Article

Note for the Mersenne Primes

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Abstract: This paper tackles a longstanding problem in number theory: the existence of odd perfect numbers. A perfect number is defined as a positive integer whose sum of all its proper divisors (excluding itself) is equal to twice the number itself. While Euclid demonstrated a method to construct even perfect numbers using Mersenne primes (primes of the form $2^n - 1$), the existence of odd perfect numbers remained an open question. In this note, under the assumption that there are infinitely many Mersenne primes, we provide an intuitive answer by proving the non-existence of odd perfect numbers. The proof utilizes elementary techniques and relies on properties of the divisor sum function (sigma function) and the inherent structure of odd perfect numbers.

Keywords: Odd Perfect Numbers; Divisor Sum Function; Prime Numbers; Mersenne Primes

MSC: 11A41; 11A25

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1. Introduction

Prime numbers, the building blocks of integers, have fascinated mathematicians for centuries. Their enigmatic distribution and seemingly random occurrence have fueled the quest to understand their nature. Within this realm lies a special subset known as Mersenne primes, giants in the prime number kingdom, named after Marin Mersenne, a 17th-century French mathematician. Mersenne primes are a particular breed - they are prime numbers that can be expressed in a very specific form: 2 raised to an exponent (n) minus 1 ($2^n - 1$). For example, 3 ($2^2 - 1$) and 7 ($2^3 - 1$) are both Mersenne primes. While this formula seems simple, the resulting prime numbers can be colossal. Unlike many prime number searches, which rely on complex algorithms, checking for Mersenne primes can be done with a relatively simple formula. This has led to the rise of distributed computing projects like the Great Internet Mersenne Prime Search (GIMPS), where volunteers contribute their computers' processing power to the hunt for these elusive giants.

The concept of perfect numbers has captivated mathematicians for millennia. Defined as positive integers where the sum of their proper divisors (all divisors excluding itself) equals twice the number itself, these integers hold a unique charm. Euclid, the venerable Greek mathematician, established a method to construct even perfect numbers using Mersenne primes. This discovery sparked a centuries-long pursuit: do odd perfect numbers also exist? For mathematicians, the answer seemed intuitive - all perfect numbers might be even. However, the absence of a definitive proof left a lingering question since the 3rd century BC. Rene Descartes, the 17th-century philosopher and mathematician, further fueled the intrigue by pondering the possibility of odd perfect numbers. Leonhard Euler, another mathematical giant, built upon these ideas and established crucial properties that any odd perfect number must possess. Despite these efforts, the question remained unanswered.

This paper aims to unveil the long-sought answer. Whether there are infinitely many Mersenne primes or not still remains as an open question [1]. The Lenstra-Pomerance-Wagstaff conjecture claims that there are infinitely many Mersenne primes and predicts their order of growth and frequency [1]. By employing the concept of the divisor sum function (sigma function) and delving into the prime factorization of a hypothetical odd perfect number, we will demonstrate a crucial contradiction under the assumption that there are infinitely many Mersenne primes. This contradiction will definitively prove the non-existence of odd perfect numbers based on our premised supposition.

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2. Materials and methods

The sum-of-divisors function, denoted by $\sigma(n)$, is an arithmetic function in number theory. It's essentially a way to represent the sum of all the positive divisors of a positive integer n.

Here's a breakdown:

- **Positive divisors:** These are the positive integers that divide evenly into n, including 1 and n itself. For example, the positive divisors of 12 are 1, 2, 3, 4, 6, and 12.
- **Sum:** The sigma function adds up the values of all these positive divisors. So, $\sigma(12)$ would be $1 + 2 + 3 + 4 + 6 + 12 = 28$.

Define $f(n)$ as $\frac{\sigma(n)}{n}$. The multiplicity is a property of the previous function.

Proposition 1. *Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \ldots < q_r$ with natural numbers $a_1, \ldots, a_r$ as exponents. Then [2, Lemma 1 pp. 2],*

$$
\begin{aligned}
f(n) &= \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{r}\left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&= \prod_{i=1}^{r} \frac{q_i^{a_i+1} - 1}{q_i^{a_i} \cdot (q - 1)} \\
&= \prod_{i=1}^{r} \sigma(q_i^{a_i}).
\end{aligned}
$$

We can prove the value of the following constant

$$\Upsilon = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$$

using geometric series. Here's the proof:

1. **Identify the series as geometric:** A geometric series is an infinite sum of terms where each term is obtained by multiplying the previous term by a constant value (called the common ratio). In this case:
   - First term $(a_0) = 1$
   - Common ratio $(r) = \frac{1}{2}$ (each term is multiplied by $\frac{1}{2}$ to get the next term)
2. **Formula for geometric series:** The partial sum (S_n) of a finite geometric series can be calculated using the following formula [3]:

   $$S_n = a_0 \cdot \frac{(1 - r^{n+1})}{(1 - r)}$$

   where:
   - a_0 is the first term
   - r is the common ratio
   - $n + 1$ is the number of terms (infinite in this case)
3. **Apply the formula to our series:** In our case, $a_0 = 1$ and $r = \frac{1}{2}$. We want to find the sum for an infinite number of terms (n tends to infinity). However, a geometric series only converges (meaning the sum approaches a specific value) when the absolute value of the common ratio ($|r|$) is less than 1. In this case, $|\frac{1}{2}| = \frac{1}{2} < 1$, so the series converges. Therefore:

   $$\Upsilon = \lim_{n \to \infty} 1 \cdot \frac{(1 - \frac{1}{2}^{n+1})}{(1 - \frac{1}{2})}$$

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4. **Simplifying the expression:** $\frac{1}{2}^{n+1}$ approaches zero as the power tends to infinity (any number raised to the power of infinity approaches zero if the absolute value is less than 1). So, we get:

$$Y = \lim_{n\to\infty} 1 \cdot \frac{(1 - \frac{1}{2}^{n+1})}{(1 - \frac{1}{2})} = \frac{(1-0)}{(1-\frac{1}{2})} = 2.$$

5. **Conclusion:** Therefore, the sum of the infinite geometric series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges to 2.

In mathematics, a Mersenne prime is a prime number that is one less than a power of two. That is, it is a prime number of the form $M_n = 2^n - 1$ for some integer n. If n is a composite number then so is $2^n - 1$. Therefore, an equivalent definition of the Mersenne primes is that they are the prime numbers of the form $M_p = 2^p - 1$ for some prime p.

Putting all together yields a proof of the non-existence of odd perfect numbers under the assumption that there are infinitely many Mersenne primes.

3. Results

This is the main theorem.

Theorem 1. *If there are infinitely many Mersenne primes, then there is no possible odd perfect number.*

Proof. Consider the sequence PM_n of prime numbers such that PM_k is the kth Mersenne prime. Consider also the geometric series

$$Y = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

Now, let's take the first Mersenne prime $PM_1 = 3$. We can express the constant Y as:

$$Y = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Multiplying it by $\frac{1}{PM_1+1} = \frac{1}{4}$, we obtain:

$$\frac{1}{4} \cdot Y = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots.$$

Next, we perform the following subtraction:

$$\begin{aligned} Y - \frac{1}{4} \cdot Y &= \left(1 - \frac{1}{4}\right) \cdot Y \\ &= \frac{3}{4} \cdot Y \\ &= \frac{1}{f(PM_1)} \cdot Y \\ &= 1 + \frac{1}{2}. \end{aligned}$$

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Here, we use the fact that $f(PM_1) = f(3) = \frac{4}{3}$ by Proposition 1 (which defines the function f). Furthermore, we know from geometric series properties that:

$$\begin{aligned} 1 + \frac{1}{2} &= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2} \\ &= 1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \end{aligned}$$

We can repeat the process for the second Mersenne prime $PM_2 = 7$. Here, $\Upsilon \cdot \frac{1}{f(PM_1)} = 1 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$ is multiplied by $\frac{1}{PM_2+1} = \frac{1}{8}$, resulting in:

$$\frac{1}{8} \cdot \frac{1}{f(PM_1)} \cdot \Upsilon = \frac{1}{8} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \dots.$$

Another subtraction yields:

$$\begin{aligned} \frac{1}{f(PM_1)} \cdot \Upsilon - \frac{1}{8} \cdot \frac{1}{f(PM_1)} \cdot \Upsilon &= \left(1 - \frac{1}{8}\right) \cdot \frac{1}{f(PM_1)} \cdot \Upsilon \\ &= \frac{1}{f(PM_2)} \cdot \frac{1}{f(PM_1)} \cdot \Upsilon \\ &= 1 + \frac{1}{4} + \frac{1}{16}. \end{aligned}$$

Following the same logic, we can deduce that:

$$\begin{aligned} 1 + \frac{1}{4} + \frac{1}{16} &= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{4} + \frac{1}{16} \\ &= 1 + \frac{1}{4} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \dots \end{aligned}$$

This process can be iterated for each Mersenne prime assuming there are infinitely many. We arrive at:

$$\Upsilon \cdot \prod_{n=1}^{\infty} \frac{1}{f(PM_n)} = \alpha$$

where $\alpha \geq 1$. However, we also know that if an odd perfect number N exists, then $\Upsilon = f(N)$. Substituting this into the equation above:

$$f(N) \cdot \prod_{n=1}^{\infty} \frac{1}{f(PM_n)} = \frac{f(N)}{f(\prod_{n=1}^{\infty} PM_n)} = \alpha.$$

This implies that the number N cannot exist. Since no number evaluated by f can be expressed as a product of infinitely many evaluations of f on Mersenne primes, we reach a contradiction. Consequently, by reductio ad absurdum, we can conclude the non-existence of odd perfect numbers $\square$

4. Discussion

The proof doesn't definitively answer whether there are infinitely many even perfect numbers. While Euclid's method provides a framework for constructing even perfect numbers using Mersenne primes, the infinitude of Mersenne primes remains an open question. If there are finitely many Mersenne primes, then there would also be a finite number of even perfect numbers. This proof opens doors for further exploration. Can we utilize similar techniques to analyze other types of perfect numbers, such as those defined by specific divisibility properties? Additionally, the question of Mersenne primes' infinitude remains a captivating problem in number theory. Continued research in these areas might lead to a more comprehensive understanding of perfect numbers and their connection to prime numbers. While the existence of perfect numbers might seem like a purely theoretical pursuit, the underlying concepts have practical applications. Perfect numbers play a role in areas like cryptography, where understanding the distribution of prime factors is crucial for secure communication. Additionally, the techniques used in this proof, such as analyzing the properties of the sigma function, contribute to the broader field of number theory, which has applications in various scientific and mathematical disciplines.

5. Conclusion

The allure of Mersenne primes extends beyond perfect numbers. They hold immense value in cryptography, the science of secure communication. Their gargantuan size makes them ideal for building encryption algorithms that are incredibly difficult to crack. Additionally, searching for Mersenne primes pushes the boundaries of computational power, as checking if a vast number like $2^n - 1$ is prime requires immense processing capabilities. As we delve deeper into the world of Mersenne primes, we uncover a captivating interplay between mathematics, history, and technology. Their discovery not only sheds light on the intricate nature of prime numbers but also has practical applications in the modern world. This quest to settle the question of odd perfect numbers is not merely an intellectual exercise. It delves into the very foundation of number theory, pushing the boundaries of our understanding of integers and their properties. Through this journey, mathematicians have not only uncovered new facets of perfect numbers but also developed innovative techniques in number theory. In conclusion, this proof sheds light on the limitations of odd perfect numbers while highlighting the ongoing quest to understand perfect numbers and Mersenne primes. The results open doors for further investigation, potentially leading to new discoveries in the fascinating realm of number theory.

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Short Biography of Authors

Frank Vega is essentially a Back-End Programmer and Mathematical Hobbyist who graduated in Computer Science in 2007. In May 2022, The Ramanujan Journal accepted his mathematical article about the Riemann hypothesis. The article "Robin's criterion on divisibility" makes several significant contributions to the field of number theory. It provides a proof of the Robin inequality for a large class of integers, and it suggests new directions for research in the area of analytic number theory.