbol{x_2}\},$$
$$\boldsymbol{x_i x_i} = \boldsymbol{x_j},\ \boldsymbol{x_i x_j} = \boldsymbol{x_i},$$
$$Y = \{y_1, y_2, y_3\},$$
$$y_i + y_i = y_{i+1},\ y_i + y_j = y_i,\ \textit{provided } y_4 = y_1,$$
$$f(\boldsymbol{x_i}) = y_{i+1},\ \textit{provided } \boldsymbol{e} = \boldsymbol{x_0},$$

and we think c takes only $c = e$, then

$$A = \{(y_1, y_2, \boldsymbol{e}),\ (y_1, y_3, \boldsymbol{e})\},$$
$$B = \{(y_1, y_2, \boldsymbol{e}),\ (y_1, y_3, \boldsymbol{e}),\ (y_1, y_1, \boldsymbol{x_2})\},$$

therefore $A \neq B$ holds.

From this point of view, we understand that the formula of Corollary 5.2 in the paper [1] would be brought out from the middle and the right side of Theorem 2.1 in the paper [2]. Normally it comes from the left side and the middle one.

3. Other supplements

We show other supplements of the paper [2] in this section. From the consideration of $A = B$ in the previous section and the proof of Theorem 2.1 in the paper [2], it is clear Theorem 2.1 depends only on the correspondence of the sets of triples. Therefore, we can exchange sums to multiples as following.

Corollary 3.1.

$$\prod_{k=1}^{n} \prod_{\substack{(a+b)c=k \\ gcd(a,b)=1}} f(a, b, c) = \prod_{k=1}^{n} \prod_{\substack{a+b=k \\ gcd(a,b)=1}} \prod_{c \leq [\frac{n}{k}]} f(a, b, c) = \prod_{k=1}^{n} \prod_{\substack{a+b=k \\ gcd(a,b)=c}} f(\frac{a}{c}, \frac{b}{c}, c).$$

Moreover, in general we can think about mappings from the sets of triples to something else, such as a set of functions. We can see some similar points with cyclotomic polynomial.

In the paper [2], we evaluate a sequence $\sum_{k=1}^{n} \varphi(k)\tau(k)$. We calculate it more precisely, and we expect it would increase faster than n^2 and slower than $n^2 \log n$. If

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this expectation is right, with Corollary 2.8 in the paper [2] $n^2 = \sum_{k=1}^{n}(\sigma(k) + r_k)$, the next inequality would hold on large enough n.

$$\sum_{k=1}^{n}\{\varphi(k)\tau(k) - \sigma(k)\} > \sum_{k=1}^{n} r_k > \sum_{k=1}^{n}\{\frac{\varphi(k)\tau(k)}{\log n} - \sigma(k)\},\ provided\ r_k = n\%k.$$

Next, we can derive the following corollary from Corollary 2.4 in the paper [2].

Corollary 3.2.

$$\sum_{k=1}^{n} \frac{\{f(gcd(k,n))\}^2 - \{f(\frac{n}{gcd(k,n)})\}^2}{\varphi(\frac{n}{gcd(k,n)})} = 0$$

Proof. We set $g(x) = f(\frac{n}{x})$ and $f(x) = g(\frac{n}{x})$ on Corollary 2.4, then

$$\sum_{k=1}^{n}\{f(d)\}^2 = \sum_{k=1}^{n} \frac{\{f(gcd(k,n))\}^2}{\varphi(\frac{n}{gcd(k,n)})} = \sum_{k=1}^{n} \frac{\{f(\frac{n}{gcd(k,n)})\}^2}{\varphi(\frac{n}{gcd(k,n)})}$$

holds. □

Similarly, from Corollary 2.5 in the paper [2],

$$\sum_{k=1}^{n} \frac{f(gcd(k,n)) - f(\frac{n}{gcd(k,n)})}{\varphi(\frac{n}{gcd(k,n)})} = 0$$

holds. Eventually,

$$\sum_{k=1}^{n} \frac{f(gcd(k,n))^l - f(\frac{n}{gcd(k,n)})^l}{\varphi(\frac{n}{gcd(k,n)})} = 0$$

also holds, provided $l \geq 1$.

When we think about Corollary 2.4, we can learn more about the structure of divisors. The motivation leads us to the definitions in the previous section, one generalization of natural numbers.

In the last, from Corollary 2.4 in the paper [2] the next corollaries are also derived.

Corollary 3.3.

$$\sum_{d|n} f(d) \cdot g(\frac{n}{d}) \cdot \varphi(d) = \sum_{k=1}^{n} f(\frac{n}{gcd(k,n)}) \cdot g(gcd(k,n))$$

$$\sum_{d|n} f(d) \cdot \varphi(d) = \sum_{k=1}^{n} f(\frac{n}{gcd(k,n)})$$

Proof. We denote $f(x) = h(x) \cdot \varphi(x)$ and then insert it to Corollary 2.4. □

Corollary 3.4.

$$\sum_{d|n} f(d) \cdot g(d) = \sum_{k=1}^{n} \frac{f(gcd(k,n)) \cdot g(gcd(k,n))}{\varphi(\frac{n}{gcd(k,n)})}$$

Proof. We denote $g(x) = h(\frac{n}{x})$ and then insert it to Corollary 2.4. Or this formula is equivalent to Corollary 2.5 in the paper [2]. □

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