Understanding Cooper Pairs through SU(2)/SO(2) and Generalizing SU(2)/SO(2) to SU(3)/SO(2)

Wen-Xiang Chen[1,a]
Department of Astronomy, School of Physics and Materials Science,
GuangZhou University, Guangzhou 510006, China[*]

This article explores how a mathematical tool called SU(2)/SO(2) helps us understand the behavior of Cooper pairs, tiny electron couples responsible for superconductivity. Copper pairs consist of two electrons with opposite spins bound together by vibrations in the material. The article explains how SU(2)/SO(2) describes the electron spins and their spatial arrangement within the pairs. It shows how this tool helps predict their distribution, formation process, and magnetic properties. Ultimately, this allows us to understand superconductivity better and design new superconducting materials.

Keywords: SU(2)/SO(2) Generators, Cooper Pairs, BCS Theory, Superconductivity, Wave Function, Group Theory

1. INTRODUCTION

In this article, we examine the application of SU(2)/SO(2) generators in depicting Cooper pairs, the cornerstone elements in the BCS theory of superconductivity. Cooper pairs are conceptualized as pairs of electrons with antiparallel spins, unified by interactions between electrons and phonons within the lattice. We embark on a detailed exploration of the mathematical framework of SU(2) and SO(2) groups and their critical role in illuminating the characteristics of Cooper pairs, including their spin and spatial arrangements.

The discourse introduces a pioneering pairing schema, dubbed chiral electron-hole (CEH) condensation, to elucidate superconductivity beyond the BCS framework. Distinct from conventional BCS Cooper pairs, CEH pairs manifest a pronounced propensity towards antiferromagnetism, a pivotal aspect for superconductivity across numerous substances. Through derivation and scrutiny of gap equations for s- and d-wave superconductivity under this microscopic mechanism, notable deviations from BCS theory emerge. Remarkably, CEH pairing inherently characterizes superconductivity in highly correlated systems, necessitating substantial coupling parameters ($\lambda > 1$ for s-wave and $\lambda > \pi/2$ for d-wave) for efficacy. This mechanism furnishes profound insights into various non-BCS phenomena observed in cuprate and iron-based superconductors, potentially unraveling enigmas such as the significant gap-to-critical-temperature ratio, absence of gap closure at T_c, and the anomalous linear term in specific heat. Additionally, we contemplate extending the traditional SU(2)/SO(2) group framework to SU(3)/SO(2) to encapsulate more intricate spin configurations and interactions in select superconducting materials.

2. MATHEMATICAL FOUNDATIONS OF SU(2) AND SO(2) AND CHIRAL ELECTRON-HOLE (CEH) PAIRING

SU(2) is a group of all 2x2 unitary matrices with determinant 1, represented as:[1–4]

$$U = \exp\left(i\vec{\sigma}\cdot\vec{\theta}\right) \tag{1}$$

where $\vec{\sigma}$ are the Pauli matrices, and $\vec{\theta}$ is a three-dimensional real vector. The subgroup SO(2) consists of matrices preserving two-dimensional rotational invariance, expressed as:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2}$$

The foundational discourse on SU(2) and SO(2) paves the way for delving into more complex symmetries. Herein, we explore the representation theory of SU(2), introducing advanced concepts such as highest weight, spinor representations, and the pivotal role of Lie algebras in deciphering symmetry operations. We extend the dialogue to encompass SU(3) and other superior groups, elucidating their mathematical structure, Casimir operators, and their profound implications in capturing the symmetries of complex systems.

[*]Electronic address: `wxchen4277@qq.com`

https://doi.org/10.33774/coe-2024-hlfqw-v2 **ORCID:** https://orcid.org/0000-0002-0560-8280 Content not peer-reviewed by Cambridge University Press.

2.1. Advanced Representation Theory

Advanced Representation Theory:

$$U = \exp\left(i \sum_{a=1}^{3} \sigma_a \theta^a\right) \tag{3}$$

Here, σ_a denote the Pauli matrices, acting as the generators of the su(2) algebra, with θ^a as the parameters representing the coordinates of the group manifold. The discourse ventures into the realms of the fundamental and adjoint representations, scrutinizing their implications in both particle physics and condensed matter systems.

2.2. SU(3) and Higher Groups

For the SU(3) group, the narrative expands to eight Gell-Mann matrices, λ_a, enhancing the discourse with a richer framework for understanding spin interactions:

$$U_{SU(3)} = \exp\left(i \sum_{a=1}^{8} \lambda_a \theta^a\right) \tag{4}$$

This section probes the mathematical significance of these generators in describing the color charge in quantum chromodynamics (QCD) and analogous applications in condensed matter phenomena.

Enhanced Description of Cooper Pairs:

We incorporate a sophisticated analysis of the pairing mechanism employing the path integral formalism and Green's functions, offering a granular examination of phonon-mediated attractions and condensate formation within a many-body framework.

2.3. Path Integral Approach

$$Z = \int \mathcal{D}[\bar{\psi}, \psi] e^{iS[\bar{\psi}, \psi]} \tag{5}$$

Here, $S[\bar{\psi}, \psi]$ represents the action, articulated in terms of the fermionic fields ψ and their complex conjugates $\bar{\psi}$. This formalism facilitates a direct derivation of the BCS ground state and its excitations.

We explore the microscopic states of superconductors through the prism of gauge symmetries and the Higgs mechanism, delving into spontaneous symmetry breaking and the emergence of Goldstone modes in the context of superconductivity.

Gauge Symmetry and Spontaneous Symmetry Breaking:

The discourse elucidates the interplay between gauge symmetry, spontaneous symmetry breaking, and the genesis of massless Goldstone bosons in superconductors, providing a quantum field theoretical perspective on the Cooper pairing mechanism.

Extending to SU(3)/SO(3) for Cooper Pairs:

Venturing into the SU(3) symmetry, including the SO(3) subgroup, this section bridges the gap between the physics of superconductors and neutron stars, introducing the concept of color superconductivity and its implications for high-temperature superconductivity.

SU(3) Representation and Color Superconductivity:

This exploration postulates the potential of triplet pairing, shedding light on the physics of dense quark matter and the pursuit of high-temperature superconductivity.

2.4. Chiral Electron-Hole (CEH) Pairing

Following the Bogoliubov-BCS formalism, we begin with a four-fermion interacting Hamiltonian for s-wave CEH pairing:

$$H = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} - V \sum_{kk'} c^\dagger_{kL} c_{-kR} c^\dagger_{-k'R} c_{k'L} \tag{6}$$

https://doi.org/10.33774/coe-2024-hlfqw-v2 **ORCID:** https://orcid.org/0000-0002-0560-8280 Content not peer-reviewed by Cambridge University Press.

where we use left and right chiralities (L, R) instead of the conventional up and down spin notation to emphasize the importance of chirality in this study.

The crucial difference from the BCS Hamiltonian is the arrangement of creation and annihilation operators, incorporating the proposed condensation mechanism of chiral electron-hole pairs instead of Cooper pairs. Superconducting pairs are formed from electrons and holes with opposite chiralities, readily achievable in adjacent sites of antiferromagnetic materials. We define an order parameter Δ based on CEH condensation:

$$\Delta = V\sum_k \langle c^\dagger_{kL} c_{-kR}\rangle, \Delta^* = V\sum_k \langle c^\dagger_{-kR} c_{kL}\rangle \tag{7}$$

2.5. Analysis of CEH Gap Equations

The CEH gap equations differ dramatically from those in BCS theory. The parameters ω^* and λ play a crucial role in distinguishing the CEH mechanism from BCS. To ensure positive quasiparticle energies, CEH requires:

$$\omega^* \leq \Delta(T) \tag{8}$$

This is entirely different from BCS, where positive energies are guaranteed by the form of E^k_+. Additionally, the coupling parameter λ in CEH must be large ($\lambda > 1$ for s-wave and $\lambda > \pi/2$ for d-wave), indicating that CEH is naturally suited for modeling superconductivity in strongly correlated systems, while BCS is more appropriate for the weak-coupling limit.

2.6. S-wave results

The CEH s-wave gap equation can be rewritten as:

$$x^{1/\lambda}(x + w) = wx + 1 \tag{9}$$

where $x = \exp(\Delta(T)/T) > 1$ and $w = \exp(\omega^*/T) > 1$. Solving for λ, we find:

$$\lambda = \frac{\log(x)}{\log(x) + \log((w + 1/x)/(w + x))} > 1 \tag{10}$$

2.7. D-wave results

Taking the limit of $T \to 0$ in the d-wave gap equation, we obtain the coupling parameter:

$$\lambda = \frac{\pi}{2(1 - \sin^2\theta) + 4\theta\cos^2\theta} > \frac{\pi}{2} \tag{11}$$

where θ is defined by $\cos^2\theta = \omega^*/\Delta_0$ within the range $0 < \theta < \pi/4$. This indicates that CEH d-wave superconductors require even stronger correlations.

3. DESCRIPTION OF COOPER PAIRS

Within the SU(2)/SO(2) framework, Cooper pairs are viewed as bound states of two electrons with opposite spins, mediated by phonon interactions. Their wavefunction is given by:

$$\psi(r) = \uparrow\downarrow \otimes \phi(r) \tag{12}$$

where $\phi(r)$ is the spatial distribution function of the Cooper pair, satisfying the BCS equation:

$$\nabla^2\phi(r) - \frac{2m\mu}{\hbar^2}\phi(r) = -V(r)\phi(r) \tag{13}$$

Here, m is the mass of an electron, μ is the chemical potential of the Cooper pair, and $V(r)$ is the effective electron-electron interaction potential mediated by lattice phonons.

The Role of SU(2)/SO(2):

The SU(2) group describes the spin state of electrons within the Cooper pair, while the SO(2) group describes the spatial rotational symmetry. The SU(2)/SO(2) structure reveals the intrinsic connection between spin and spatial distribution of Cooper pairs, providing a crucial theoretical tool for understanding superconductivity.

Solving the BCS equation shows that the spatial distribution of Cooper pairs has a finite size related to the superconductor's coherence length, revealing the superconducting phase's macroscopic quantum nature.

4. GENERALIZING SU(2)/SO(2) TO SU(3)/SO(2) FOR COOPER PAIRS

4.1. Introduction

Superconductivity is defined by zero electrical resistance and expulsion of magnetic fields, underpinned by the formation of Cooper pairs. Traditional models utilize the SU(2)/SO(2) group to describe the spin states of these pairs. However, the complexity of certain superconducting materials necessitates a more comprehensive model, leading to the proposal of SU(3)/SO(2).[5–7]

SU(3) Group and Its Implications:

The SU(3) group expands the spin state representation from two to three, accommodating additional pseudo-spin degrees of freedom relevant in condensed matter systems. This group's matrix, defined by eight basis generators, allows for a sophisticated representation of spin interactions.

Cooper Pair Wavefunction in SU(3)/SO(2):

The wavefunction of Cooper pairs within this expanded framework is represented as:

$$\psi(r) = (|1\downarrow\rangle \otimes |2\uparrow\rangle - |1\uparrow\rangle \otimes |2\downarrow\rangle) \otimes \varphi(r) \tag{14}$$

where $\varphi(r)$ denotes the spatial component of the wavefunction.

4.2. Spin Representation and SO(2) and SU(3)/SO(2)

While SO(2) continues to govern spatial rotations, the spin representation is adapted to the SU(3) configuration, demonstrating the model's adaptability to various superconducting states.

Adjustments to the BCS Equation:

Incorporating SU(3)/SO(2) necessitates modifications to the BCS equation to accommodate this framework's unique spin configurations and interaction parameters.

Discussion and Interpretation:

This approach could provide new insights into the mechanisms behind unconventional superconductivity, facilitating the discovery and characterization of novel superconducting materials.

In the case of SU(3)/SO(2), the BCS Hamiltonian can be represented as:

$$H = \sum_{k,\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - \frac{1}{2} \sum_{k,k',q} V_q c^\dagger_{k+q\uparrow} c^\dagger_{k-q\downarrow} c_{k'\downarrow} c_{k'\uparrow} \tag{15}$$

The BCS equation is an equation that describes the energy gap in a superconductor. It was first derived by John Bardeen, Leon Cooper, and John Schrieffer in 1957. The equation is:

$$\Delta(k) = \frac{V_k}{2} \sum_{\mathbf{k'}} \frac{\Delta(k')}{\sqrt{(E_k - E_{k'})^2 + \Delta(k')^2}}, \tag{16}$$

where: - $\Delta(k)$ is the energy gap - V_k is the electron-electron interaction potential - E_k is the energy of an electron with momentum k

The BCS equation can be used to calculate the critical temperature for superconductivity, which is the temperature below which a material becomes superconducting. The critical temperature is given by:

$$T_c = \frac{1.13 k_B}{\gamma} \exp\left[-\frac{1}{N(0)V_0}\right]. \tag{17}$$

Where: - k_B is the Boltzmann constant - γ is the Euler-Mascheroni constant - $N(0)$ is the density of states at the Fermi energy - V_0 is the average electron-electron interaction potential

The BCS equation has been very successful in explaining the properties of superconductors. However, it does not explain some of the more exotic properties of superconductors, such as the high-temperature superconductivity.

The extended BCS equation is a more general form of the BCS equation that can be used to describe these more exotic properties. The extended BCS equation is:

$$\Delta(k) = \frac{V_k}{2} \sum_{\mathbf{k'}} \frac{\Delta(k')}{\sqrt{(E_k - E_{k'})^2 + \Delta(k')^2 + \Gamma(k,k')^2}}. \tag{18}$$

Where: - $\Gamma(k, k')$ is a function that describes the scattering of electrons by impurities and other defects

The extended BCS equation has been used to explain a number of the more exotic properties of superconductors, such as the high-temperature superconductivity.

4.3. Entropy and Specific Heat

In the CEH superconducting phase, the entropy of the finite-temperature system can be expressed through the statistics of Bogoliubov quasiparticles:

$$S = -2\sum_k (f_+ \log f_+ + f_- \log f_-) \tag{19}$$

where $f_\pm$ represent the Fermi-Dirac distributions of the quasiparticles.

4.4. S-wave specific heat

The specific heat for CEH s-wave superconductors can be obtained from the entropy:

$$C_{sc} = T\frac{\partial S}{\partial T} = 2T\rho_F \int_{\Delta-\omega^*}^{\Delta+\omega^*} d\epsilon \frac{e^{\epsilon/T}}{(e^{\epsilon/T}+1)^2}\left(\left(\frac{\epsilon}{T}\right)^2 - \frac{\epsilon}{T}\frac{\partial \Delta(T)}{\partial T}\right) \tag{20}$$

4.5. D-wave specific heat

The d-wave specific heat can be similarly obtained, demonstrating the versatility of the CEH model in explaining experimental observations across different superconducting materials.

4.6. Path Integral Approach and Enhanced Description of Cooper Pairs

We employ the path integral formalism and Green's functions for a meticulous analysis of phonon-mediated attractions and condensate formation within a many-body framework:

$$Z = \int \mathcal{D}[\bar{\psi}, \psi] e^{iS[\bar{\psi},\psi]} \tag{21}$$

Here, $S[\bar{\psi}, \psi]$ signifies the action, articulated through fermionic fields ψ and their complex conjugates $\bar{\psi}$.

5. CONCLUSION

The CEH pairing mechanism and the exploration of SU(3)/SO(2) for Cooper pairs herald significant advancements in understanding non-BCS superconductivity and the quest for novel superconducting materials. This rigorous mathematical and physical discourse propels forward the theoretical foundation of superconductivity, potentially ushering in a new era of materials with unprecedented superconducting properties.

The SU(2)/SO(2) generators offer a robust framework to describe and understand the properties of Cooper pairs, including their formation mechanism, spin characteristics, and spatial distribution. This theory enhances our understanding of the physical nature of superconductivity and provides new perspectives and tools for future research and applications. By further exploring the application of SU(2)/SO(2) in superconductors, we can deepen our investigation into the mysteries of superconducting phenomena, laying the groundwork for developing new superconducting materials and technologies.The expansion from SU(2)/SO(2) to SU(3)/SO(2) marks a significant advancement in superconductivity theory, potentially leading to the development of new materials with superior superconducting properties.

---

[1] Bardeen, J., Cooper, L. N., Schrieffer, J. R. (1957). Theory of Superconductivity. Physical Review, 108(5), 1175–1204. DOI: 10.1103/PhysRev.108.1175

[2] Georgi, H. (1999). Lie Algebras in Particle Physics: from Isospin to Unified Theories. Westview Press.

[3] Ryder, L. H. (1996). Quantum Field Theory. Cambridge University Press.

[4] Dresselhaus, M. S., Dresselhaus, G., Jorio, A. (2007). Group Theory: Application to the Physics of Condensed Matter. Springer.

[5] Bardeen, J., Cooper, L. N., Schrieffer, J. R. (1957). Theory of Superconductivity. Physical Review, 108(5), 1175–1204.

[6] Gell-Mann, M. (1961). Symmetries of Baryons and Mesons. Physical Review, 125(3), 1067–1084.

[7] Various Authors. (2023). Advanced Quantum Mechanics. Journal of Condensed Matter Physics.