

Unified Description of Dirac and Proca Equations by Entangled Spin Fields

Alexandru Gabriel Mitruț *

Universitatea din București

March 14, 2024

Abstract

In this paper was proved the equivalence of Dirac equation for mass m with two entangled Proca fields of mass $2m$ and is proposed an equation for entangled spin fields.

1 Introduction

In addition to the well known Dirac, Weyl and Majorana representations of gamma matrices, we define a new one as

$$\gamma^0 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (1)$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2)$$

Next we solve Dirac equation to get two plane wave solutions $\Phi_1(p)$ and $\Phi_2(p)$ of positive energy,

$$\mathbf{H}\Phi_1(p) = +\varepsilon \Phi_1(p) \quad , \quad \mathbf{H}\Phi_2(p) = +\varepsilon \Phi_2(p) \quad , \quad (3)$$

and two plane wave solutions $\Psi_1(p)$ and $\Psi_2(p)$ of negative energy,

$$\mathbf{H}\Psi_1(p) = -\varepsilon \Psi_1(p) \quad , \quad \mathbf{H}\Psi_2(p) = -\varepsilon \Psi_2(p) \quad , \quad (4)$$

where $\mathbf{H} = \gamma^0(-i\gamma^i\partial_i + m)$ denotes the Hamiltonian, and $\varepsilon = +(\mathbf{p}^2 + m^2)^{1/2}$. Explicitly, the plane wave solutions in this representation are

$$\Phi_1(p) = \frac{1}{[2\varepsilon(\varepsilon + p_3)]^{1/2}} \begin{pmatrix} m \\ -i(\varepsilon + p_3) \\ p_1 - ip_2 \\ 0 \end{pmatrix} e^{-ipx} \quad , \quad \Phi_2(p) = \frac{1}{[2\varepsilon(\varepsilon - p_3)]^{1/2}} \begin{pmatrix} 0 \\ p_2 - ip_1 \\ \varepsilon - p_3 \\ m \end{pmatrix} e^{-ipx} \quad , \quad (5)$$

and

$$\Psi_1(p) = \frac{1}{[2\varepsilon(\varepsilon + p_3)]^{1/2}} \begin{pmatrix} p_1 + ip_2 \\ 0 \\ m \\ -\varepsilon - p_3 \end{pmatrix} e^{+ipx} \quad , \quad \Psi_2(p) = \frac{1}{[2\varepsilon(\varepsilon - p_3)]^{1/2}} \begin{pmatrix} -i(\varepsilon - p_3) \\ m \\ 0 \\ p_2 + ip_1 \end{pmatrix} e^{+ipx} \quad . \quad (6)$$

*amitru@gmail.com

2 Parity, Chirality and Charge Conjugation Operators

Alongside the chirality operator γ_5 , we define $\mathbf{P} = -i\gamma^1\gamma^2$ the parity operator and \mathbf{C} charge (parity and chirality) conjugation operator as

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \mathcal{C} \quad (7)$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and \mathcal{C} is the complex conjugation operator. Chirality and parity conjugation operator \mathbf{C} is antilinear and anticommutes with the Hamiltonian,

$$\mathbf{C}^2 = 1, \quad \mathbf{H}\mathbf{C} = -\mathbf{C}\mathbf{H} \quad (8)$$

therefore if Ψ is a positive(negative) energy solution then $\Phi = \mathbf{C}\Psi$ is a negative (positive) solution. While charge conjugation operators transform a spinor into another of opposite energy leaving its four-momentum unchanged, \mathbf{P} transform a spinor with four-momentum $p^\alpha = (\varepsilon, p_1, p_2, p_3)$ into the same spinor but with four-momentum $\gamma_5 p = (\varepsilon, -p_1, -p_2, p_3)$. The commutation relation $[\gamma_5, \mathbf{P}] = 0$ imply that eigenvalues of chirality γ_5 and parity \mathbf{P} can be used to classify their common eigenstates. The -1 and $+1$ eigenvalues of γ_5 will be denoted L and R i.e. -1 correspond to left chiral L and $+1$ to right chiral R, while the eigenvalues -1 and $+1$ of \mathbf{P} will be denoted as $-$ and $+$ i.e. $-$ for negative parity eigenvalue -1 and $+$ for positive parity $+1$. We define the eight common eigenstates of chirality γ_5 and parity \mathbf{P} using the solutions Φ and Ψ of opposite energy

$$\begin{aligned} \Psi_{\pm L} &= \frac{1 \pm \mathbf{P}}{2} \frac{1 - \gamma_5}{2} \Phi, & \Psi_{\pm R} &= \frac{1 \pm \mathbf{P}}{2} \frac{1 + \gamma_5}{2} \Phi, \\ \Phi_{\pm L} &= \frac{1 \pm \mathbf{P}}{2} \frac{1 - \gamma_5}{2} \Psi, & \Phi_{\pm R} &= \frac{1 \pm \mathbf{P}}{2} \frac{1 + \gamma_5}{2} \Psi, \end{aligned} \quad (9)$$

Charge (chirality and parity) conjugation operator \mathbf{C} transform Φ eigenstates of chirality and parity into Ψ eigenstates with inverse eigenvalues of chirality and parity, and vice-versa

$$\begin{aligned} \Psi_{+L} &= \mathbf{C}\Phi_{-R}, & \Phi_{-R} &= \mathbf{C}\Psi_{+L}, \\ \Psi_{-L} &= \mathbf{C}\Phi_{+R}, & \Phi_{+R} &= \mathbf{C}\Psi_{-L}, \\ \Psi_{+R} &= \mathbf{C}\Phi_{-L}, & \Phi_{-L} &= \mathbf{C}\Psi_{+R}, \\ \Psi_{-R} &= \mathbf{C}\Phi_{+L}, & \Phi_{+L} &= \mathbf{C}\Psi_{-R}. \end{aligned} \quad (10)$$

Next, we define left X_L and right X_R eigenstates of chiral operator by superposition of all left chiral and respectively right eigenstates

$$X_L = \Phi_{-L} + \Phi_{+L} + \Psi_{-L} + \Psi_{+L}, \quad X_R = \Phi_{-R} + \Phi_{+R} + \Psi_{-R} + \Psi_{+R}, \quad (11)$$

which form an orthogonal set

$$\begin{aligned} \bar{X}_L X_L &= 0, & \bar{X}_R X_R &= 0, \\ \bar{X}_L X_R &= 0, & \bar{X}_R X_L &= 0, \end{aligned} \quad (12)$$

mix positive (negative) energy solutions with negative (positive) energy solutions of Dirac equation

$$\begin{aligned} X_L &= \frac{1 - \gamma_5}{2} (\Psi + \Phi), & X_R &= \frac{1 + \gamma_5}{2} (\Psi + \Phi), \\ \bar{X}_L &= (\bar{\Psi} + \bar{\Phi}) \frac{1 + \gamma_5}{2}, & \bar{X}_R &= (\bar{\Psi} + \bar{\Phi}) \frac{1 - \gamma_5}{2}, \end{aligned} \quad (13)$$

which satisfy Proca equation

$$(\square + m^2)X_L = 0, \quad (\square + m^2)X_R = 0, \quad (14)$$

and the following partial differential coupled equations

$$i\partial_\alpha \gamma^\alpha X_L - mX_R = 0, \quad i\partial_\alpha \gamma^\alpha X_R - mX_L = 0. \quad (15)$$

3 Conserved four-currents

We introduce left and right conserved four-currents (probability densities) for eigenstates of the same chirality defined by 11

$$j^\alpha(X_L, X_L) = \bar{X}_L \gamma^\alpha X_L \quad , \quad j^\alpha(X_R, X_R) = \bar{X}_R \gamma^\alpha X_R \quad , \quad (16)$$

which are equal and real defined

$$\mathcal{C}j^\alpha(X_L, X_L) = j^\alpha(X_L, X_L) \quad , \quad \mathcal{C}j^\alpha(X_R, X_R) = j^\alpha(X_R, X_R) \quad . \quad (17)$$

We interpret the conservation of the left and right four-current

$$\partial_\alpha j^\alpha(X_L, X_L) = 0 \quad , \quad \partial_\alpha j^\alpha(X_R, X_R) = 0 \quad , \quad (18)$$

and the zero current between eigenstates of left and right chirality

$$j^\alpha(X_L, X_R) = j^\alpha(X_R, X_L) = 0 \quad , \quad (19)$$

as the existence of chiral particles L and R. The two chiral particles are transformed one into another by charge (chirality and parity) operator \mathbf{C}

$$X_L = \mathbf{C}X_R \quad , \quad X_R = \mathbf{C}X_L \quad , \quad (20)$$

The particle described by $\Psi + \Phi = X_L + X_R$, which has no definite chirality nor parity and is an eigenstate of chirality and conjugation operator, i.e. is transformed into itself by chirality and parity operator,

$$\mathbf{C}(\Psi + \Phi) = \Psi + \Phi \quad , \quad (21)$$

therefore it is a Majorana particle. All three particles mix positive (negative) energy solutions of Dirac equation with negative (positive) energy solutions, and their currents are explicitly given by

$$j^\alpha(X_L, X_L) = (\bar{\Psi} + \bar{\Phi})\gamma^\alpha \frac{1 - \gamma_5}{2}(\Psi + \Phi) \quad , \quad j^\alpha(X_R, X_R) = (\bar{\Psi} + \bar{\Phi})\gamma^\alpha \frac{1 + \gamma_5}{2}(\Psi + \Phi) \quad , \quad (22)$$

and

$$j^\alpha(\Psi + \Phi, \Psi + \Phi) = (\bar{\Psi} + \bar{\Phi})\gamma^\alpha(\Psi + \Phi) \quad . \quad (23)$$

Out of 64 currents between all 8 common eigenstates of chirality and parity defined by 9, 32 of them, between L and R or R and L are zero. The 32 non-zero currents are between eigenstates of the same chirality, but are not conserved.

4 Parity Violation

Next, we calculate in the ultrarelativistic regime $\varepsilon \gg m$ the helicity of L and R particles moving along the z axis. The helicity operator is given by

$$\mathbf{h} = \frac{1}{p}(p_1 S^1 + p_2 S^2 + p_3 S^3) \quad , \quad (24)$$

where $p = (p_1^2 + p_2^2 + p_3^2)^{1/2}$ is momentum and $S^1 = \Sigma^{23}$, $S^2 = \Sigma^{31}$, $S^3 = \Sigma^{12}$ are spatial components of spin, related to the generators of Lorentz transformations of gamma matrices by $\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Helicity commutes with parity and chirality operator and for a particle moving along z axis we get $\mathbf{h} = -\frac{1}{2}\mathbf{P}$. For each of the four solution of Dirac equations we calculate the action of helicity operator on ultrarelativistic left and right particles moving along z axis and get correct predictions

$$\mathbf{h}X_L = -\frac{1}{2}X_L \quad , \quad \mathbf{h}X_R = +\frac{1}{2}X_R \quad , \quad (25)$$

when $\varepsilon \gg m$. We can now identify the neutrino as the left chiral particle X_L having helicity $-\frac{1}{2}$ and the antineutrino as the right chiral particle X_R having helicity $+\frac{1}{2}$. Conserved currents are

formed only by left X_L and right X_R eigenstates of chirality, not by the parity eigenstates. The superposition of all negative $\Psi_{-L} + \Psi_{-R} + \Phi_{-L} + \Phi_{-R}$ parity eigenstates and of all positive parity eigenstates $\Psi_{+L} + \Psi_{+R} + \Phi_{+L} + \Phi_{+R}$ gives currents which are not conserved, i.e. we can not built conserved currents using linear combinations of parity eigenstates. Therefore, it is the chirality which is the fundamental symmetry of nature, because the left chiral and right chiral currents are conserved and identified with L and right R particle, i.e. neutrino and respectively antineutrino and the question is if chirality, not parity, is violated, i.e. if there is a field (particle) which couple with different strengths to L and R currents. The property of parity operator, in Weyl basis, to transform the left-handed Weyl spinor into the right-handed Weyl spinor and vice-versa requires that right-handed neutrino and left-handed antineutrino should also exist. The absence of right-handed neutrino and left-handed antineutrino had led to the conclusion that parity is not conserved. The mistake was to associate left-handed Weyl spinor, which gives a four current that is not conserved, to neutrino, and similarly, a right-handed Weyl spinor, which gives a four current that is not conserved, to antineutrino. Our definition of currents associated to left and right eigenstates of chirality are conserved 18 and transformed one into another by the chirality and parity conjugation operator 10, not by the parity operator as in Weyl representation. Therefore, the puzzling absence of positive helicity neutrino and respectively negative helicity antineutrino, dubbed as "parity violation" is a consequence wrong association of spinors to particles. The $-1/2$ helicity of the neutrino and $+1/2$ helicity of antineutrino are a result of their ultrarelativistic regime in which eigenstates X_L and X_R of chirality operator γ_5 are also eigenstates of parity operator \mathbf{P} .

5 Entangled spin fields equation

We construct conserved currents with first order derivatives of left eigenstates $\partial^\mu X_L$ and symmetrical conserved currents with first order derivatives of right eigenstates $\partial^\mu X_R$,

$$\begin{aligned} j_{(1)}^\alpha(X_L, X_L) &= \partial_\mu \bar{X}_L \gamma^\alpha \partial^\mu X_L \quad , & j_{(1)}^\alpha(X_R, X_R) &= \partial_\mu \bar{X}_R \gamma^\alpha \partial^\mu X_R \quad , \\ \partial_\alpha j_{(1)}^\alpha(X_L, X_L) &= 0 \quad , & \partial_\alpha j_{(1)}^\alpha(X_R, X_R) &= 0 \quad , \end{aligned} \quad (26)$$

as well as conserved currents with second order derivatives of left and respectively right eigenstates

$$\begin{aligned} j_{(2)}^\alpha(X_L, X_L) &= \partial_\mu \partial_\nu \bar{X}_L \gamma^\alpha \partial^\mu \partial^\nu X_L \quad , & j_{(2)}^\alpha(X_R, X_R) &= \partial_\mu \partial_\nu \bar{X}_R \gamma^\alpha \partial^\mu \partial^\nu X_R \quad , \\ \partial_\alpha j_{(2)}^\alpha(X_L, X_L) &= 0 \quad , & \partial_\alpha j_{(2)}^\alpha(X_R, X_R) &= 0 \quad , \end{aligned} \quad (27)$$

and also conserved currents with derivatives of higher order than 2. For both left and right particles, only j^α and $j_{(1)}^\alpha$ are independent, all other currents built with derivatives of higher order than one can be expressed in term of j^α or $j_{(1)}^\alpha$ as follows

$$\begin{aligned} j_{(2)}^\alpha(X_L, X_L) &= m^4 j^\alpha(X_L, X_L) \quad , & j_{(2)}^\alpha(X_R, X_R) &= m^4 j^\alpha(X_R, X_R) \quad , \\ j_{(3)}^\alpha(X_L, X_L) &= m^4 j_{(1)}^\alpha(X_L, X_L) \quad , & j_{(3)}^\alpha(X_R, X_R) &= m^4 j_{(1)}^\alpha(X_R, X_R) \quad , \end{aligned} \quad (28)$$

and so on. The currents $j^\alpha(X_L, X_L)$ and $j_{(1)}^\alpha(X_L, X_L)$ satisfy coupled second order differential equations

$$\begin{aligned} \square j^\alpha(X_L, X_L) &= -2m^2 j^\alpha(X_L, X_L) + 2j_{(1)}^\alpha(X_L, X_L) \quad , \\ \square j_{(1)}^\alpha(X_L, X_L) &= -2m^2 j_{(1)}^\alpha(X_L, X_L) + 2m^4 j^\alpha(X_L, X_L) \quad , \end{aligned} \quad (29)$$

and similar equations holds for $j^\alpha(X_R, X_R)$ and $j_{(1)}^\alpha(X_R, X_R)$. We define four vector potentials A_L^α , A_R^α , B_L^α , and B_R^α

$$\begin{aligned} A_L^\alpha &= \frac{1}{2} \left[j^\alpha(X_L, X_L) - m^{-2} j_{(1)}^\alpha(X_L, X_L) \right] \quad , & A_R^\alpha &= \frac{1}{2} \left[j^\alpha(X_R, X_R) - m^{-2} j_{(1)}^\alpha(X_R, X_R) \right] \quad , \\ B_L^\alpha &= \frac{1}{2} \left[j^\alpha(X_L, X_L) + m^{-2} j_{(1)}^\alpha(X_L, X_L) \right] \quad , & B_R^\alpha &= \frac{1}{2} \left[j^\alpha(X_R, X_R) + m^{-2} j_{(1)}^\alpha(X_R, X_R) \right] \quad . \end{aligned} \quad (30)$$

Using 29 find that A_L^α and A_R^α fields satisfy Proca equation for a particle with mass $2m$ and the Lorenz gauge is automatically held

$$\begin{aligned} (\square + 4m^2)A_L^\alpha &= 0 \quad , & (\square + 4m^2)A_R^\alpha &= 0 \quad , \\ \partial_\alpha A_L^\alpha &= 0 \quad , & \partial_\alpha A_R^\alpha &= 0 \quad , \end{aligned} \quad (31)$$

while B_L^α and B_R^α satisfy Maxwell equations

$$\begin{aligned}\square B_L^\alpha &= 0 \quad , & \square B_R^\alpha &= 0 \quad , \\ \partial_\alpha B_L^\alpha &= 0 \quad , & \partial_\alpha B_R^\alpha &= 0 \quad .\end{aligned}\tag{32}$$

Using 22 the A and B fields are expressed as

$$\begin{aligned}A_L^\alpha &= \frac{1}{2} \left[(\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 - \gamma_5}{2} (\Psi + \Phi) - \frac{1}{m^2} \partial_\lambda (\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 - \gamma_5}{2} \partial^\lambda (\Psi + \Phi) \right] \quad , \\ A_R^\alpha &= \frac{1}{2} \left[(\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 + \gamma_5}{2} (\Psi + \Phi) - \frac{1}{m^2} \partial_\lambda (\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 + \gamma_5}{2} \partial^\lambda (\Psi + \Phi) \right] \quad .\end{aligned}\tag{33}$$

$$\begin{aligned}B_L^\alpha &= \frac{1}{2} \left[(\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 - \gamma_5}{2} (\Psi + \Phi) + \frac{1}{m^2} \partial_\lambda (\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 - \gamma_5}{2} \partial^\lambda (\Psi + \Phi) \right] \quad , \\ B_R^\alpha &= \frac{1}{2} \left[(\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 + \gamma_5}{2} (\Psi + \Phi) + \frac{1}{m^2} \partial_\lambda (\bar{\Psi} + \bar{\Phi}) \gamma^\alpha \frac{1 + \gamma_5}{2} \partial^\lambda (\Psi + \Phi) \right] \quad .\end{aligned}\tag{34}$$

For each solution of Dirac equation we compute the Maxwell fields $B_{L,R}^\alpha$ fields and find that all are constant in spacetime and equal with p^α/ε .

$$B_{L,R}^\alpha(\Psi_1) = B_{L,R}^\alpha(\Phi_2) = B_{L,R}^\alpha(\Psi_2) = B_{L,R}^\alpha(\Phi_1) = \frac{p^\alpha}{\varepsilon}.\tag{35}$$

This result should be compared with Higgs Kibble cumbersome mechanism that postulates a scalar field ϕ which couples to a massless boson to obtain a factor in the wave equation for W_- boson which play the same role as a mass term, and it is assumed that the scalar field is constant in space, while in our study we have proved, not postulated, the existence of a vector field, derived from Dirac equation, that is constant in spacetime. The Proca A fields for Ψ_1 and Φ_1 are equal

$$\begin{aligned}A_{L,R}^0(\Psi_1) &= A_{L,R}^0(\Phi_1) = +\frac{m}{\varepsilon(\varepsilon + p_3)} [p_1 \cos(2px) - p_2 \sin(2px)], \\ A_{L,R}^1(\Psi_1) &= A_{L,R}^1(\Phi_1) = +\frac{m}{\varepsilon} \cos(2px), \\ A_{L,R}^2(\Psi_1) &= A_{L,R}^2(\Phi_1) = -\frac{m}{\varepsilon} \sin(2px), \\ A_{L,R}^3(\Psi_1) &= A_{L,R}^3(\Phi_1) = -\frac{m}{\varepsilon(\varepsilon + p_3)} [p_1 \cos(2px) - p_2 \sin(2px)],\end{aligned}\tag{36}$$

while Proca A fields for Ψ_2 and Φ_2 are equal up to sign

$$\begin{aligned}A_{L,R}^0(\Psi_2) &= -A_{L,R}^0(\Phi_2) = +\frac{m}{\varepsilon(\varepsilon - p_3)} [p_1 \cos(2px) + p_2 \sin(2px)], \\ A_{L,R}^1(\Psi_2) &= -A_{L,R}^1(\Phi_2) = +\frac{m}{\varepsilon} \cos(2px), \\ A_{L,R}^2(\Psi_2) &= -A_{L,R}^2(\Phi_2) = +\frac{m}{\varepsilon} \sin(2px), \\ A_{L,R}^3(\Psi_2) &= -A_{L,R}^3(\Phi_2) = +\frac{m}{\varepsilon(\varepsilon - p_3)} [p_1 \cos(2px) + p_2 \sin(2px)].\end{aligned}\tag{37}$$

The four spinors $\Psi_{1,2}$ and $\Phi_{1,2}$ for a particle with mass m generate two Proca potentials of mass $2m$ given by 36 and 37 further denoted by S_1^α and S_2^α , defined by

$$S_1^\alpha = \varepsilon A_L^\alpha(\Psi_1) = \varepsilon A_L^\alpha(\Phi_1), \quad S_2^\alpha = \varepsilon A_L^\alpha(\Psi_2) = -\varepsilon A_L^\alpha(\Phi_2),\tag{38}$$

which will be called spin fields due to the following relations of orthogonality to the four-momentum $p^\alpha = (\varepsilon, p_1, p_2, p_3)$

$$p_\alpha S_1^\alpha = 0, \quad p_\alpha S_2^\alpha = 0.\tag{39}$$

The spin fields are solutions of the following 8 nonlinear, entangled partial differential equations which unifies Dirac and Proca equations

$$S_2^\alpha \partial_\alpha S_1^\beta = 0 \quad , \quad S_1^\alpha \partial_\alpha S_2^\beta = 0 \quad . \quad (40)$$

The spin fields S_1 and S_2 are entangled, the source of S_1 field is S_2 and vice versa, and the mass is not explicitly contained into the equations. The squared norm of the spin field is related to the mass, therefore the solutions S_1 and S_2 of coupled equations gives the mass spectrum

$$S_1^\alpha S_\alpha^1 = -m^2 \quad , \quad S_2^\alpha S_\alpha^2 = -m^2 \quad . \quad (41)$$

6 Discrete Symmetries Algebra

The parity operator \mathbf{P} transform the spin fields S_1 and S_2 one into another

$$\mathbf{P}S^1(p, x) = S^2(\mathbf{P}p, \mathbf{P}x) \quad , \quad \mathbf{P}S^2(p, x) = S^1(\mathbf{P}p, \mathbf{P}x) \quad , \quad (42)$$

in the same way that charge operator \mathbf{C} change left and right particle spinors

$$X_L(p, x) = \mathbf{C}X_R(p, x) \quad , \quad X_R(p, x) = \mathbf{C}X_L(p, x) \quad . \quad (43)$$

Two commuting, antilinear and antiunitary operators $\mathbf{P}\mathbf{C} = \mathbf{T}$ and $\gamma_5\mathbf{C} = \tilde{\mathbf{T}}$ can be defined within the symmetry, which satisfy $\mathbf{T}^2 = \tilde{\mathbf{T}}^2 = -1$, as required for time reversal operator and their action on spinors is

$$\begin{aligned} \mathbf{T}\Phi_1(p, x) &= \gamma_5\Psi_1(p, x) \quad , & \mathbf{T}\Phi_2(p, x) &= i\mathbf{P}\Psi_2(p, x) \quad , \\ \tilde{\mathbf{T}}\Phi_1(p, x) &= \mathbf{P}\Psi_1(p, x) \quad , & \tilde{\mathbf{T}}\Phi_2(p, x) &= i\gamma_5\mathbf{P}\Psi_2(p, x) \quad , \end{aligned} \quad (44)$$

and on parity and chirality common eigenstates

$$\begin{aligned} \mathbf{T}\Psi_{+L} &= -\Phi_{-R} \quad , & \tilde{\mathbf{T}}\Psi_{+L} &= +\Phi_{-R} \quad , \\ \mathbf{T}\Psi_{-L} &= +\Phi_{+R} \quad , & \tilde{\mathbf{T}}\Psi_{-L} &= +\Phi_{+R} \quad , \\ \mathbf{T}\Psi_{+R} &= -\Phi_{-L} \quad , & \tilde{\mathbf{T}}\Psi_{+R} &= -\Phi_{-L} \quad , \\ \mathbf{T}\Psi_{-R} &= +\Phi_{+L} \quad , & \tilde{\mathbf{T}}\Psi_{-R} &= -\Phi_{+L} \quad , \end{aligned} \quad (45)$$

shows no linear combination with real coefficients of parity and chirality eigenstates is left invariant by action of \mathbf{T} and $\tilde{\mathbf{T}}$, therefore no conserved currents exists and no particles associated. The simultaneous application of charge conjugation, parity and \mathbf{T} or $\tilde{\mathbf{T}}$ is of fundamental importance

$$\mathbf{CPT} = -1 \quad , \quad \mathbf{CPT}\tilde{\mathbf{T}} = +\gamma_5\mathbf{P} \quad , \quad (46)$$

because the together with the expected $\text{diag}(-1, -1, -1, -1)$ we had the unexpected result $\gamma_5\mathbf{P} = \text{diag}(1, -1, 1, -1)$ is a consequence of charge conjugation, which change both chirality and parity. The set of 8 operators $\{1, \gamma_5, \mathbf{P}, \mathbf{C}, \gamma_5\mathbf{P}, \gamma_5\mathbf{C}, \mathbf{PC}$ and $\gamma_5\mathbf{PC}\}$ has 4 significant subsets listed below which span the algebra of split-quaternions, namely, linear combinations with real coefficients of four basis elements $\{1, e_1, e_2, e_3\}$ that satisfy the following product rules: $e_1^2 = -1, e_2^2 = e_3^2 = 1, e_1e_2 = e_3 = -e_2e_1$

$$\begin{aligned} \{1, \gamma_5\mathbf{C}, \mathbf{C}, \gamma_5\}, & & \{1, \mathbf{PC}, \mathbf{C}, \mathbf{P}\}, \\ \{1, \gamma_5\mathbf{C}, -\mathbf{P}, \gamma_5\mathbf{PC}\}, & & \{1, \mathbf{PC}, \gamma_5\mathbf{PC}, \gamma_5\}. \end{aligned} \quad (47)$$

Changing the base to $K_+ = \frac{1}{2}(e_1 + e_2)$, $K_- = \frac{1}{2}(e_1 - e_2)$ and $K_3 = \frac{1}{2}e_3$ we get the algebra of $SO(2, 1)$

$$[K_3, K_+] = +K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = 2K_3, \quad (48)$$

7 Conclusions

This result, that Dirac equation is equivalent with two entangled Proca fields with mass $2m$ and two constant massless fields, suggest that this representation unifies fermion and bosons fields, its algebra of 8 operators could be used for study of weak and strong interactions. While the gauge symmetry $U(1) \times SU(2) \times SU(3)$ of the Standard Model is exact only when the particles are massless, the set of 8 operators defined in our study have the mass built in, i.e. it require neither cumbersome mechanism with arbitrary chosen fields and parameters nor spontaneous breaking symmetry to generate mass. In our model, the solutions of the equations $S_1^\alpha \partial_\alpha S_2^\beta = 0$ and $S_2^\alpha \partial_\alpha S_1^\beta = 0$ gives the mass of the field $S_1^\alpha S_\alpha^1 = -m^2$ and $S_2^\alpha S_\alpha^2 = -m^2$. The algebra of entangled spin is $SO(2,1)$.

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