# Relationship Between The Ellipse Curve and The Sine Wave Using Circular Cylinder Section

**ABSTRACT**

We can create an elliptical curve by intersecting a plane with a cylinder at an angle. A circle is an ellipse whose major and minor axes are identical. This study aimed to investigate the relationship between the ellipse and the sine wave curve using a cylinder section. The study found that we can derive the formula for a sine wave curve from the cylinder section, the ellipse formula can come from the sine wave curve formula, and both the sine wave curve and the ellipse have the same perimeter.

Keywords: cylinder section; ellipse dilation; wave curve; ellipse formula;

## A. INTRODUCTION

An ellipse is a plane shape that is interesting to research. We can do several ways to get an ellipse curve. Crossing a circular cone with a plane creates an ellipse, as does transforming the coordinates of a circle (Rohman, 2022). Apart from that, ellipses are also related to circles. The article (Rohman & Jupri, 2019a) shows some properties of the relationship between ellipses and circles.

Lockhart (2012) and Wells (2018) describe an ellipse as a circle experiencing stretching that is pulled in opposite directions horizontally or vertically (Lockhart, 2012; Wells, 2018). In geometric terms, this stretching can be explained by dilation (Lockhart, 2012; Mazer, 2010). Figure 1 shows the horizontal stretching of a circle into an ellipse.

Archimedes used transformations for coordinates to transform a circle into an ellipse (Archimedes, 2010). Through this transformation, Archimedes got the equation and area of an ellipse. Lockhart (2019) showed that this is a dilation transformation of a circle so that it becomes an ellipse (Lockhart, 2012). This dilation does not produce anything regarding the relationship between the perimeter of a circle and an ellipse. We can find the area of the ellipse by using this dilation (Lockhart, 2012; Rohman & Jupri, 2019a).

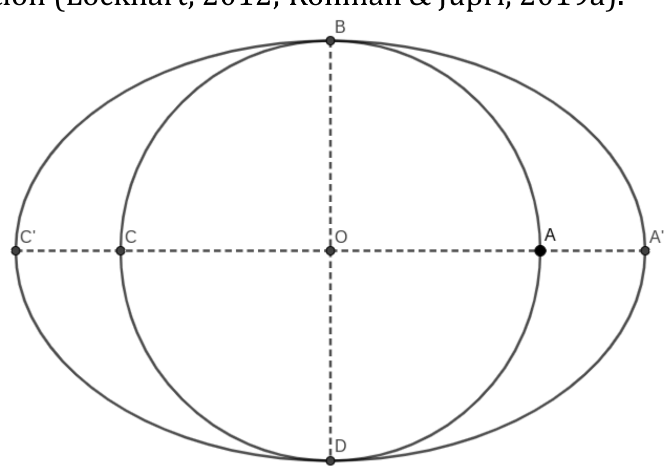


**Figure 1**. The Circle Stretches Horizontally In The Opposite Direction

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Dong (1821) and Germinal Pierre Dandelin (1794-1847) gave the idea that the result of cutting a cylinder by an inclined plane produces an ellipse curve (Bréard, 2019; Dwijayanti, 2014; Hilbert & Cohn-Vossen, 1991). Pierce Morton (1803 - 1859) discovered the focal-directive properties of ellipses from conical intersections from Dandelin's ideas. An ellipse's focus-directive characteristic is $m\overline{BF_1} + m\overline{BF_2} = m\overline{P_1P_2}$ with $m\overline{P_1P_2}$ always constant, and both $F_1$ and $F_2$ are the focal point of the ellipse (Alsina & Nelsen, 2015; Hilbert & Cohn-Vossen, 1991). This proof can show that the curve is an ellipse.

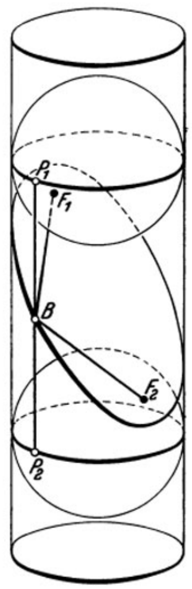


**Figure 2**. Source:Hilbert and Cohn-Vossen. Geometry and the Imagination. Chelsea Publishing Company

There are various ways to derive the formula for an ellipse apart from proving Pierce Morton's theory. One method involves using Cartesian coordinates to derive the formula for the ellipse resulting from a cylinder section. This method has been carried out by (Lockhart, 2012; Rohman & Jupri, 2019a). Another method is to use Affine Transformation, as demonstrated by (Brannan, 2007). Both methods result in the dilation of the circle's coordinates and produce an ellipse formula on the Cartesian coordinate plane.

Look at Figure 4 to understand the derivation of the formula by Rohman (2019). The derivation of this formula uses congruence, namely ΔJKL~ΔMFC. If $m\overline{MC} = b$ as the length mayor of ellipse *ABCD* and $m\overline{MF} = a$ as the length radius of circle *BDEF* but as the length minor of ellipse *ABCD.* Then, compare the distance between $m\overline{JL}/m\overline{JK} = b/a$ which is always constant. Scimone (2015) refers to this as the golden ratio ellipses, which have several properties related to their comparison (Rashed, 2014; Rohman & Jupri, 2019a; Scimone, 2015). By using $b/a$ and placing the ellipse *BFDE* and circle *ABCD* in Cartesian coordinates with the same center, we will get the ellipse formula.

By using the through cylinder section, we can transform the ellipse into the sine (Ferréol, 2023; García, 2018). During the study of Pitot (1724), he found that when the cylinder containing an ellipse rolls on a flat plane, its trace takes the form of a sine curve.

The study of the relationship between the ellipse and sine curve began about six years later (Boyer, 2012). On the site https://en.etudes.ru/models/sine-wave/, there is an animation of how the trace of the ellipse curve takes the form of a sine curve, and https://www.geogebra.org/m/njdk3fs9 (Mathematical Etudes Foundation, 2023; Mentrard, 2023). Figure 3 shows that the cylinder section is a sine wave on a plane.

**Figure 3**. Curved surface span (source: https://medium.com/@mail.nirmal.r/so-an-ellipse-is-a-sine-wave-in-disguise-312b32026d4)

We can map a curved surface containing an ellipse curve into Cartesian coordinates as in Figure 5. Note, that Figure 4 shows that the cylinder base or cylinder cover is circular. If we expand the curved surface into a rectangle, we get the rectangle whose length is equal to the circle's perimeter, and its width is the height of the cylinder. Figure 5 shows how the curved surface changes to the rectangle, and the ellipse curve becomes a sine wave.

Proving the relationship between the sine wave and the ellipse can be done by: The first way, the ellipse equation converted into a sine wave equation (Kumar, 2021). The second method involves transforming the coordinates of an elliptical point resulting from the intersection of a cylinder and a plane from 3D to 2D. To simplify, imagine flattening a curved surface onto a plane (Alsina & Nelsen, 2015; Toth, 2021).

Even though this study is not new, it is relatively simple to find the sine formula from this cylinder section. In this research, the derivation of the sine formula involves the similarity of triangles and trigonometry. Figure 4 shows right-angled triangles formed by the intersection of an ellipse curve and a circle. Meanwhile, the first step is to map the sine formula into Cartesian coordinates to get the ellipse formula from the sine formula. In this way, we get the ellipse formula in Cartesian coordinates.

Based on previous research, this article will describe how to derive the sine wave equation using the cylinder section and curved surface. Additionally, it will explain how to obtain the ellipse equation from the sine equation, and how to prove that the length of the wave curve is equal to the ellipse's perimeter.

## B. METHODS

For this particular study, I conducted a literature review. During this process, I carefully selected several articles and books that were relevant to my study. The primary objective of this study was to provide direct evidence of the wave formula from cylinder sections using triangular congruence and trigonometry. As part of the research, I also attempted to derive the ellipse formula from the wave formula in the Cartesian coordinate plane. Furthermore, after conducting an integral analysis, I established that the wave curve and the ellipse have the same length. Overall, the study aimed to provide a detailed and comprehensive analysis of the wave formula and its relationship with the ellipse formula.

## C. RESULT

Look again at Figure 4. We can always form $\Delta MFC \sim \Delta JKL$ between the space bounded by the circle BDEF and the ellipse ABCD. The corresponding sides of the two triangles are parallel: $\overline{JK} \parallel \overline{MF}$, $\overline{KL} \parallel \overline{FC}$, and $\overline{JL} \parallel \overline{MC}$. When two triangles have parallel sides, their corresponding angles are the same. We can say that $\Delta JKL \sim \Delta MFC$ by using the Angle-Angle-Angle Theorem. Proof of theorems related to the similarity of triangles can be seen in (Fuat et al., 2020; Jupri et al., 2021; Meilantifa, Herfa M.D. Sewardini, Mega Teguh Budiarto,Janet T. Many et al., 2018).

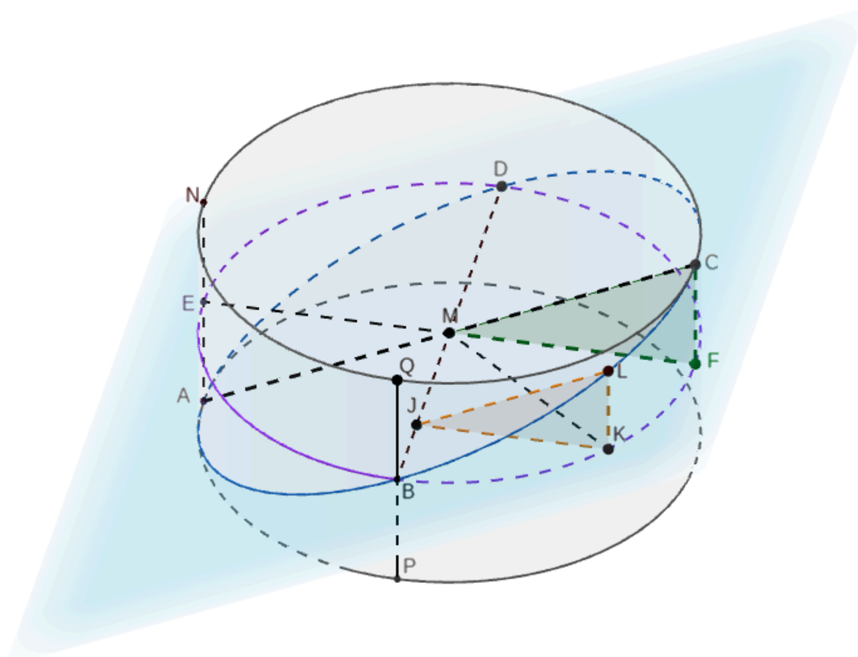


**Figure 4**. Cylinder Section with an Inclined Plane

### 1. Sine Wave

Figure 4 shows that $\overline{KL}$ can move along circle BFDE and ellipse BCDA. We get that $\Delta JKL \sim \Delta MFC$. The corresponding side lengths of ΔJKL and ΔMFC are

$m\overline{FC} = h$ corresponds to $m\overline{KL} = h'$

$m\overline{MC} = b$ corresponds to $m\overline{JL} = b'$

$m\overline{MF} = a$ corresponds to $m\overline{JK} = a'$

By using the Pythagorean theorem on $\Delta MFC$, we obtain the relationship $m\overline{MC}^2 = m\overline{MF}^2 + m\overline{FC}^2$. We can write as

$$b^2 = a^2 + h^2 \quad (1)$$

Meanwhile, in $\Delta JKL$, we get a relationship $m\overline{JL}^2 = m\overline{JK}^2 + m\overline{KL}^2$. We can write as

$$b'^2 = a'^2 + h'^2 \quad (2)$$

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Note that $m\overline{KL} = h'$ is the distance between the ellipses $ABCD$ with the circle $BDEF$. Based on equation (2) we obtain

$$h'^2 = b'^2 - a'^2 \quad (3)$$

We get $m\overline{KL}$ maximum occurs when $h' = m\overline{KL} = m\overline{CF} = h$ .

Note if $m\angle BMK = \theta$ then we get a relationship $m\overline{JK}= m\overline{MK}\ \sin(\theta)$, Because $m\overline{JK} = a'$ whereas $m\overline{MK} = m\overline{MF} = a$ is the length of the radius of the circle $BDEF$ for

$$a'= a\ \sin(\theta) \quad (4)$$

By using the congruence theorem, because $\Delta JKL \sim \Delta MKF$ then we get the appropriate side comparison

$$h'/h = a'/a \quad (5)$$

Substitute equation (4) into equation (5)

$$h'/h = a\ sin(\theta)/a \quad (6)$$

and we get the distance between the ellipses $ABCD$ with a circle $BDEF$ that is

$$h' = h\ sin\ (\theta) \text{ with } 0 \leq \theta \leq 2\pi \quad (7)$$

If we open the curved surface by cutting along $\overline{PQ}$ and then spread it out into a plane shape, it will look like in Figure 5 below. We get the equation $h' = h\ sin(\theta)$, which is the height of the sine wave along the circle BFDE.

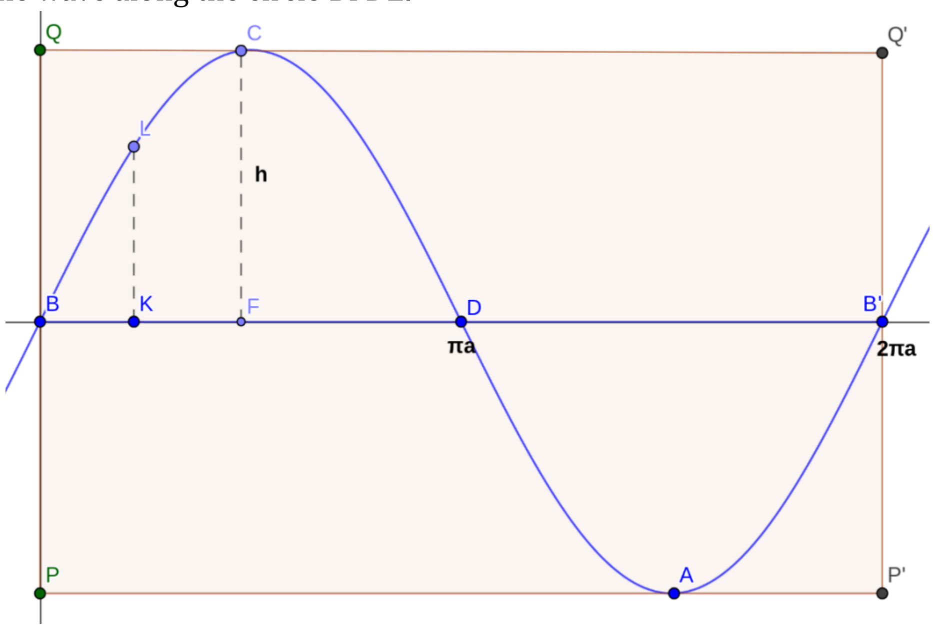


**Figure 5**. Elliptical Curves on Curved Surface Spans

Look at Figure 5. We can write the wavelength equation as follows: $l = a\theta$ with $0 \leq \theta \leq 2\pi$. As we already know, $m\overline{BB'}$ is the circle's perimeter, so get $m\overline{BD} = \pi a$ and $m\overline{BF} = \pi a/2$. The wave with length $m\overline{BB'}$ has a maximum height of h, which occurs when the angle is $\theta = \pi/2$.

If we increase $a$, the sine wave will appear smoother, provided that the maximum height, $h$, remains unchanged. If $a$=$b$ then the wave is a straight line because $h^2 = b^2 - a^2 = 0$. If $a = h$ for $b^2 = h^2 + a^2 = h^2 + h^2 = 2h^2$. We also get $h = b/\sqrt{2}$.

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## 2. The Ellipse Curve

If we put the ellipse *ABCD* and circle *BFDE* on Cartesian coordinates with the same center point, M(0,0), we get a pair of parallel and coincident segments, namely $\overline{JK}//\overline{JL}$ and $MF\overline{MF}//\overline{MC}MC$, as seen in Figure 6 below.

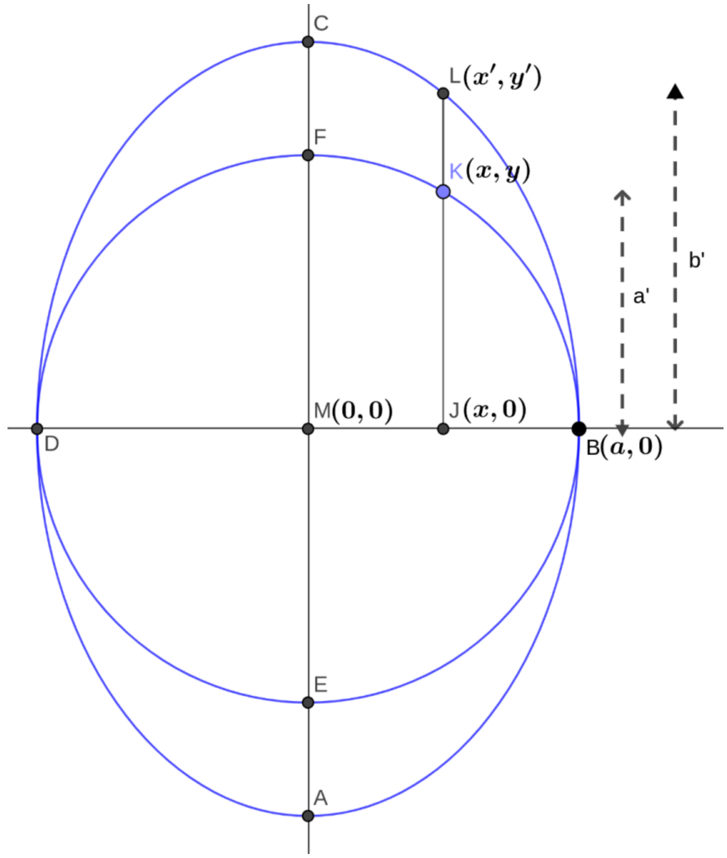


**Figure 5**. Circle *BFDE* And Ellipse *ABCD* in Cartesian Coordinate Plane

Based on Figure 4, we have found that the equation $b' = \sqrt{h'^2 + a'^2}$. Because $h' = h\, sin(\theta)$ then

$$b' = \sqrt{h'^2 + a'^2} = \sqrt{(h^2 sin^2\theta + a^2 sin^2\theta)} = \sqrt{(h^2 + a^2)} sin\theta \qquad (8)$$

And because $h = \sqrt{(b^2 - a^2}$ then we get equation (8) to be

$$b' = \sqrt{(h^2 + a^2)} sin\theta = \sqrt{(b^2 - a^2 + a^2)} sin\theta = b\, sin\theta$$

Let's write it simply

$$b' = b\, sin\theta \qquad (9)$$

Look again at Figure 4, because $m\angle BMK = \theta$ then we get the form of equation (4) to be $sin(\theta) = m\overline{JK}/m\overline{MK}$ or we write as

$$sin(\theta) = a'/a \qquad (10)$$

We substitute equation (10) into equation (9) and we get

$$b' = b(a'/a)$$

in other words

$$b' = (b/a)a' \tag{11}$$

Looking at Figure 6, we get that $a' = m\overline{JK}$ which we get from the point distance $J(x, 0)$ the $K(x, y)$ that is

$$m\overline{JK} = a' = \sqrt{(y - 0)^2 + (x - x)^2}$$

in other words

$$m\overline{JK} = a' = \sqrt{y^2} \tag{12}$$

And $b' = m\overline{JL}$ namely the distance of the point $J(x, 0)$ to $L(x', y')$ obtained from

$$m\overline{JL} = b' = \sqrt{(y' - 0)^2 + (x' - x)^2}$$

Because $x' = x$ then

$$m\overline{JK} = a' = \sqrt{(y - 0)^2 + (x - x)^2} \tag{13}$$

We input equations (12) and (13) into equation (11) and we obtain the coordinate relationship between the ellipse and the circle, namely $m\overline{JL} = b' = \sqrt{y'^2}$

in other words

$$y' = (b/a)y \tag{14}$$

As we already know that $y^2 = a^2 - x^2$ and we substitute it into the equation $y'^2 = (b/a)^2 y^2$, we obtain

$$y'^2 = (b/a)^2 (a^2 - x^2)$$

Because $x' = x$ then $y'^2/b^2 = 1/a^2 (a^2 - x'^2)$. We can change the form of the equation to:

$$y'^2/b^2 = 1 - x'^2/a^2$$

in other words

$$y'^2/b^2 + x'^2/a^2 = 1 \tag{15}$$

Thus it is proven that by using the sine wave equation, we can derive the ellipse equation.

## 3. Length of the ellipse curve and sine wave

We can calculate the length of a sine wave curve using an integral that equals the ellipse's perimeter. When an elliptical curve stretches into a sine curve, it does not change in length. We can get this proof: if the measured length of the sine curve is $L$

$$L = \int_0^{2\pi} \sqrt{(dh/d\theta)^2 + (dx/d\theta)^2} d\theta \tag{16}$$

Because $h' = h \sin(\theta)$ the $dh/d\theta = h\cos(\theta)$, and $x = \theta a$ then $dx/d\theta = a$. Thus

$$L = \int_{0}^{2\pi} \sqrt{(h\, sin\,(\theta))^2 + (a)^2} d\theta \qquad (17)$$

Because $h = \sqrt{b^2 - a^2}$ then

$$L = \int_{0}^{2\pi} \sqrt{(b^2 - a^2)\, sin^2\theta + a^2} d\theta$$

or

$$L = \int_{0}^{2\pi} \sqrt{(b^2 sin^2\theta + a^2 - a^2 sin^2\theta} d\theta$$

because $a^2 - a^2 sin^2\theta = a^2(1 - sin^2\theta) = a^2 cos^2(\theta)$ then

$$L = \int_{0}^{2\pi} \sqrt{b^2 sin^2\theta + a^2 cos\theta^2} d\theta \qquad (18)$$

We can see that L is the length of the sine curve equal to the ellipse perimeter.

## D. DISCUSSION

This study explores the relationship between the ellipse and the sine curve. Unlike previous studies using Cartesian coordinates, this work derives the sine wave formula from the cylinder section. We obtain the wave height, $h\, sin\theta$, and the wavelength, $a\theta$ with $0 \leq \theta \leq 2\pi$. However, both previous research and this study involve mapping an ellipse from the spatial dimension into a sine curve in a flat dimension (Alsina & Nelsen, 2015; Boyer, 2012; Toth, 2021).

In this study, the ellipse formula is derived from the sine wave formula resulting from mapping the ellipse, which is the intersection of the cylinder. Several previous studies have shown directly the derivation of ellipse formulas from cylindrical intersections as in the articles (Alsina & Nelsen, 2015; Boyer, 2012; Hilbert & Cohn-Vossen, 1991; Rashed, 2014; Rohman & Jupri, 2019b). Kumar used an ellipse formula to generate a sine wave (Kumar, 2021). Everything proves that ellipse and sine curves are the same but exist in different planes.

This study also shows that the length of the sine wave and the elliptical curve are of equal length for $0 \leq \theta \leq 2\pi$ as evidenced by the same integral between them to find their circumferences (Weir et al., 2008). We can transform a curve into another curve based on the plane it occupies without altering its size. However, finding the exact integral value is computationally challenging (Weir et al., 2008).

If we could change the shape of an ellipse curve into a straight line on a plane, we could easily calculate its length. To calculate the distance traveled for one rotation of an object, we can roll an elliptical shape on a plane and measure the trace left behind. However, this method has limitations and may lead to calculation errors.

We can easily explain the concept of transforming a circle into an ellipse or vice versa by applying coordinate dilation, as described in various articles (Archimedes, 2010; Rohman, 2022; Rohman & Jupri, 2019b). We can transform an ellipse into a circle by stretching and compressing it. According to Rohman (2022), this method aims to maintain the circumference of the ellipse while transforming it into a circle (Rohman, 2022). However using this method only yields the perimeter limits of the ellipse (E. Pfiefer, 1988).

Although the derivation of this formula is nothing new, this research provides a simple method of deriving the wave formula directly through a tube slice. This problem can be solved using the similarity of triangles instead of Cartesian coordinates. The formula for the ellipse can derived from the wave formula. This approach differs from earlier approaches presented in the works (Hilbert & Cohn-Vossen, 1991; Kumar, 2021). The wave and the ellipse curve are equivalent, differing only in the plane they occupy.

## E. CONCLUSION AND SUGGESTIONS

We can create an ellipse by cylinder section and transform it into a wave curve. First, We stretch the curved surface to a flat, rectangular shape. By examining the similarity of triangles on the cylinder section, we can derive the formula for wave curves. From this formula, we can obtain the ellipse formula. Interestingly, the length of the ellipse and the sine wave on a plane are equal.

For future research, one possible method to calculate the perimeter of an ellipse is to transform it into a circle while preserving the perimeter of the circle equal to the perimeter of the ellipse. Alternatively, we can transform the ellipse into another curve or line. For instance, we can map an ellipse to a sine wave and consider it as mapping to a straight segment. This way, we can measure the length of the ellipse by measuring the length of the segment.

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